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MULTIDIMENSIONAL QUANTIZATION AND $U(1)$-COVERING

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Introduction

In order to find irreducible unitary representations of a connected and simply connected Lie group G, Kirillov's method of orbit furnishes a procedure of quantization starting from linear bundles over a G-homogeneous symplectic manifold (see [5, §15]). In [1] and [2] Do Ngoc Diep has proposed a new procedure of quantization for the general case, starting from arbitrary irreducible G-bundles associated with the given Hamiltonian mechanical system. In [4] M. Duflo proposed three methods for constructing large subsets of the unitary dual of Lie groups. To avoid the Mackey's obstructions when reducing Kirillov's orbits method to special contexts, M. Duflo lifted the character of the stabilizer to $\mathbf{Z}_2$-covering by using metaplectic structures.

In this paper, using the technique of P.L. Robinson and J.H. Rawnsley (see [6]) we shall lift the character to $U(1)$-covering via Mp^c-structures instead of Mp-structures. Our purpose is also to eliminate the Mackey's obstructions to obtain linear representations from the projective ones. First of all, in the Bargmann-Segal model we present an account of the character of the stabilizer. Secondly, by using new results of P.L. Robinson and J.H. Rawnsley (see [6]) and a new notion of positive polarizations we shall construct unitary induced representations of G denoted by $\mathrm{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}, \sigma_0, \rho)$. Finally, they will be illustrated as representations obtained from the procedure of geometrical

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multidimensional quantization which is a natural generalization of Kirillov - Kostant - Souriau's procedure of quantization.

1. Positive polarization - The character of the stabilizer.

1.1. Positive polarization. Let G be a connected and simply connected Lie group. Denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ its dual space. The group G acts on $\mathcal{G}$ by the adjoint representation Ad, and on $\mathcal{G}^*$ by the coadjoint representation Ad^*. We shall simply call it K-representation. Let $F \in \mathcal{G}^*$ be an arbitrary point in a K-orbit Ω, and G_F the stabilizer of this point. Denote by $\mathcal{G}_F$ its Lie algebra. The bilinear form B_F on $\mathcal{G}$ given by $B_F(X,Y) = < F, [X,Y] >$ has Ker $B_F = \mathcal{G}_F$. We write $\widetilde{B}_F$ for the symplectic form on $\mathcal{G}/\mathcal{G}_F$ induced from B_F and B_Ω the Kirillov 2-form on Ω. Denote by $Sp(\mathcal{G}/\mathcal{G}_F, \widetilde{B}_F)$ or simply $Sp(\mathcal{G}/\mathcal{G}_F)$ the sympletic group.

DEFINITION 1.1: A complex subalgebra $\mathcal{N}$ of $\mathcal{G}^{\mathbf{C}} = \mathcal{G} \otimes \mathbf{C}$ is called a positive polarization iff

i) $\mathcal{N}$ is invariant under the action Ad of G_F.

ii) The subspace $\mathcal{N}/\mathcal{G}_F^{\mathbf{C}}$ denoted by L in $\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}} \cong (T_F\Omega)^{\mathbf{C}}$ satisfies the following three conditions :

$$(\alpha)\ \dim_{\mathbf{C}} \mathcal{N}/\mathcal{G}_F^{\mathbf{C}} = \frac{1}{2}\dim_{\mathbf{R}} \Omega,$$

$$(\beta)\ \widetilde{B}_F(X,Y) = 0 \text{ for all } X, Y \in L,$$

$$(\gamma)\ i\widetilde{B}_F(X,\overline{X}) \geq 0 \text{ for all } X \in L,$$

where $\overline{X}$ is the conjugation of X in the complex space $\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}}$.

iii) $\mathcal{N}$ is G-invariant in the sense that for all $g \in G$, $F' = K(g)F$, so $Adg^{-1}\mathcal{N}/\mathcal{G}_F^{\mathbf{C}}$ satisfies (γ) in the space $\mathcal{G}^{\mathbf{C}}/\mathcal{G}_{F'}^{\mathbf{C}}$.

We say that $\mathcal{N}$ is strictly positive if the inequality (γ) is strict for nonzero $X \in L$.

1.2. The integral kernel of the character χ_F. Let $\mathcal{N}$ be a positive polarization. We write $\mathcal{N} \cap \overline{\mathcal{N}} = \mathcal{H}^{\mathbf{C}}$, $\mathcal{H} = \mathcal{N} \cap \mathcal{G}$, $\mathcal{M}^{\mathbf{C}} = \mathcal{N} + \overline{\mathcal{N}}$, $\mathcal{M} =$

$(\mathcal{N}+\overline{\mathcal{N}})\cap G$. Let $H = G_F H_0$ and M be the corresponding analytic subgroups of $\mathcal{H}$ and $\mathcal{M}$, where H_0 is the connected component of the identity of the subgroup H.

Let $\hbar$ be a fixed positive number, say the normalized Planck's constant, and $h = 2\pi\hbar$ the Planck's constant.

DEFINITION 1.2: A K-orbit Ω is called an integral orbit iff there exists a unitary character $\chi_F : G_F \to T = S^1$ such that its differential $d\chi_F$ satisfies the condition

$$\begin{aligned}(d\chi_F)(X) &\stackrel{\text{def}}{=} \frac{d}{dt}(\chi_F(\exp tX))\mid_{t=0}\\ &= \frac{i}{\hbar} < F, X >, \ \forall X \in \mathcal{G}_F.\end{aligned}$$

The above definition is equivalent to the condition that the form $\frac{B\Omega}{\hbar}$ belongs to an integral de Rham's cohomology class.

