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BANACH HYPERBOLICITY AND EXISTENCE OF HOLOMORPHIC MAPS IN INFINITE DIMENSION

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Introduction

Finite dimensional hyperbolic analysis has been investigated by several authors, in particular by Kobayashi, Kwack and recently by Noguchi, Zaidenberg. The obtained results have been used to study some important problems of complex analysis and number theory. In recent years Bart, Lempert, Vesentini have obtained important results concerning the hyperbolicity of convex domains in Banach spaces. The aim of the present paper is to study some questions concerning the extension of holomorphic maps in Banach hyperbolic analysis.

In Section 1 we will solve the Kobayashi problem [7] for proper holomorphic maps between Banach analytic spaces. Applying Brody's characterization of compact hyperbolic spaces, Urata [10] and independently Zaidenberg [12] have solved this problem in the finite dimensional case. In Section 2 it is shown that for a convex domain in a Banach space the Kwack extension theorem holds if and only if this domain contains no complex lines. Moreover, in the finite dimensional case we show that the above statement is equivalent to the H^∞-extendibility. Finally, in Section 3 we extend results of Hirschowitz [5] and Sibony [9] on the extension of holomorphic maps with values in complete C-spaces. We prove that every holomorphic map from a Riemann domain Ω over a topological vector space into a Banach manifold, modelled by open sets in an injective Banach space which is complete for the distance of Caratheodory, can be extended holomorphically to $\widehat{\Omega}_\infty$, the envelope of holomorphy of Ω for the set $H^\infty(\Omega)$ of bounded holomorphic functions on Ω.

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Let X be a Banach analytic space in the sense of Mazet [8]. By d_X we denote the Kobayashi pseudodistance on X. Unlike the finite dimensional case, there exists a Banach manifold X on which the Kobayashi pseudodistance is a distance but it does not define the topology in the manifold X. We say that X is hyperbolic if d_X is a distance defining the topology of X.

1. Inverse invariance of Banach hyperbolicity.

In this section we investigate the inverse invariance of Banach hyperbolicity and in particular, the Kobayashi problem [7]. First we prove the following theorem which extends a result of Eastwood [4] to the Banach case.

THEOREM 1.1. *Let $\theta : X \to Y$ be a holomorphic map between Banach analytic spaces. If Y is hyperbolic and every point of Y has a neighbourhood V such that $\theta^{-1}(V)$ is hyperbolic, then X is also hyperbolic.*

PROOF: Let $\{x_n\} \subset X$ and $d_X(x_n, x_0) \to 0$, $x_0 \in X$. We have to prove that $x_n \to x_0$. Since Y is hyperbolic and $d_Y(\theta x_n, \theta x_0) \leq d_X(x_n, x_0)$ it follows that $\{\theta x_n\}$ converges to θx_0. Put $y_0 = \theta x_0$. By the hypothesis we can find a neighbourhood V of y_0 such that $\theta^{-1}(V)$ is hyperbolic. On the other hand, since d_Y defines the topology of Y, there exists a neighbourhood W of y_0 such that $d_Y(W, \partial V) > 0$. Then there exists $\delta > 0$ such that $f(\delta\Delta) \subset V$ for every holomorphic map f from Δ into Y such that $f(0) \in W$, where Δ denotes the open unit disc in C. We may assume that the neighbourhood W has the form

$$W = \{y \in Y : d_Y(y_0, y) < r\}$$

and $x_n \in \theta^{-1}(W)$ for all $n \geq 1$.

