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ON k-SURFACES MINIMIZING A FUNCTIONAL WITH A CONVEX LAGRANGIAN IN R^n

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0. Introduction

The problem of minimal currents and surfaces in Riemannian manifolds was studies by A.T. Fomenko [1], Dao Trong Thi [2], H. Federer, W.H. Fleming [3], and others.

The aim of this paper is to investigate some properties of k-surfaces minimizing a functional given by a lagrangian. In the case when $k = 1$, minimal curves were described in [4].

§1. Preliminaries

Let R^n be the n-–dimensional Euclidean space, $\wedge_k R^n$ and $\wedge^k R^n$ be the vector spaces of k-vectors and k-covectors on R^n respectively. Let M be a Riemannian manifold. Denote by $E^k M$ and $E_k M$ the vector spaces of differential k-forms and k-currents. Consider a functional J on $E_k M$. A k-current S is called absolutely minimal with respect to J if $J(S) \leq J(S')$ for any k-current S' such that $S - S'$ is closed.

A lagrangian L of degree k is a mapping $L : \wedge_k M \to R$ such that its restriction to each fibre $\wedge_k M_k$ is positively homogeneous, where M_x is the tangent space to M at x. Each lagrangian L of degree k on M defines a positively homogeneous functional J on $E_k M$ by the formula

$$J(S) = \int L(\vec{S}_x) d\, ||S||(x), \ S \in E_k M,$$

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where the k-vector $\vec{S}_x$ is defined by S, $||S||$ is the variational measure given by S (see [2]), $||\vec{S}_x|| = 1$.

Each oriented compact variety V in M can be identified with a k-current $[V]$ by the formula

$$[V] \quad (\varphi) = \int_V \varphi \,, \qquad \varphi \in E^k M.$$

Then the oriented tangent space V_x at x can be identified with the k-vector $[\vec{V}]_x$.

In this paper, assume that the lagrangian L is parallel, i.e. not dependent on x. Put

$$C_L = \{\xi \in \wedge_k R^n_x \,, \; L(\xi) \leq 1\},$$

and

$$S_L = \{\xi \in \wedge_k R_x \,, \; L(\xi) = 1\}.$$

We also assume that C_L is a convex polyhedron of dimension $\binom{n}{k}$ in the space $\wedge_k R^n_x = \wedge_k R^n = R^{\binom{n}{k}}$. This lagrangian is called convex polyhedral. For a set Z in a finite dimensional Euclidean space R^N, a hyperplane H^* is called supporting at $x \in Z$ if there is a linear form ω on R^N such that $\omega(\xi) = h$ for every $\xi \in H^*$ and $\omega(\xi) \leq h$ for every $\xi \in Z$, where $h \in R$. Then the set $H = H^* \cap Z$ is a face of Z. For any Z in R^N denote by CZ the set

$$CZ = \{t\xi; \; \xi \in Z, \; t > 0\}.$$

From Theorem 3.6 and 3.7 in [2] we have

THEOREM 1. *Let J be a functional on $E_k R^n$ given by a convex polyhedral lagrangian L. The k-current S is absolutely minimal with respect to*

J if and only if there is a face H of C_L such that $\vec{S}_t \in CH$ for almost every $t \in R^n$ in the sense of the measure $||S||$.

The k-surfaces minimizing J in the class of all k-currents with the same boundary are described by Theorem 1. Next we shall find conditions for the minimality of k-surfaces in the class of all oriented compact k-surfaces with the same boundary.

§2. On minimal k-surfaces

Given a lagrangian L of degree k in R^n as above. Denote by $G(k,n)$ the set of all oriented k-planes passing through the origin in R^n. Each k-plane of them can be identified with a simple k-vector in R^n, the norm of which equals to unit. Thus, $G(k,n)$ is contained in the unit sphere in the space $R^{\binom{n}{k}} = \wedge_k R^n$. Put

$$CG(k,n) = \{t\xi;\ t > 0,\ \xi \in G(k,n)\}.$$

If $1 < k < n-1$, then the set $CG(k,n) \cap C_L$ is not convex in $\wedge_k R^n$. By using faces of the set $CG(k,n)$ we can obtain a sufficient condition for the minimality of k-surfaces in R^n.