Suppose that there exists an extension U of the character χ_F on H such that the following equality

$$U(\exp X) = \chi_F(\exp X) = \exp\{\frac{i}{\hbar} < F, X)\}$$

satisfies in a neighbourhood of the identity of H (see [5, §15]).

We shall construct the space of vacuum states on which the subgroup H acts (see [6, §3]). Let $L = \mathcal{N}/\mathcal{G}_F^{\mathbb{C}}$ be a positive polarization in the symplectic space $(\mathcal{G}/\mathcal{G}_F, \widetilde{B}_F)$. Set $D = \mathcal{H}/\mathcal{G}_F$. We have $L \cap \overline{L} = D^{\mathbb{C}}$, $D^{\perp} = \mathcal{M}/\mathcal{G}_F = ((\mathcal{N}+\overline{\mathcal{N}})\cap\mathcal{G})/\mathcal{G}_F$, where $D^{\perp}$ is given by

$$D^{\perp} = \{\widetilde{X} \in \mathcal{G}/\mathcal{G}_F / \widetilde{Y} \in D \Longrightarrow \widetilde{B}_F(\widetilde{X}, \widetilde{Y}) = 0\}.$$

Then $D \subset D^{\perp}$, or D is an isotropic subspace of $(\mathcal{G}/\mathcal{G}_F, \widetilde{B}_F)$. By the isotropic reduction, $\widetilde{B}_F$ descends to a symplectic structure on the space $D^{\perp}/D$ denoted by $\widetilde{B}_{F,D}$ (see [6, §4]). We have

$$D^{\perp}/D = (\mathcal{M}/\mathcal{G}_F) / (\mathcal{H}/\mathcal{G}_F) \cong \mathcal{M}/\mathcal{H}.$$

Thus there exists a symplectic structure on the space $\mathcal{M}/\mathcal{H}$ with the strictly positive polarization Γ_D of $(D^{\perp}/D, \widetilde{B}_{F,D})$ in the sense of P.L. Robinson and J.H. Rawnsley (see [6, §3]), which is defined as follows :

$$\Gamma_D = (\mathcal{N}/\mathcal{G}_F^{\mathbf{C}}) \;/\; (\mathcal{H}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}}) \cong \mathcal{N}/\mathcal{H}^{\mathbf{C}}.$$

PROPOSITION 1.1. *The subgroup H acts on the Bargmann space $\mathsf{H}(\mathcal{M}/\mathcal{H})$ and preserves the one-dimensional space of vacuum states $\{\epsilon'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})\}^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}}$*

PROOF: Since $\mathcal{M}/\mathcal{H}$ is a symplectic space, we have a rigged Hilbert space in the sense of Gelfand as follows :

$$\epsilon(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}) \subset \mathsf{H}(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}) \subset \epsilon'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})$$

(see [6, §1]). In this case,

$$\{\mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})\}^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}} = \{f \in \mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}) / \dot{W}(v)f = 0,\ \forall v \in \mathcal{N}/\mathcal{H}^{\mathbf{C}}\},$$

where $\dot{W} : \mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}} \to \mathrm{End}\, \mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})$ is defined by

$$(\dot{W}(v_1 + iv_2)f)(z) = -df_z(v_1 + Jv_2) + \frac{1}{2\hbar} < z, v_1 - Jv_2 > f(z).$$

Here $< .,. >$ is the scalar product in $\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}$ and J denotes the multiplication by i on $\mathcal{M}/\mathcal{H}$.

Since $\Gamma_D = \mathcal{N}/\mathcal{H}^{\mathbf{C}}$ is a strictly positive polarization, $\{\mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})\}^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}}$ is a complex line with basis vector f_D (see [6, §4]). The action of H on $\{\mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})\}^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}}$ is given by

$$(U(h)f_D)(v) = U(h)f_D(v),\ v \in \mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}.$$

In a certain neighbourhood of the identity of the subgroup H, we have

$$\chi_F(\exp X).f_D = e^{(i/\hbar)F(X)}.f_D.$$

We may therefore consider the action of H on $\mathsf{H}(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})$ given by

$$\exp X \mapsto e^{(i/\hbar)F(X)}.I,$$

where $I = id(\mathsf{H}(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}))$. Then H preserves $\{\mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})\}^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}}$.

PROPOSITION 1.2. *$e^{(i/\hbar)F(X)}.I$ is a unitary operator of $\mathsf{H}(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})$ and its integral kernel $u(X,z,w)$ is given by*

$$u(X,z,w) = u(z,w) = \exp\frac{1}{\hbar}\{iF(X) + \frac{1}{2} < z,w > -\frac{1}{4} < w,w >\},$$

where $z, w \in \mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}$.

PROOF: Since the identity operator $I = \mathrm{id}(\mathsf{H}(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}))$ has its integral kernel given by kernel $I(z,w) = (Ie_w)(z) = e_w(z) = \exp(\frac{1}{2\hbar} < z,w > -\frac{1}{4\hbar} < w,w >)$, the operator

$$(e^{\frac{i}{\hbar}F(X)}.I)(f) = [\exp(\frac{i}{\hbar}F(X))].f$$

is unitary, and its kernel is

$$u(z,w) = \exp(\frac{i}{\hbar}F(X) + \frac{1}{2\hbar} < z,w > -\frac{1}{4\hbar} < w,w >).$$

REMARKS: 1) If $\tilde{\sigma} : G_F \to U(\widetilde{V})$ is an irreducible unitary representation of G_F in the Hilbert space $\widetilde{V}$, then

$$(\tilde{\sigma} \otimes \chi_F^{U(1)})(\tilde{v}, f) = \tilde{\sigma}(\tilde{v}) \otimes \chi_F^{U(1)}(f)$$

has an integral kernel given by

$$u(z,w,\tilde{v}) = \tilde{\sigma}(\tilde{v}).\exp\{\frac{i}{\hbar}F(X) + \frac{1}{2\hbar} < z,w > -\frac{1}{4\hbar} < w,w >\}.$$