Put

$$W' = \{y \in Y : d_Y(y_0, y) < r/2\}$$

To prove that $d_{\theta^{-1}(W)}(x_n, x_0) \to 0$ and hence $x_n \to x_0$, we only need to prove that there exist positive numbers c, s such that

$$d_X(p, q) \geq \min\{s, c d_{\theta^{-1}(W)}(p, q)\} \quad \text{for all} \quad p, q \in \theta^{-1}(W')$$

Consider a holomorphic chain joining p and q: $\{f_i\}_{i=1}^k$, $f_i : \Delta \to X$ are holomorphic, $f_i(0) = p_{i-1}$, $f_i(a_i) = p_i$ $i = 1, ..., k$, where $p_0 = p$, $p_k = q$; $a_1, ..., a_k \in \Delta$. There are only two cases :

(i) $p_j \notin \theta^{-1}(W)$ for some $j = 1, ..., k$. Then

$$\sum_{i=1}^{k} d_\Delta(0, a_i) \geq \sum_{i=1}^{k} d_Y(\theta p_{i-1}, \theta p_i) \geq \sum_{i=1}^{j} d_Y(\theta p_{i-1}, \theta p_i)$$
$$\geq d_Y(\theta p, \theta p_j) \geq d_Y(\theta p_j, y_0) - d_Y(\theta p, y_0) \geq r - r/2 = r/2.$$

(ii) $p_0, ..., p_k \in \theta^{-1}(W)$. Then $\theta f_i(\delta\Delta) \subseteq V$ for all $i = 1, .., k$. If $a_j \notin (\delta/2)\Delta$ for some $j = 1, ..., k$, we have

$$\sum_{i=1}^{k} d_\Delta(0, a_i) \geq d_\Delta(0, \delta/2).$$

If $a_i \in (\delta/2)\Delta$ for $i = 1, ..., k$, we find $c > 0$ such that

$$d_\Delta(y, z) \geq c d_{\delta\Delta}(y, z) \quad \text{for all} \quad y, z \in (\delta/2)\Delta.$$

Thus

$$\sum_{i=1}^{k} d_\Delta(0, a_i) \geq c \sum_{i=1}^{k} d_{\delta\Delta}(0, a_i) \geq c \sum_{i=1}^{k} d_{\theta^{-1}(W)}(f_i(0), f_i(a_i))$$
$$= c \sum_{i=1}^{k} d_{\theta^{-1}(W)}(p_i, p_{i-1}) \geq c d_{\theta^{-1}(W)}(p, q).$$

So there exist $c, s > 0$ having the required property.

We say that a map $\theta : X \to Y$ is locally proper if for every $x \in X$, there exist two neighbourhoods U and V of x and $\theta(x)$ respectively such that $\theta : U \to V$ is proper. We note that in the case where $\dim X < \infty$ and θ has discrete fibers, θ is locally proper.

THEOREM 1.2. *Let $\theta : X \to Y$ be a locally proper holomorphic map with discrete fibers. If Y is hyperbolic, then X is also hyperbolic.*

PROOF: Let $d_X(x_n, x) \to 0$. It follows from the hyperbolicity of Y that $\{\theta(x_n)\}$ converges to θx_0. Assume that $x_n \not\to x_0$. Take two neighbourhoods U

and V of x and $\theta(x)$ respectively such that $\theta : U \to V$ is proper. Without loss of generality we may assume that $x_n \notin \overline{U}$ $\forall n \geq 1$ and $\theta^{-1}(\theta(U)) \cap \theta^{-1}(\theta(x)) = \{x\}$. Let W be a neighbourhood of x such that $\overline{W} \subset U$. Since $x_n \notin \overline{U}$ there exists $\tilde{x}_n \in \partial W \subset U$ such that

$$d_X(\tilde{x}_n, x) \leq d_X(x_n, x) \to 0.$$

This yields

$$\theta(\tilde{x}_n) \to \theta(x) \quad \text{as} \quad n \to \infty.$$

By the property of $\theta : U \to V$, one can conclude that $\{\tilde{x}_n\}$ contains a subsequence $\{z_n\}$ converging to $z \in \partial W$. It is obvious that $\theta(z) = \theta(x)$. But this is impossible because of the relation

$$\theta^{-1}(\theta(U)) \cap \theta^{-1}(\theta(x)) = \{x\}.$$

The main result of this section is the following

THEOREM 1.3. *Let $\theta : X \to Y$ be a proper holomorphic map from a Banach manifold X having C^1-partitions of unity into a Banach analytic space Y. Assume that $\theta^{-1}(y)$ is hyperbolic for every $y \in Y$ and that Y is hyperbolic. Then X is also hyperbolic.*