THEOREM 2. *Let J be the function on $E_k R^n$ given by the lagrangian L. Let S be a compact oriented k-surface of dimension k in R^n. If there is a face H of $CG(k,n) \cap S_L$ such that $\vec{S}_x \in CH$ for every $x \in S$, then S is minimizing the function J in the class of all compact oriented k-surfaces with the same boundary.*

PROOF: Let H be a face of $CG(k,n) \cap S_L$ defined by a hyperplane H^* in $R^{\binom{n}{k}}$. We shall show that H^* does not contain the origin 0 in $R^{\binom{n}{k}}$. First of all, assume that $0 \in H^*$. There are two cases.

1) There is a k-vector $\xi \in (CG(k,n) \cap S_L) \setminus H^*$. Then there exists $t > 0$ such that $-t\xi \in CG(k,n) \cap S_L$. Assume that $\omega(\xi) = h$ is the equation of H^* and $\omega(\xi) \leq h$ for every $\xi \in CG(k,n) \cap S_L$. Since $0 \in H^*$, it follows $h = 0$. For a point $\xi \in (CG(k,n) \cap S_L) \setminus H^*$ we have $\omega(\xi) < 0$. Then for $t > 0$, $\omega(-t\xi) = -t\omega(\xi) > 0$. It follows $-t\xi \notin CG(k,n) \cap S_L$ for any $t > 0$. That is a contradiction.

2) There is no k-vector in $(CG(k,n) \cap S_L) \setminus H^*$. Then $CG(k,n) \cap S_L \subset H^*$. It follows that $CG(k,n) \cap C_L \subset H^*$ and $CG(k,n) \subset H^*$. But in $CG(k,n)$ there are $\binom{n}{k}$ linearly independent k-vectors and $\dim H^* = \binom{n}{k} - 1$. Hence we obtain a contradiction.

Thus, we can assume that H^* does not contain the origin 0 in $R^{\binom{n}{k}}$. Hence, there is a linear form ω on $R^{\binom{n}{k}}$ such that H^* has the equation $\omega(\xi) = 1$ and $\omega(\xi) \leq 1$ for every $\xi \in CG(k,n) \cap S_L$. Then $\omega(\xi) = L(\xi) = 1$ for every $\xi \in H$. It follows that $\omega(\xi) \leq L(\xi)$ for every $\xi \in CG(k,n)$ and equality holds if and only if $\xi \in CH$.

Denote by $\overline{\omega}$ the constant-coefficient differential k-form corresponding to ω, i.e. $\overline{\omega}_x = \omega$ for every $x \in R^n$. It is easy to see that $\overline{\omega}$ is closed, hence exact, i.e. there is a differential $(k-1)$-form θ such that $d\theta = \omega$.

If H is a face of $CG(k,n) \cap S_L$ that satisfies the assumption of the theorem, then

$$(1) \qquad \omega(\vec{S}_x) = L(\vec{S}_x)$$

for every $x \in S$. Hence

$$(2) \qquad \int_S \overline{\omega} = \int_S L.$$

On the other hand,

$$(3) \qquad \int_S L = \int L(\vec{S}_x) d\, \|S\|(x) = J(S),$$

and

(4) $$\int_S \overline{\omega} = \int_S d\theta.$$

It follows that

(5) $$J(S) = \int_S d\theta$$

By the Stokes's theorem, we have

(6) $$\int_S d\theta = \int_{\partial S} \theta.$$

Let S' be an arbitrary compact oriented k-surface which has the same boundary with S. Then

(7) $$J(S') = \int_S L = \int L(\vec{S'}_x) d||S'||(x).$$

From the inequality $L(\xi) \geq \overline{\omega}(\xi)$ it follows

(8) $$\int_{S'} L \geq \int_{S'} \overline{\omega}.$$

By the Stokes' theorem, we have

(9) $$\int_{S'} \overline{\omega} = \int_{\partial S'} \theta = \int_{\partial S} \theta = J(S).$$

From (7), (8), (9) it follows

$$J(S') \geq J(S)$$

Thus, the minimality of S is proved and the proof of the theorem is completed.

By the above theorem, for a face H of the set $CG(k,n) \cap S_L$ there is a class of minimal k-surfaces which satisfy the assumption of the theorem. Denote it by $F(H)$. We shall describe $F(H)$.

REMARK: If L is the norm in $R^{\binom{n}{k}}$ induced by the Euclidean norm in R^n then $CG(k,n) \cap S_L = G(k,n)$ and we obtain a condition for the volume-minimality of a k-surface.