2) If we use the H-principal bundle $H \mapsto M \to M \setminus H$ and the representation $U : H \to \mathcal{E}'(\mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}})^{\mathcal{N}/\mathcal{H}^{\mathbf{C}}}$, we can also construct the unitary induced representation of M. In this case the scalar product of the basis vector f_D is defined by

$$(f_D, f_D) = \int_{\mathcal{M}/\mathcal{H}} |f_D|^2 d\mu,$$

where μ is the Gaussian measure on $\mathcal{M}/\mathcal{H}$ with the density function θ given by

$$\theta(z) = h^{-m}\exp(-\frac{|z|^2}{2\hbar}),\ z \in \mathcal{M}/\mathcal{H},\ m = \dim_{\mathbf{C}} \mathcal{M}/\mathcal{H}.$$

2. A lift of the character to $U(1)$-covering

2.1. $U(1)$-covering of stabilizer G_F. The symplectic group $Sp(\mathcal{G}/\mathcal{G}_F)$ consists of all the real automorphisms that preserve the symplectic form $\widetilde{B}_F$.

If $\mathcal{G}_F \neq \mathcal{G}$, i.e. $\mathcal{G}/\mathcal{G}_F$ is not trivial, then $Sp(\mathcal{G}/\mathcal{G}_F)$ has an $U(1)$-connected covering $Mp^c(\mathcal{G}/\mathcal{G}_F, \widetilde{B}_F)$ or simply $Mp^c(\mathcal{G}/\mathcal{G}_F)$ (see [6, §2). If $U \in Mp^c(\mathcal{G}/\mathcal{G}_F)$, then U has parameters (λ, g) such that $|\lambda^2 \det Cg| = 1$ for some $\lambda \in \mathbb{C}$, $g \in Sp(\mathcal{G}/\mathcal{G}_F)$ and $Cg = \frac{1}{2}(g - igi)$. We have a surjective group homomorphism σ from $Mp^c(\mathcal{G}/\mathcal{G}_F)$ to $Sp(\mathcal{G}/\mathcal{G}_F)$ defined by $\sigma(U) = g$ with Ker σ being precisely the unitary scalar operator $U(1)$. Thus we have a central short exact sequence

$$1 \to U(1) \to Mp^c(\mathcal{G}/\mathcal{G}_F) \xrightarrow{\sigma} Sp(\mathcal{G}/\mathcal{G}_F) \to 1$$

If $\mathcal{G}_F = \mathcal{G}$, then taking $Mp^c(\mathcal{G}/\mathcal{G}_F) = U(1)$ we have a split short exact sequence

$$1 \to U(1) \to Mp^c(\mathcal{G}/\mathcal{G}_F) \xrightarrow{\sigma} 1 \to 1$$

PROPOSITION 2.1. *There exists a group homomorphism j from G_F to $Sp(\mathcal{G}/\mathcal{G}_F)$.*

PROOF: Let $g \in G_F$. Then $Adg^{-1} : \mathcal{G} \to \mathcal{G}$ is a real automorphism. Denote by $\widetilde{A}dg^{-1} : \mathcal{G}/\mathcal{G}_F \to \mathcal{G}/\mathcal{G}_F$ the real automorphism induced from Adg^{-1}. We have

$$\begin{aligned}\widetilde{B}_F(\widetilde{A}dg^{-1}\widetilde{X},\ \widetilde{A}dg^{-1}\widetilde{Y}) &= \widetilde{B}_F(Adg^{-1}X, Adg^{-1}Y)\\ &= < K(g)F,\ [X,Y] > = < F, [X,Y] > =\\ &= < F, Adg^{-1}[X,Y] > = \widetilde{B}_F(\widetilde{X},\widetilde{Y}),\end{aligned}$$

for all $g \in G_F$. Hence $j(g) = \widetilde{A}dg^{-1}$ is the desired homomorphism.

Denote by $G_F^{U(1)}$ the Lie subgroup of the cartesian product of Lie groups $G_F \times Mp^c(\mathcal{G}/\mathcal{G}_F)$ consisting of all pairs (g, U) such that $\sigma(U) = \widetilde{A}dg^{-1}$, i.e. g and U have the same image in $Sp(\mathcal{G}/\mathcal{G}_F)$.

PROPOSITION 2.2. *There exists a commutative diagram of the form*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G_F^{U(1)} & \xrightarrow{\sigma_j} & G_F & \longrightarrow & 1 \\ & & \downarrow id & & \downarrow k & & \downarrow j & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & Mp^c(\mathcal{G}/\mathcal{G}_F) & \xrightarrow{\sigma} & Sp(\mathcal{G}/\mathcal{G}_F) & \rightarrow & 1 \end{array}$$

PROOF: We have $G_F^{U(1)} = \{(g, U) \mid \sigma(U) = j(g) = \widetilde{A}dg^{-1}\}$, where $U \in Mp^c(\mathcal{G}/\mathcal{G}_F)$ has parameter (λ, f) with $f \in Sp(\mathcal{G}/\mathcal{G}_F)$ and $\lambda \in \mathbb{C}$ such that $|\lambda^2 \text{Det } C_f| = 1$ with $C_f = \frac{1}{2}(f - ifi)$ commuting with $i \in \mathbb{C}$ $(i^2 = -1)$.