Cover X by a locally finite system of coordinates $\{(U_i, \varphi_i)\}$. Let $\{V_i\}$ be an open cover of X such that $V_i \subset U_i$, dist $(V_i, \partial U_i) > 0$ and $\varphi_i(V_i)$ is isomorphic to a ball in a Banach space for every i. By the hypothesis there exists a C^1-partition of unity $\{h_i\}$ such that $h_i = 1$ on V_i and supp $\{h_i\} \subset U_i$ for every i. Let $\pi : TX \to X$ be the tangent bundle of X. For each $u \in TX$, put

$$||u|| = \sum h_i(\pi u)||D\varphi_i(\pi u)(u)||$$

By ρ_X we denote the integral distance on X associated to $||\cdot||$.

LEMMA 1.4. *ρ_U defines the topology in U, where U is the unit ball in a Banach space B.*

PROOF: We have

$$\begin{aligned} ||x-y|| &= \sup\{|x^*(x)-x^*(y)| : x^* \in B^*,\ ||x^*|| \le 1\} \\ &\le 2\sup\{|f(x)-f(y)| : f \in H^\infty(2U),\ ||f|| \le 1\} \\ &= C_{2U}(x,y), \end{aligned}$$

where B^* denotes the dual space of B and C_{2U} denotes the Caratheodry distance on $2U$. On the other hand, for the differential Caratheodory metric γ_{2U} of $2U$ we have by [11]

$$\begin{aligned} C_{2U}(x,y) &\le \inf\{\int_0^1 \gamma_{2U}(\sigma'(t))dt : \sigma \in \Omega_{x,y}(2U)\} \\ &\le \inf\{\int_0^1 \gamma_{2U}(\sigma'(t))dt : \sigma \in \Omega_{x,y}(U)\} \\ &\le \inf\{\int_0^1 ||\sigma'(t)||dt : \sigma \in \Omega_{x,y}(U)\} = \rho_U(x,y) \end{aligned}$$

for all $x, y \in U$, where $\Omega_{x,y}(U)$ is the set of C^1-paths joining x and y in U. Thus ρ_U defines the topology of U.

LEMMA 1.5. *Let X be as in Theorem 1.3. Then ρ_X defines the topology in X.*

PROOF: Let $\{x_n\} \subset X, \rho_X(x_n, x) \to 0$. Take i_0 such that $x \in V_{i_0}$. For each $n \ge 1$ take $\sigma_n \in \Omega_{x_n,x}(X)$ such that

$$\begin{aligned} \rho_X(x_n,x) &\ge \int_0^1 ||\sigma_n'(t)||dt - 1/n \\ &= \int_0^1 \sum_i h_i(\sigma_n(t))||D\varphi_i(\sigma_n(t)(\sigma_n'(t))||dt - 1/n \\ &\ge \int_0^{\varepsilon_n} ||D\varphi_{i_0}(\sigma_n(t))(\sigma_n'(t))||dt - 1/n \\ &= \int_0^1 ||D\varphi_{i_0}(\beta_n(s))(\beta_n'(s))||dt - 1/n \end{aligned}$$

$$\geq \rho\varphi_{i_0}(V_{i_0})(\varphi_{i_0}(x_n), \varphi_{i_0}(x)) - 1/n$$

where

$$\varepsilon_n = \sup\{r > 0 : \sigma_n([0,r]) \subset V_{i_0}\}$$

$$s = t/\varepsilon_n \text{ and } \beta_n(s) = \sigma_n(\varepsilon_n s) \text{ for } s \in [0,1]$$

This implies

$$\rho\varphi_{i_0}(V_{i_0})(\varphi_{i_0}(x_n), \varphi_{i_0}(x)) \to 0.$$

Hence $x_n \to x$ by Lemma 1.4.