LEMMA 1. *Let ξ_1 and ξ_2 be two noncollinear simple k-vectors. The straight line $< \xi_1, \xi_2 >$ consists of simple k-vectors if and only if there exist linearly independent vectors $e_1, \cdots, e_k, e_{k+1}$ in R^n such that*

$$\xi_1 = e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k, \qquad (11)$$
$$\xi_2 = e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}.$$

PROOF: 1) Assume that ξ_1 and ξ_2 have the form (11). For any $\xi \in < \xi_1, \xi_2 >$, $\xi = t\xi_1 + (1-t)\xi_2$, where $t \in R$. It follows

$$\xi = te_1 \wedge \cdots \wedge e_{k-1} \wedge e_k + (1-t)e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}$$
$$= e_1 \wedge \cdots \wedge e_{k-1} \wedge (te_k + (1-t)e_{k+1}).$$

Hence ξ is a simple k-vector.

2) Now assume that the straight line $< \xi_1, \xi_2 >$ consists of simple k-vectors. Denote by $V(\xi_1)$, $V(\xi_2)$ the vector spaces associated to ξ_1, ξ_2 respectively, i.e.

$$e \in V(\xi_i) \Longleftrightarrow e \wedge \xi_i = 0; \; i = 1, 2.$$

Assume that $\dim V(\xi_1) \cap V(\xi_2) = 1$. Then there exist vectors $e_1, \cdots, e_\ell, \cdots, e_k$ in $V(\xi_1)$ and $e_1, \cdots, e_\ell, \cdots, e_{k+1}, \cdots, e_{2k-1}$ in $V(\xi_2)$ such that the system $\{e_1, \cdots, e_k, \cdots, e_{2k-1}\}$ is linearly independent and

$$\xi_1 = e_1 \wedge \cdots \wedge e_1 \wedge \cdots \wedge e_k,$$

$$\xi_2 = e_1 \wedge \cdots \wedge e_1 \wedge e_{k+1} \wedge \cdots \wedge e_{2k-1}.$$

By the assumption, $\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2$ is a simple k-vector. It follows that $\xi_1 + \xi_2$ is a simple k-vector. Assume that $\xi_1 + \xi_2 = f_1 \wedge \cdots \wedge f_k$, where $f_i \in R^n$, $i = 1, 2, \cdots, k$.

Let us choose vectors $e_{2k-1+1}, \cdots, e_n$ such that the system $\{e_1, e_2, \cdots, e_n\}$ is a basic of the space R^n. We have

$$f_i = \sum_{j=1}^{n} x_{ij} e_j, \ i = 1, 2, \cdots, k,$$

and

$$(14) \qquad f_1 \wedge \cdots \wedge f_k = \sum_{i_1 < \cdots < i_k} det \begin{bmatrix} x_{1i_1} \cdots x_{1i_k} \\ x_{ki} \cdots x_{ki_k} \end{bmatrix} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

On the other hand,

$$(15) \qquad f_1 \wedge \cdots \wedge f_k = e_1 \wedge \cdots \wedge e_k + e_1 \wedge \cdots e_1 \wedge e_{k+1} \wedge \cdots \wedge e_{2k-1}.$$

Since the system $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ is linearly independent, by (14), (15) all coefficients of $e_{i_1} \wedge \cdot \wedge e_{i_k}$ in (14 are equal to zero except the coefficients of $e_1 \wedge \cdots \wedge e_k$ and $e_1 \wedge \cdots \wedge e_1 \wedge e_{k+1} \wedge \cdots \wedge e_{2k-1}$.

Now assume that $1 < k - 1$, then

$$(16) \qquad det \begin{bmatrix} x_{11} \cdots x_{1k-1} x_{1k+i} \\ \cdots \\ x_{k1} \cdots x_{kk-1} x_{kk+i} \end{bmatrix} = 0, \ i = 1, 2, \cdots, k-1.$$

Consider $\alpha_1, \alpha_2, \cdots, \alpha_{2k-1} \in R^k$ given by $\alpha_j = (x_{1j}, \cdots, x_{kj})$, $i = 1, 2, \cdots, 2k-1$. If the system $\{\alpha_1, \cdots, \alpha_{k-1}\}$ is linearly dependent then the system $\{\alpha_1, \cdots, \alpha_{k-1}, \alpha_k\}$ is also linearly dependent. Hence

$$det \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{bmatrix} = 0$$

and it follows that the coefficient of $e_1 \wedge \cdots \wedge e_k$ in (14) equals to zero. This is a contradiction to (15).