Since $\sigma(U) = f = \widetilde{A}dg^{-1}$, every member of $G_F^{U(1)}$ has the form $(g; (\lambda, \widetilde{A}dg^{-1}))$ such that

$$|\lambda^2 \text{Det } C_{\widetilde{A}dg^{-1}}| = 1.$$

Put $k(g; (\lambda, \widetilde{A}dg^{-1})) = (\lambda, \widetilde{A}dg^{-1}) \in Mp^c(\mathcal{G}/\mathcal{G}_F)$. Then k is a group homomorphism from $G_G^{U(1)}$ to $Mp^c(\mathcal{G}/\mathcal{G}_F)$, and $\sigma_j(g, (\lambda, \widetilde{A}dg^{-1})) = g$. It is easily seen that this homomorphism k makes the above diagram commutative.

From Proposition 2.2 we obtain the short exact sequence

$$1 \longrightarrow U(1) \longrightarrow G_F^{U(1)} \xrightarrow{\sigma_j} G_F \longrightarrow 1.$$

Hence $G_F^{U(1)}$ is a $U(1)$-covering of G_F.

2.2. Lifting the character of G_F^0 to $G_F^{U(1)}$. We do not assume that Ω passing $F \in \mathcal{G}^*$ is an integral orbit, i.e. there does not exist a character χ_F of G_F. Since

$$1 \rightarrow U(1) \rightarrow G_F^{U(1)} \xrightarrow{\sigma_j} G_F \rightarrow 1$$

is a short exact sequence, we have a split short exact sequence of the corresponding Lie algebras

$$0 \longrightarrow U(1) \longrightarrow \text{Lie } G_F^{U(1)} \longrightarrow \mathcal{G}_F \longrightarrow 0.$$

Thus the Lie algebra of $G_F^{U(1)}$ is $\mathcal{G}_F \oplus U(1)$ (see [6, §5]).

DEFINITION 2.1: The K-orbit Ω_F is called $U(1)$-admissible iff there exists a unitary character $\chi_{F,k}^{U(1)} : G_F^{U(1)} \to S^1$ such that $(d\chi_{F,k}^{U(1)})(X,\varphi) = \frac{i}{\hbar}(F(X)+k\varphi)$, where $(X,\varphi) \in \mathcal{G}_F \oplus U(1)$, $k \in \mathbf{Z}$.

REMARK: The case $k = 0$ is considered in the previous section. For $k \neq 0$, it is enough to consider the case $k = 1$. Set $\chi_{F,1}^{U(1)} = \chi_F^{U(1)}$. If Ω_F is an integral orbit then it is $U(1)$-admissible, but the reverse does not hold.

PROPOSITION 2.3. *In a neighbourhood of the identity of* $G_F^{U(1)}$ *we have* $\chi_F^{U(1)}(g; \lambda, \widetilde{A}dg^{-1})) = \exp\{\frac{i}{\hbar}(F(X)+\varphi)\}$, *where* $\varphi \in \mathbf{R}$ *satisfies the relation* λ^2 *Det* $C_{\widetilde{A}dg^{-1}} = e^{\frac{i}{\hbar}\varphi}$. *The integral kernel of* $\chi_F^{U(1)}$ *is given by the formula*

$$u(z,w) = \exp\{\frac{i}{\hbar}(F(X)+\varphi) + \frac{i}{2\hbar} < z,w > \; - \frac{1}{4\hbar} < w,w >\},$$

where $z, w \in \mathcal{M}^{\mathbf{C}}/\mathcal{H}^{\mathbf{C}}$.

PROOF: In the special case $F_0 = 0$, we have $G^{U(1)} = G \times U(1)$ $\chi_{F_0}^{U(1)}(\exp\ X, \varphi) = e^{\frac{i}{\hbar}}(F_0(X)+\varphi) = e^{\frac{i}{\hbar}\varphi}$. Hence

$$\begin{aligned}\frac{d}{dt}\chi_F^{U(1)}(\exp\ X, t\varphi)|_{t=0} &= \frac{d}{dt}[\exp\{\frac{i}{\hbar}tF(X)\}.\exp\frac{i}{\hbar}t\varphi]|_{t=0}\\ &= \frac{i}{\hbar}F(X) + \frac{i}{\hbar}\varphi = \frac{i}{\hbar}(F(X)+\varphi).\end{aligned}$$

In the general case, the character $\chi_F^{U(1)}$ in a neibourhood of the identity $(1,(1,1))$ of $G_F^{U(1)}$ is given by

$$\chi_F^{U(1)}(g;(\lambda,\widetilde{A}dg^{-1})) = \exp\{\frac{i}{\hbar}(F(X)+\varphi)\},$$

where $g = \exp\ X$ and $\varphi \in \mathbf{R}$ such that λ^2 Det $C_{\widetilde{A}dg^{-1}} = e^{\frac{i}{\hbar}\varphi}$.

Similarly as in the proof of Proposition 1.2 in Section 1, we can show that the integral kernel of $\chi_F^{U(1)}$ is given by the stated formula.

2.3. Extension of the character on $H^{U(1)} = G_F^{U(1)} \propto H_0$. Let H_0 be the connected component of the identity in H then H_0 is a normal subgroup in H, and G_F has the adjoint action on H_0. Moreover, we have the homomorphism

$\sigma_j : G_F^{U(1)} \to G_F$. Thus $G_F^{U(1)}$ acts on H_0, and we can define the semidirect product $G_F^{U(1)} \ltimes H_0$. The following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G_F^{U(1)} & \xrightarrow{\sigma_j} & G_F & \longrightarrow & 1 \\ & & \uparrow id & & \uparrow P_{H_0} & & \uparrow P_{H_0} & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & G_F^{U(1)} \ltimes H_0 & \longrightarrow & G_F H_0 & \longrightarrow & 1 \end{array}$$

where P_{H_0} is the quotient homomorphism. Hence $G_F^{U(1)} \ltimes H_0$ is the $U(1)$-covering of $H = C_F H_0$. Set $H^{U(1)} = G_F^{U(1)} \ltimes H_0$. The Lie algebra of $H^{U(1)}$ is $\mathcal{H} \oplus u(1)$.