Since X is a manifold, it is easy to see that the identity map $X \to (X, \rho_X)$ is continuous. Hence ρ_X defines the topology of X.

LEMMA 1.6. *Let X be a Banach manifold such that*

$$\sup\{||f'(0)|| : f \in Hol(\Delta, X)\} < \infty$$

where Hol (Δ, X) *denotes the space of holomorphic maps from Δ into X. Then X is hyperbolic.*

PROOF: Let $d_X(x_n, x) \to 0$. For each $n \geq 1$ there exists a holomorphic chain joining x_n and x, $(f_1^n, ... f_{k_n}^n;\ a_1^n, ..., a_{k_n}^n)$ such that

$$\sum_{j=1}^{k_n} d_\Delta(0, a_j^n) \to 0.$$

By the hypothesis we have

$$a = \sup\{||f'(z)|| : f \in \text{Hol}(\Delta, X),\ |z| < r\} < \infty,$$

where $0 < r < 1$ is choosen such that $|a_j^n| < r$ for $j = 1, ..., k_n$. Then

$$\begin{aligned} \rho_X(p_i^n, p_{i-1}^n) &\leq \int_0^1 ||(f_i^n \sigma_i^n)'(t)||dt \\ &\leq a \int_0^1 ||\sigma_i^n(t)||dt \\ &= a d_\Delta(0, a_i^n), \end{aligned}$$

where $p_i^n = f_i^n(a_i^n)$ and $\sigma_i^n(z) = a_i^n z$.

Thus $\rho_X(x_n, x) \to 0$. By Lemma 1.5 we have $x_n \to x$.

LEMMA 1.7. *Let Z be a compact Banach analytic space in a Banach manifold X such that Z contains no complex lines, i.e. every holomorphic map from C into Z is contant. Then there exists a hyperbolic neighbourhood of Z in X.*

PROOF: By Lemma 1.6 it suffices to show that there exists a neighbourhood W of Z in X such that

$$\sup\{\|f'(0)\| : f \in \mathrm{Hol}(\Delta, W)\} < \infty.$$

If not, for each n we can find $f_n \in \mathrm{Hol}(\Delta, W_n)$ such that $\|f'(0)\| = r_n \uparrow \infty$, where $\{W_n\}$ is a decreasing neighbourhood basis of Z in X. By the parametrization lemma of Brody [3] there exists for each n a holomorphic map φ_n from $(r_n/2)\Delta$ into W_n such that

$$\|\varphi_n'(0)\| = 1$$

and

$$\|\varphi_n'(z)\| \leq r_n^2/r_n^2 - |z|^2 \quad \text{for} \quad |z| < r_n/2.$$

This yields

$$\|\varphi_n'(z)\| \leq 4/3 \quad \text{for} \quad |z| < r_n/2$$

and hence $\{\varphi_n\}$ is equicontinuous. By the compactness of Z and by the relation $Z = \cap W_n$, it follows that $\{\varphi_n\}$ cofntains a subsequence $\{\psi_n\}$ converging to $\psi \in$ Hol (C, Z). Obviously, $\psi \neq$ const. This is impossible because Z contains no complex lines.

PROOF OF THEOREM 1.3: By the hypothesis and by Lemma 1.7, for each $y \in Y$ there exists a hyperbolic neighbourhood W of $\theta^{-1}(y)$. We can find a neighbourhood V of y such that $\theta^{-1}(V) \subset W$. Thus $\theta^{-1}(V)$ is hyperbolic. Hence, by Theorem 1.1 X is also hyperbolic.

2. Kwack holomorphic extendibility of convex domains.

Let X be a Banach analytic space. We say that X has the Kwack holomorphic property if every holomorphic map from $M\backslash A$ into X, where M is a Banach manifold and A is a hypersurface of M, can be extended holomorphically to M.