If the system $\{\alpha_1, \alpha_2, \cdots, \alpha_{k-1}\}$ is linearly independent, then

(16) $$\alpha_{k+i} = t_{i1}\alpha_1 + \cdots + t_{ik-1}\alpha_{k-1}$$

for $i = 1, 2, \cdots, k-1$.

It easy to see that the system $\alpha_1, \cdots, \alpha_1, \alpha_{k+1}, \cdots, \alpha_{2k-1}$ is linearly dependent. It follows

$$det \begin{bmatrix} x_{11} & \cdots & x_{1\ell} & x_{1k+1} & \cdots & x_{12k-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{k1} & \cdots & x_{k\ell} & x_{kk+1} & \cdots & x_{k2k-1} \end{bmatrix} = 0.$$

Thus, the coefficient of $e_1 \wedge \cdots \wedge e_1 \wedge e_{k+1} \wedge \cdots \wedge e_{2k-1}$ in (14) equals to zero. That is a contradiction to (15). Hence we obtain $\ell = k-1$ and the proof of the lemma is completed.

LEMMA 2. *Let $\xi_0, \xi_1, \cdots, \xi_m$ be linearly independent simple k-vectors and $< \xi_0, \xi_1, \cdots, \xi_m >$ be the m-plane in $R^{\binom{n}{k}}$ defined by $\xi_0, \xi_1, \cdots, \xi_m$. Assume that $< \xi_0, \xi_1, \cdot, \xi_m >$ consists of only simple k-vectors. Denote by $V(\xi_0), \cdots, V(\xi_m)$ the vector spaces associated to $\xi_0, \cdots, \xi_m$ respectively. Then*

$$dim\ [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_m)] \leq k + m.$$

PROOF: Since $< \xi_0, \xi_1, \cdots, \xi_m >$ consists of only simple k-vectors, the straight line $< \xi_i, \xi_j >$ consists of only simple k-vectors. By Lemma 1,

(19) $$dim\ V(\xi_i) \cap V(\xi_j) = k-1 \quad \text{for} \quad i \neq j.$$

On the other hand,

$$dim\ [V(\xi_o) + V(\xi_1)] = dim\ V(\xi_0) + dim\ V(\xi_1) -$$

$$-dim\ V(\xi_0) \cap V(\xi_1) = k+1$$

and

$$dim\ [V(\xi_0) + V(\xi_1) + V(\xi_2)] = dim\ [V(\xi_0) + V(\xi_1)] +$$

$$dim\ V(\xi_2) - dim\ [V(\xi_0) + V(\xi_1)] \cap V(\xi_2).$$

It follows $dim\ [V(\xi_0)+V(\xi_1)+V(\xi_2)] \leq k+2$. Analogously, we obtain

$$dim\ [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_m)] \leq k+m.$$

Thus, the lemma is proved.

LEMMA 3. *Let H be an m-dimensional face of $CG(k,n) \cap S_L$ which is contained in an m-plane. Let $\xi_0, \xi_1, \cdots, \xi_m$ be linearly independent k-vectors in H. Then for every $\xi \in H$,*

$$V(\xi) \subset [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_m)]. \tag{20}$$

PROOF: By the asssumption, it easy to see that $H \subset \, < \xi_0, \xi_1, \cdots, \xi_m >$. Denote by $< \xi_0, \cdots, \xi_i >$ the i-plane defined by $\xi_0, \xi_1, \cdots, \xi_i$. We shall show by induction that for any $\xi \in < \xi_0, \xi_1, \cdots, \xi_i >$,

$$V(\xi) \subset [V(\xi_0) + \cdots + V(\xi_i)]$$

For $i = 1$, by Lemma 1 we have

$$V(\xi) \subset [V(\xi_0) + V(\xi_1)]. \tag{21}$$

By the induction hypothesis,

$$V(\xi) \subset [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_{m-1})]. \tag{22}$$

for every $\xi \in < \xi_0, \xi_1, \cdots, \xi_{m-1} >$.

Let ξ' be a k-vector in H. Without loss of generality, we may assume that $< \xi_m, \xi' > \cap < \xi_0, \xi_1, \cdots, \xi_{m-1} > = \xi$ then $V(\xi) \subset [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_{m-1})]$. On the other hand, by Lemma 1

$$V(\xi') \subset [V(\xi_m) + V(\xi)].$$

Hence we obtain

$$V(\xi') \subset [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_m)].$$

Thus, the proof of the lemma is completed.