We now assume that the K-orbit Ω is $U(1)$-admissible and $\mathcal{N}$ is a positive polarization in $\mathcal{G}^{\mathbb{C}}$. Let $\tilde{\sigma}$ be a fixed irreducible unitary representation of G_F in a separable Hilbert space $\widetilde{V}$ such that the restriction of $\chi_F^{U(1)}.(\tilde{\sigma}\sigma_j)$ to $(G_F^0)^{U(1)}$ is a multiple of the character $\chi_F^{U(1)}$, where σ_j is the homomorphism defined in Proposition 2.2. The polarization $\mathcal{N}$ is called closed if all the subgroups H_0, M_0 and $H = G_F H_0$, $M = G_F M_0$ are closed in G. In what follows we assume that $\mathcal{N}$ is closed.

Definition 2.2: The triplet $(\mathcal{N}, \rho. \sigma_0)$ is called a $(\tilde{\sigma}, \chi_F^{U(1)})$-polarization if

i) σ_0 is an irreducible unitary representation of the subgroup $(H_0^{U(1)})$ in a Hilbert space V such that

$$\sigma_0 | G_F^{U(1)} \cap (H_0^{U(1)}) = (\tilde{\sigma}\sigma_j).\chi_F^{U(1)}).$$

ii) ρ is a representation of the complex Lie algebra $\mathcal{N} \oplus u(1)$ in V such that

$$d\sigma_0 = \rho|_{\mathcal{H}\oplus u(1)}.$$

Proposition 2.4. *If Ω_F is $U(1)$-admissible and $(\mathcal{N}, \rho, \sigma_0)$ is a $(\tilde{\sigma}, \chi_F^{U(1)})$-polarization, then there exists a unique irreducible representation $\sigma : H^{U(1)} \to U(V)$ such that*

$$\sigma_0 | G_F^{U(1)} = (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}) \quad \text{and} \quad d\sigma = \rho|_{\mathcal{H}\oplus u(1)}.$$

PROOF: Similarly as in the proof of Theorem 2 in [1], using the fact that $\sigma_0|G_F^{U(1)} \cap (H_0^{U(1)}) = (\tilde{\sigma}\sigma_j)\chi_F^{U(1)})$ and $d\sigma_0 = \rho|_{\mathcal{H}\oplus u(1)}$ we can extend σ_0 uniquely to an irreducible representation as in the above statement.

3. Induced representations

3.1. Induced respresentation obtained from the $(\tilde{\sigma}, \chi_F^{U(1)})$ - polarization. If we choose $F_0 = 0 \in \mathcal{G}^*$, then $G_{F_0} = G$. According to the definition of $Mp^c(\mathcal{G}/\mathcal{G}_{F_0}, \widetilde{B}_{F_0}) = U(1)$, we put $G^{U(1)} = G \times U(1)$.

PROPOSITION 3.1.

$$G \underset{H}{\times} H^{U(1)} \cong G \underset{G_F}{\times} G_F^{U(1)} \cong G^{U(1)}$$

in the category of principal bundles.

PROOF: Fix a connection $\overline{\Gamma}$ on the principal bundle $H \mapsto G \to H \setminus G$, and let $Pr_{\overline{\Gamma}} : G \to H$ be the projection induced from $\overline{\Gamma}$. We can construct the covering map $\widetilde{Pr}_{\overline{\Gamma}}$ from $G^{U(1)}$ to $H^{U(1)}$ by the following commutative diagram (in the category of topological spaces and continuous maps)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G^{U(1)} & \xrightarrow{\sigma_j} & G & \longrightarrow & 1 \\ & & \downarrow id & & \downarrow \widetilde{Pr}_{\overline{\Gamma}} & & \downarrow Pr_{\overline{\Gamma}} & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & H^{U(1)} & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

Obviously, the maps $Pr_{\overline{\Gamma}}$ and $\widetilde{Pr}_{\overline{\Gamma}}$ induce the isomorphisms of the statement.

Because of the isomorphisms of Proposition 3.1 we have a principal $H^{U(1)}$-bundle on $H \setminus G$ and a principal $G_F^{U(1)}$-bundle on $G_F \setminus G$ as follows

$$\begin{array}{ccc} H^{U(1)} \rightarrowtail G^{U(1)} & & G_F^{U(1)} \rightarrowtail G^{U(1)} \\ \downarrow & & \downarrow \\ H \setminus G & \xleftarrow{\pi} & G_F \setminus G, \end{array}$$

where π is the natural projection from $G_F \setminus G$ to $H \setminus G$.

Denote by $E_{\sigma,\rho} = G^{U(1)} \underset{H^{U(1)},\sigma}{\times} V$ the vector bundle on $H \setminus G$ associated with the representation $\sigma : H^{U(1)} \to U(V)$. The inverse image bundle $\pi^* E_{\sigma,\rho}$ and $G^{U(1)} \underset{G_F^{U(1)},\sigma | G_F^{U(1)}}{\times} V$ are equivalent in the category of smooth vector bundles.