THEOREM 2.1. *Let θ be a proper holomorphic map from a Banach manifold X having C^1-partitions of unity into a Banach analytic space Y. If Y and the fibers of θ have the Kwack holomorphic extension property, then X has the Kwack holomorphic extension property.*

PROOF: Given a holomorphic map $f : M\backslash A \to X$, where M is a Banach manifold and A is a hypersurface of M. By the hypothesis, θf can be extended to holomorphic map g from M into Y. Let $x_0 \in A$. First consider the case where x_0 is a regular point of A. Then x_0 has a neighbourhood of the form $U = \Delta^* \times D$, where $\Delta^* = \Delta \setminus \{0\}$. By Theorem 1.3 we can find a hyperbolic neighbourhood of $\theta^{-1}(gx_0)$ of the form $\theta^{-1}(V)$. Assume that $g(U) \subset V$. Then by the property of $\theta : \theta^{-1}(V) \to V$, for each $z \in D$, $f(.,z)$ can be extended to a holomorphic map $\hat{f}(.,z)$ from Δ into X as in the finite dimensional case. From the hyperbolicity of $\theta^{-1}(V)$ it is easy to see that $\hat{f}$ is continuous on U. This yields the holomorphicity of $\hat{f}$. Now assume that $x_0 \in S(A)$, the singular locus of A. Since codim $S(A) \geq 2$ [8] we can find a neighbourhood U of x_0 and a holomorphic function h on U such that $S(A) \cap U \subseteq h^{-1}(0)$ and x_0 is a regular point of $h^{-1}(0)$. By (i), $f|U \setminus h^{-1}(0)$ can be extended holomorphically at x_0.

PROPOSITION 2.2. *Let D be a convex domain in a Banach space B. The following two conditions are equivalent:*

(i) D contains no complex lines

(ii) D has the Kwack holomorphic extension property.

PROOF: Without loss of generality we may assume that $0 \in D$. Then D can be written in the form

$$D = \cap\{Re x_i^* < \varepsilon_i : i \in I\},\ \varepsilon_i > 0 \quad \text{for all} \quad i \in I.$$

(i) $\Rightarrow$ (ii). Given a holomorphic map $f : M \setminus A \to D$, where M is a Banach manifold and A is a hypersurface of M. Let $x_0 \in A$. As is Theorem 2.1 we may assume that x_0 is a regular point of A. Thus x_0 has a neighbourhood of the form $\Delta^* \times V$. For each $i \in I$, the map $x^*f : \Delta^* \times X \to H_i$, where $H_i = \{z \in C : Rez < \varepsilon_i$, can be extended to a holomorphic map $(x_i^* f)^\wedge$ from $\Delta \times X$ into H_i [9]. We write

$$f(t,z) = \sum_{j=-\infty}^{\infty} a_j(z)t^j \quad \text{for} \quad (t,z) \in \Delta^* \times V,$$

where a_j are holomorphic functions on V with values in B. Since $(x_i^* f)^\wedge$ is holomorphic on $\Delta \times V$, it follows that $x_i^*(a_j(z)) = 0$ for all $i \in I$, $z \in V$ and $j < 0$. This yields the holomorphicity of f on $\Delta \times V$.

(ii) $\Rightarrow$ (i). Let f be a holomorphic map from C into D. By the hypothesis f can be extended holomorphically to CP^1. It follows from the compactness of CP^1 that f is a constant map

PROPOSITION 2.3. *Let* $\dim B < \infty$. *Then the conditions (i) and (ii) of Proposition 2.2 are equivalent to the following condition*

(iii) Every holomorphic map from a Riemann domain Ω *over a Banach space into* D *can be extended holomorphically to* $\widehat{\Omega}_\infty$.

PROOF: The implication (iii) $\Rightarrow$ (ii) is trivial.