THEOREM 3. *Let H be an m-dimensional face of $CG(k,n) \cap S_L$ which is contained in an m-place. Let S be a minimal k-surface corresponding to H (i.e. $S \in F(H)$). Assume that S is arcwise connected. Let $\xi_0, \xi_1, \cdots, \xi_m$ be linearly independent k-vectors in H and W be the $(k+m)$-plane in R^n defined by vector space $V(\xi_0) + \cdots + V(\xi_m)$ and a point $x_0 \in S$. Then $S \subset W$.*

PROOF: Since S is contained in $F(H)$, $\vec{S}_x \in CH$ for every $x \in S$, where $\vec{S}_x$ is k-vector defined by the tangent space S_x at $x \in S$. By Lemma 3 we have

$$V(\vec{S}_x) \subset [V(\xi_0) + V(\xi_1) + \cdots + V(\xi_m)]$$

Let y be an arbitrary point on S. Since S is arcwise connected, there is a differentiable curve γ passing through x_0, y. Then every tangent vector to γ is contained in $V(\xi_0) + \cdots + V(\xi_m)$. Hence γ is contained in W.

The theorem is proved.

§3. On the convexity of C_L

In the case when C_L is convex, every k-plane K in R^n is minimal with respect to J. Actually, from the convexity of C_L it follows that for each

$\xi \in S_L$ there is a face H containing ξ. Obviously, $\vec{K}_x$ is a fixed k-vector for every x. There exists $t > 0$ such that $t\vec{K}_x \in S_L$. Hence there is a face H of C_L such that $\vec{K}_x \in CH$. By Theorem 1, the k-plane K is minimal.

In the case when C_L is not convex, the above statement is not true. For example, assume that $k = 1$. Since C_L is not convex, there are two points $a \in S_L$, $b \in S_L$ such that the straight segment $[a, b]$ is not contained in C_L. Hence there is a point $\xi \in [a, b] \setminus C_L$. Then there is a point ξ_1 in $[0, a]$ such that $[\xi, \xi_1] \parallel [0, b]$. Let ξ_2 be a point in $[0, b]$ such that $[\xi, \xi_2] \parallel [0, a]$. Then $\xi = \xi_1 + \xi_2$. Moreover $\xi_1 = t_1 a$, $\xi_2 = (1 - t_1)b$, where $t_1 \in R$. Denote by $[0, \xi_2, \xi]$ the broken line passing through $0, \xi_2, \xi$. Then we have

$$\begin{aligned} J([0,\xi_2,\xi]) &= J([0,\xi_2]) + J([\xi_2,\xi]) \\ &= L(\xi_2 - 0) + L(\xi - \xi_2) = L(\xi_2) + L(\xi_1) \\ &= L(t_1 a) + L((1 - t_1)b) = t + (1 - t) = 1 \end{aligned} \tag{24}$$

On the other hand, putting $\xi' = [0, \xi] \cap S_L$ we have

$$J([0,\xi]) = L(\xi), \tag{25}$$

$$J([0,\xi']) = L(\xi') = 1.$$

Since $\xi' = t\xi$, $0 < t < 1$ it follows

$$L(\xi') = tL(\xi). \tag{26}$$

From (25) we have

$$L(\xi) > 1. \tag{27}$$

From (24), (25), (27) we obtain

$$J([0,\xi]) > J([0,\xi_2,\xi]). \tag{28}$$

Thus, the broken line $[0, \xi_2, \xi]$ is shorter than the straighty segment $[0, \xi]$. Hence $[0, \xi]$ is not minimal with respect to the functional J.

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REFERENCES

1. Dao Trong Thi and A.T. Fomenko, *Minimal surfaces and Plateau's problem*, Nauka 1987.
2. Dao Trong Thi, *Global minimal currents and surfaces in Riemannian mar[illegible]*, Acta Math. Vietnamica **10** (1985), 296–332.
3. H. Federer and W.H. Fleming, *Normal and integral currents*, Ann. Math. **72** (1[illegible]0) 458–520.
4. Tran Viet Dung, *On curves minimizing polyhedral functionals in R^n*; Acta Math. Vietnamica **15** (1990), 103–110.

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