Denote by $\mathcal{S}(\pi^* E_{\sigma,\rho})$ the space consisting of all smooth sections of the bundle $\pi^* E_{\sigma,\rho}$. The connection $\overline{\Gamma}$ induces a connection Γ on $G_F^{U(1)} \rightarrowtail G^{U(1)} \rightarrowtail \Omega$. We obtain an affine connection ∇^Γ on the associated bundle $G^{U(1)} \underset{G_F^{U(1)},(\tilde{\sigma}\sigma_j)\chi_F^{U(1)}}{\times} V$. Put

$$\mathcal{S}_{\widetilde{L}}(\pi^* E_{\sigma,\rho}) = \{ s \in \mathcal{S}(\pi^* E_{\sigma,\rho}) \mid \nabla_\xi^\Gamma = 0, \ \forall \xi \in \widetilde{L} \}.$$

The natural representation of G in $\mathcal{S}_{\widetilde{L}}(\pi^* E_{\sigma,\rho})$ denoted by $\operatorname{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}, \sigma_0, \rho)$ is called the induced representation (see [5, §13]).

3.2. Induced unitary representation. To obtain a unitary representation we shall apply the pairing used in [6].

For $v \in \mathcal{G}/\mathcal{G}_F$, a unitary operator $W(v)$ on $\mathsf{H}(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ is defined by

$$(W(v)f)(z) = \exp\{\frac{1}{4\hbar} < 2z - v, v > f(z - v)\},$$

where $f \in \mathsf{H}(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ and $z \in \mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}}$.

The map $W : \mathcal{G}/\mathcal{G}_F \to \operatorname{Aut}\mathsf{H}(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ is an irreducible projective unitary representation of the vector group of $\mathcal{G}/\mathcal{G}_F$ with multiplier $\exp\{\frac{1}{i\hbar}\widetilde{B}_F\}$. The differentiated projective representation $\dot{W}$ of $\mathcal{G}/\mathcal{G}_F$ on $\mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ complexifies to yield

$$\dot{W} : (\mathcal{G}/\mathcal{G}_F)^{\mathbf{C}} \longrightarrow \operatorname{End} \mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$$

with

$$(\dot{W}(v_1 + iv_2)f)(z) = -df_z(v_1 + Jv_2) + \frac{1}{2\hbar} < z, v_1 - Jv_2 > f(z)$$

for $z, v_1, v_2 \in \mathcal{G}/\mathcal{G}_F$ and $f \in \mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$.

For convenience, in what follows we denote by P the principal $G_F^{U(1)}$-bundle $G_F^{U(1)} \twoheadrightarrow G^{U(1)} \rightarrowtail G_F \setminus G$. By the group homomorphism $k : G_F^{U(1)} \to$

$Mp^c(\mathcal{G}/\mathcal{G}_F)$ and the metaplectic representation $\mu : Mp^c(\mathcal{G}/\mathcal{G}_F) \to \text{End}\, \mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ (see [6, §2]), we obtain a bundle associated with P via the homomorphism $\mu \circ k$ with $\mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ as the typical fiber. We consider the homomorphism $(\tilde{\sigma}\sigma_j\chi_F^{U(1)})(\mu \circ k)$ defined by

$$(\tilde{\sigma}\sigma_j\chi_F^{U(1)})(\mu \circ k) : G_F^{U(1)} \to \text{End}\,\{\widetilde{V} \otimes \mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})\}$$

$$(g, U) \mapsto (\tilde{\sigma}\sigma_j\chi_F^{U(1)})(g, U) \otimes (\mu \circ k)(g, U)$$

Denote by $\mathcal{E}'_{\sigma,\rho}(P)$ the vector bundle associated with P via the homomorphism $(\tilde{\sigma}\sigma_j\chi_F^{U(1)})(\mu \circ k)$ with $\widetilde{V} \otimes \mathcal{E}'(\mathcal{G}^{\mathbf{C}}/\mathcal{G}_F^{\mathbf{C}})$ as the typical fibre.

PROPOSITION 3.2. *For each $F' \in \Omega$, there is a canonical linear map*

$$\dot{W}_{F'} : (T_{F'}\Omega)^{\mathbf{C}} \to End\,(\mathcal{E}'_{\sigma,\rho}(P))_{F'}$$

such that if $\widetilde{X}, \widetilde{Y} \in (T_{F'}\Omega)^{\mathbf{C}}$ then

$$[\dot{W}_{F'}(\widetilde{X}, \dot{W}_{F'}(\widetilde{Y})] = -\frac{i}{\hbar}\widetilde{B}_{F'}(\widetilde{X}, \widetilde{Y}).$$

PROOF: Each $p \in P_{F'}$ determines the isomorphisms

$$p : \widetilde{V} \otimes \mathcal{E}'(\mathcal{G}/\mathcal{G}_F) \to \mathcal{E}'_{\sigma,\rho}(P)_{F'}$$
$$p : (\mathcal{G}/\mathcal{G}_F)^{\mathbf{C}} \to (T_{F'}\Omega)^{\mathbf{C}}.$$

We define

$$\dot{W}_{F'}(\widetilde{X}) = p_0\dot{W}(p^{-1}(\widetilde{X}))_0 p^{-1} \text{ for } \widetilde{X} \in (T_{F'}\Omega)^{\mathbf{C}}.$$

This definition is independent of the choice of p (see [6, §3]). The commutation relations for $\dot{W}_{F'}$ come from those for $\dot{W}$. Since $\tilde{\sigma}\sigma_j\chi_F^{U(1)} \in (G_F^{U(1)})^\wedge$ is a representation such that its restriction to $(G_F^0)^{U(1)}$ is just a multiplier of $\chi_F^{U(1)}$, the commutation relations for $\dot{W}$ can be provided by a direct computation as done in [1-3].