(i) $\Rightarrow$ (iii). Given a holomorphic map f from Ω into D, where D is defined as in Proposition 2.2. By the hypothesis we have $\cap\{\mathrm{Ker} x_i^* : i \in I\} = 0$. Since $\dim B < \infty$, we have span $\{x_i^* : i \in B\} = B^*$. Now, for each $i \in I$ the map $x_i^* f$ is extended to a holomorphic map $(x_i^* f)^\wedge : \widehat{\Omega}_\infty \to H_i$. Thus for $x^* \in B^*$, $x^* = \sum \lambda_j x_{i_j}^*$, the form

$$(x^* f)^\wedge(z) = \sum_j \lambda_{i_j} (x_{i_j}^* f)^\wedge(z)$$

defines a holomorphic extension of $x^* f$ to $\widehat{\Omega}_\infty$. Then

$$\hat{f}(z)(x) = (x^* f)^\wedge(z)$$

is a holomorphic extension of f to $\widehat{\Omega}_\infty$. On the other hand, since $(x_i^* f)^\wedge(z) \in H_i$ for every $i \in I$, it follows that $\hat{f}(\widehat{\Omega}_\infty) \subseteq D$.

A domain D in a Banach space B is called completely circular if $x \in D$ implies $tx \in D$ for all $t \in \Delta$.

PROPOSITION 2.4. *Let D be a completely circular domain in a Banach space B. Then D is hyperbolic if and only if D is bounded.*

PROOF: The sufficiency is trivial. Assume that D is hyperbolic but unbounded. Then there exists a sequence $\{x_n\} \subset D$ such that $||x_n|| \to \infty$. Since D is completely circular for each $n \geq 1$, we can consider the holomorphic map $h_n \in \text{Hol}\,(\Delta, D)$ defined by $h_n(z) = zx_n$. Take $\delta > 0$ such that

$$\{x \in B : ||x|| < \delta\} \subseteq D.$$

Putting

$$y_n = (\delta/||x_n||)x_n \quad \text{and} \quad z_n = \delta/||x_n||$$

we have

$$h_n(z_n) = y_n \quad \text{and} \quad h_n(0) = 0.$$

Hence

$$d_D(0, y_n) = d_D(h_n(0), h_n(z_n) \leq d_\Delta(0, z_n) \to 0.$$

Since D is hyperbolic, $y_n \to 0$. This is impossible because of the relation $||y_n|| = \delta \; \forall n \geq 1$.

Note that the case $\dim B < \infty$ of Proposition 2.4 was proved by Kodama [6].

REMARK 2.5: 1) For every infinite dimensional Banach space B there exists a balanced convex domain such that its Kobayashi pseudodistance is a distance but does not define the topology. Indeed, since $\dim B = \infty$, there exists a continuous norm p on B such that $U_p = \{x \in B : p(x) < 1\}$ is unbounded. Obviously d_{U_p} is a distance, but by Proposition 2.4 it does not define the topology of U

2) There exists an injective holomorphic map from a Banach non - hyperbolic manifold into a Banach hyperbolic manifold. For example, consider the identity map $id : U_p \to \widehat{U}_p$, where $\widehat{U}_p$ is the unit ball in $\widehat{B}_p$, the completion of (B, p). Then U_p is hyperbolic and $\widehat{U}_p$ is not hyperbolic.

3. H-extendibility of Banach complete C-manifolds.

Let X be a Banach analytic space. We say that X has the H^∞ - extendibility if every holomorphic map from a Riemann domain Ω over a topological vector space can be extended holomorphically to $\widehat{\Omega}_\infty$.

Given a Banach analytic space X. By c_X we denote the Caratheodory pseudo-distance on X defined by

$$c_X(x, y) = \sup\{|f(x) - f(y)| : f \in H^\infty(X),\ ||f|| \leq 1\}.$$

Then X is called a C-space if c_X defines the topology of X. Moreover, if c_X is complete, then X is called a complete C-space.

Finally, a Banach space Q is said to be injective if every continuous linear map from a subspace of a Banach space B into Q can be extended to a continuous linear map from B into Q.