We assume that $\dim_{\mathbf{R}} \Omega = 2m$. Recall that the canonical bundle of $\widetilde{L}$ is the top exterior power $K^{\widetilde{L}} = \wedge^m(\widetilde{L}^0)$ of its annihilator $(\widetilde{L})^0 \subset (T\Omega_{\mathbf{C}})^*$. Then $K^{\widetilde{L}}$ is a complex line bundle in $\wedge^m(T\Omega_{\mathbf{C}})$ with the basis vector denoted by $\mathcal{K}_{\widetilde{L}}$.

Put $\mathcal{E}'_{\sigma,\rho}(P)^{\widetilde{L}}_{F'} = \{f \in \mathcal{E}'_{\sigma,\rho}(P)_{F'} \mid \widetilde{X} \in \widetilde{L}_{F'} \Rightarrow \dot{W}_{F'}(X)f = 0\}$. Then $\mathcal{E}'_{\sigma,\rho}(P)^{\widetilde{L}}$ is a tensor product of the complex line bundle $\mathcal{E}'(P)^{\widetilde{L}}$ associated with the representation $P(\tilde{\sigma}\sigma_j\chi_F^{U(1)})$. Arguing as in Theorem (6.9) of [6] we obtain the following result.

Proposition 3.3. *There exists a canonical isomorphism of complex bundles*

$$\mathcal{E}'_{\sigma,\rho}(P)^{\widetilde{L}} \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}} \to P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}).$$

In our multidimentional situation, by putting

$$Q(P)^{\widetilde{L}} = \mathcal{E}'_{\sigma,\rho}(P)^{\widetilde{L}} \otimes K^{\widetilde{L}},$$

we have

$$Q(P)^{\widetilde{L}} = P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}}.$$

Hence,

$$(Q(P)^{\widetilde{L}})^2 = P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}} \otimes K^{\widetilde{L}}.$$

Taking Proposition 3.3 into account we get

$$\begin{aligned}(Q(P)^{\widetilde{L}})^2 &= P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes \mathcal{E}'_{\sigma,\rho}(P)^{\widetilde{L}} \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}} \otimes K^{\widetilde{L}} \\ &= [P(\tilde{\sigma}\sigma_j\chi_F^{U(1)})]^2 \otimes K^{\widetilde{L}}.\end{aligned}$$

As in the above section, ∇^Γ is the connection in $P(\tilde{\sigma}\sigma_j\chi_F^{U(1)})$. Indeed, $\nabla^\Gamma \otimes I + I \otimes L$ (where L is the usual Lie derivation) gives a connection in $[P((\tilde{\sigma}\sigma_j)\chi_F^{U(1)})]^2$ and so uniquely defines a connection in $Q(P)^{\widetilde{L}}$ denoted by $\nabla^{\widetilde{L}}$. Denote by $\mathcal{S}_{\widetilde{L}}(Q(P)^{\widetilde{L}})$ the space of all sections of $Q(P)^{\widetilde{L}}$ for which $\nabla^{\widetilde{L}}_\xi s = 0$, $\forall \xi \in \widetilde{L}$. Put $H_{\widetilde{L}} = \{s \in \mathcal{S}_{\widetilde{L}}(Q(P)^{\widetilde{L}}) \mid$ the density $< s, s >_{\widetilde{L}}$ has compact support $\}$. An inner product is defined on $H_{\widetilde{L}}$ by integrating the density $< s, t >_{\widetilde{L}}$ over the leaf space $\Omega_{\widetilde{L}}$. Denote by $\mathcal{H}_{\widetilde{L}}$ the Hilbert space which is the completion of the space $H_{\widetilde{L}}$ (see [6, §8]).

Now we summarize the obtained results as follows.

THEOREM 1. *With any* $(\tilde{\sigma}, \chi_F^{U(1)})$*-polarization* $(\mathcal{N}, \rho, \sigma_0)$ *there exists a natural unitary representation of* G *in* $\mathcal{H}_{\widetilde{L}}$ *written by* $\mathrm{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j\chi_F^{U(1)}, \sigma_0, \rho)$ *as in Section 3.1.*

4. Unitary representation arising in the procedure of multidimensional quantization.

In this section, applying the procedure of multidimentional quantization of [3], we shall propose a mechanical interpretation of the representation $\mathrm{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}, \sigma_0, \rho)$.

4.1 The conditions of a procedure of quantization. As a model of the quantum system we choose the Hilbert space $\mathcal{H}_{\widetilde{L}}$ of Section 3.2. We shall use the bundle $Q(P)^{\widetilde{L}} = P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}}$ to construct the procedure of quantization :

$$\begin{array}{ccc} \widetilde{V} \otimes \mathcal{E}'(\mathcal{G}^{\mathbb{C}}/\mathcal{G}_F^{\mathbb{C}})^{\widetilde{L}} \otimes K^{\widetilde{L}} & \rightarrowtail & P(\tilde{\sigma}\sigma_j\chi_F^{U(1)}) \otimes \mathcal{E}'(P)^{\widetilde{L}} \otimes K^{\widetilde{L}} \\ & & \nabla^{\widetilde{L}} \downarrow \\ & & \Omega \end{array}$$

More precisely, we define the procedure of quantization as follows

$$\widehat{()} : C^\alpha(\Omega) \to \mathcal{L}(\mathcal{H}_{\widetilde{L}})$$
$$f \mapsto \hat{f} = f + \frac{\hbar}{i}\nabla^{\widetilde{L}}_{\xi_f},$$

where $\mathcal{L}(\mathcal{H}_{\widetilde{L}})$ is the space of all (unbounded Hermitian operators on $\mathcal{H}_{\widetilde{L}}$ and $\nabla^{\widetilde{L}}_{\xi_f}$ is the coveriant derivation associated with the connection $\nabla^{\widetilde{L}}$ on the G-bundle $Q(P)^{\widetilde{L}}$. We recall that $\nabla^{\widetilde{L}}_{\xi_f}$ is defined by the formula

$$\nabla^{\widetilde{L}}_{\xi_f} = L_{\xi_f} + \frac{i}{\hbar}\alpha(\xi_f),$$

where the 1-form $\frac{i}{\hbar}\alpha$ is the connection form of $\nabla^{\widetilde{L}}, L_{\xi_f}$ is the Lie derivation along the strictly Hamiltonian vector field ξ_f corresponding to f (see [5]).