THEOREM 3.1. *If X is a Banach complete C-manifold modelled by open subsets of an injective Banach space, then X has the H^∞-extendibility.*

PROOF: (i) Consider the canonical map $\delta : X \to (H^\infty(X))^*$ defined by

$$\delta(x)(h) = h(x) \quad \text{for } x \in X,\ h \in H^\infty(X).$$

By the hypothesis, δ is a homeomorphism onto Im δ and Im δ is closed in $(H^\infty(X))^*$. Moreover, for each $x_0 \in X$ we can find a neighbourhood U of x_0 biholomorphic to the unit ball in an injective Banach space Q such that

$$c_X(x_0, x_0 + h) \geq A||h|| \quad \text{for} \quad h \in Q,\ ||h|| < \varepsilon,\ \varepsilon > 0,$$

where A is a constant independent on $h \in Q$, $||h|| < \varepsilon$ [2]. Thus, for every $h \in Q$, $||h|| < \varepsilon$, there exists $\sigma \in H^\infty(X)$, $||\sigma|| \leq 1$, such tha

$$|\sigma(x_0 + h) - \sigma(x_0)| \geq A||h||.$$

Consequently, for every $\xi \in \Delta$ and $||h|| < \varepsilon$ there exists $\sigma \in H^\infty(X)$, $||\sigma|| \leq 1$, and $\eta \in \Delta$ such that

$$|D\sigma(x_0 + \eta\xi h)(h)| = |\sigma(x_0 + \xi h) - \sigma(x_0)| \geq A||h||.$$

Hence

$$\sup\{|D\sigma(x_0 + \xi h)(h)| : ||\sigma|| \leq 1, \ |\xi| \leq 1\} \geq A||h||.$$

So, for $0 < \alpha \leq 1$ and $||h|| < \varepsilon$ we have

$$\begin{aligned} &\sup\{|D\sigma(x_0 + \xi h)(h)| : ||\sigma|| \leq 1, \ |\xi| \leq \alpha\} \\ =&\sup\{|D\sigma(x_0 + \xi/\alpha \ \alpha h)(\alpha h)|1/\alpha : ||\sigma|| \leq 1, \ |\xi| \leq \alpha\} \\ =&(1/\alpha)\sup\{|D\sigma(x_0 + \eta\alpha h)(\alpha h)| : ||\sigma|| \leq 1, \ |\eta| \leq 1\} \\ \geq&(1/\alpha)A||\alpha h|| = A||h|| \end{aligned}$$

By the Monteless of $H(X)$ we then have

$$\sup\{|D\sigma(x_0)(h)| : ||\sigma|| \leq 1\} \geq A||h||$$

for every $h \in Q$. Therefore $D\delta(x_0) : T_{x_0}X \to \text{Im } D\delta(x_0)$ is an isomorphism. Since $T_{x_0}X \cong Q$ is injective, by the implicit function theorem we deduce that Im δ is a submanifold of $(H^\infty(X))^*$ and δ is a biholomorphism onto Im δ.

(ii) Given a holomorphic map f from a Riemann domain Ω over a topological vector space into X. For each $b \in H^\infty(X)$ consider the bounded holomorphic function $b\delta f$ on Ω. Let $(b\delta f)^\wedge$ be a holomorphic extension of $b\delta f$ to $\widehat{\Omega}_\infty$. Define the map $\hat{f}$ from $\widehat{\Omega}_\infty$ into $(H^\infty(X))^*$ by

$$\hat{f}(z)(b) = (b\delta f)^\wedge(z) \quad \text{for} \quad z \in \widehat{\Omega}_\infty \quad \text{and} \quad b \in H^\infty(X).$$

Since $\hat{f}$ is bounded and holomorphic in z and b separately, it follows that $\hat{f}$ is holomorphic on $\widehat{\Omega}_\infty$.

(iii) From (i) and from the inclusion $\hat{f}(\Omega) \subseteq \mathrm{Im}\,\delta$ it follows that $\hat{f}(\widehat{\Omega}_\infty) \subseteq \mathrm{Im}\,\delta$. Thus, $\delta^{-1}\hat{f}$ is a holomorphic extension of f to $\widehat{\Omega}_\infty$.

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