By a similar argument as in [3], we obtain the following result.

Proposition 4.1. *The following three conditions are equivalent :*

i) The application $f \mapsto \hat{f}$ is a procedure of quantization,
ii) Curv $(\nabla^{\widetilde{L}})(\xi,\eta) = -\frac{i}{\hbar}B_\Omega(\xi,\eta).I$,
iii) $d_{\nabla^{\widetilde{L}}}\alpha(\xi,\eta) = -B_\Omega(\xi,\eta).I$.

Proof: By the Leibniz rule, $\nabla^{\widetilde{L}} \otimes I + I \otimes L$ is the connection on $(Q(P)^{\widetilde{L}})^2$, and we have

$$\text{curv}\,(\nabla^{\widetilde{L}} \otimes I + I \otimes L)(\xi,\eta) = \text{curv}\,(\nabla^{\widetilde{L}})(\xi,\eta).$$

Moreover, by the definition of $\nabla^{\widetilde{L}}$ we have

$$\begin{aligned}
&\text{curv}\,(\nabla^{\widetilde{L}})(\xi,\eta) = \text{curv}\,(\nabla^{\Gamma} \otimes I + I \otimes L)(\xi,\eta) = \\
&= [(\nabla^{\Gamma} \otimes I + I \otimes L)(\xi), (\nabla^{\Gamma} \otimes I + I \otimes L)(\eta)] - (\nabla^{\Gamma} \otimes I + I \otimes L)_{[\xi,\eta]} \\
&= [\nabla^{\Gamma}_\xi + L_\xi, \nabla^{\Gamma}_\eta + L_\eta] - \nabla^{\Gamma}_{[\xi,\eta]} - L_{[\xi,\eta]} \\
&= \text{curv}\,(\nabla^{\Gamma})(\xi,\eta).
\end{aligned}$$

Hence, if we assume that curv $(\nabla^{\Gamma})(\xi,\eta) = -\frac{i}{\hbar}B_\Omega(\xi,\eta).I$, then repeating the proof of [3] we get the assertion.

4.2. Representation arising in the procedure of quantization. Having the above procedure of quantization, we obtain the following representation of the Lie algebra $\mathcal{G}$ in the space $\mathcal{L}(\mathcal{H}_{\widetilde{L}})$

$$\begin{aligned}
\wedge : \mathcal{G} &\to \mathcal{L}(\mathcal{H}_{\widetilde{L}}) \\
X &\mapsto \wedge(X) = \frac{i}{\hbar}\hat{f}_X,
\end{aligned}$$

where $X \in \mathcal{G}$ and $f_X \in C^\infty(\Omega)$ is the generating function of the hamiltonian field ξ_X corresponding to X. If G is connected and simply connected, we obtain a unitary representation T of G defined by

$$T(\exp X) = \exp(\wedge(X)),\;\; X \in \mathcal{G}.$$

We say that it is the representation of G arising in the procedure of multidimensional quantization. As in [2] we also have the following result.

PROPOSITION 4.2. *The representation $\wedge$ arising from the procedure of multidimensional quantization coincides with the representation*

$$\begin{aligned} \psi : &\mathcal{G} \to \mathcal{L}(\mathcal{H}_{\widetilde{L}}) \\ &X \mapsto \psi(X) = L_{\xi X} + \frac{i}{\hbar}\{\varphi_X + \alpha(\xi_X)\}, \end{aligned}$$

*where α is the 1-form such that $\frac{\hbar}{i}\alpha$ is the connection form of $\nabla^{\widetilde{L}}$ and $\varphi_X : \Omega \to$ **R** is defined by $\varphi_X(F) = F(X)$.*

PROOF: We have

$$\begin{aligned} \frac{i}{\hbar}\hat{\varphi}_X &= (\varphi_X + \frac{i}{\hbar}\nabla^{\widetilde{L}}_{\xi_X}) = \frac{i}{\hbar}\varphi_X + \nabla^{\widetilde{L}}_{\xi_X} \\ &= \frac{i}{\hbar}\varphi_X + L_{\xi_X} + \frac{i}{\hbar}\alpha(\xi_X) \\ &= L_{\xi_X} + \frac{i}{\hbar}\{\varphi_X + \alpha(\xi_X)\} \\ &= \psi(X). \end{aligned}$$

REMARK: It is easy to see that in the group level this representation ψ lifts to the representation $\mathrm{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}, \sigma_0, \rho)$.

Summing up, we have proved the following result.

THEOREM 2. *With any $(\tilde{\sigma}, \chi_F^{U(1)})$-polarization $(\mathcal{N}, \rho, \sigma_0)$ the multidimensional quantization procedure gives us a representation ψ which lifts to the representation* $\mathrm{Ind}(G, \mathcal{N}, (\tilde{\sigma}\sigma_j)\chi_F^{U(1)}, \sigma_0, \rho)$.

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