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ON THE LANGLANDS TYPE DISCRETE GROUPS. II THE THEORY OF EISENSTEIN SERIES

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0. Introduction

Let G be a reductive Lie group, K a maximal compact subgroup of G, Γ a Langlands type discrete subgroup (see, for example, [4], [14]) and (τ, V) some unitary Γ-module. Our final aim is to study the Eilenberg-MacLane cohomology groups $H^*(\Gamma, V)$. If Γ acts freely on the contractible space $X = K \setminus G$, the quotient X/Γ is a smooth manifold and is also an Eilenberg-MacLane space $K(\Gamma, 1)$. Therefore $H^*(\Gamma, V)$ are isomorphic to the continuous cohomology groups $H^*(X/\Gamma, \mathcal{F}V)$ with coefficient in a local coefficient sheaf system $\mathcal{F}V$, associated with the induced representation $Ind^G(\tau, V)$.

The homogeneous space X has a natural Riemannian structure with negative defined scalar curvature. Thus for continuous cohomology classes with compact support one can develop the ordinary Hodge theory. From the long exact sequence for the pair of space X/Γ and its boundary, the "suplementary" part is closely related to the boundary of X/Γ. Following A. Borel and J.P. Serre, in the first paper [4], we have constructed compactification $\overline{X}_{cusp}/\Gamma$ of X/Γ, the boundary of which is homotopic to the quotient by Γ of the cuspidal part $\overline{X}_{cusp}$ of the Tits building, consisting of all cuspidal parabolic subgroups.

With each cuspidal subgroup R.P. Langlands [13] associated a family of Eisenstein series. The spectral theory of Eisenstein series provided

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us (see [1], [13]) a spectral decomposition of the (twisted) regular representation $Ind^G(\tau, V)$. It is convenient to recall that the spectral theory of Eisenstein series was first studied by Selberg [17]. It was then completed by R.P. Langlands in a manuscript which was twelve years unpublished until its appearance in the Springer Lecture Notes in Mathematics [13] in 1976. The Langlands theory was for a long time very difficult to understand and to use, see for example Godement's talk [7] in Bourbaki Seminar. Harish-Chandra, in his lectures at the Institute for Advanced Studies [10] refined the theory for semi-simple algebraic Lie groups with arithmetically defined discrete subgroups in a more comprehensive form by using the Maas-Selberg relations. In other papers [11] he developed an analogous theory of Eisenstein integrals to study the Plancherel measure. This was an excellent work of Harish-Chandra, showing the (second) important application of the theory to Harmonic Analysis on real reductive groups, closely connected with the famous Langlands programme. Its (third) application to Algebraic Topology was done in the works of G. Harder [8], [9]. For parabolic rank one arithmetically defined discrete groups, G. Harder described the cohomology classes at infinity. J. Schwermer [16] has considered the case of SL_n as well, as K.F. Lai [15] has generalized some results to SP_{2n} case. In all these theories the Eisenstein series was constructed as finite-dimensional vector-valued ones.

Near 1980, M. Duflo [5], [6] in one hand and the author [1]–[3] in the other hand, have seen the necessity to construct the Multidimensional Orbit Method. In general, we must consider Hilbert induced bundles of finite or infinite dimension. There appears a question whether the Eisenstein series can be constructed as the (in)finite-dimensional vector-valued ones. This paper is devoted to developing such a theory of (in)finite dimensional vector-valued Eisenstein series related to cuspidal parabolic subgroups. Our main results are : 1) finiteness of the spectrum of $Ind^G(\tau, V)$ on the space $\mathcal{A}_V(G/\Gamma, \sigma, \chi)$ of automorphic (in)finite dimensional vectorvalued forms, 2) a construction and some properties of Eisenstein series. We intend also to study the cohomological usage of this theory [12].

We keep therefore all the notations appeared in our first paper [4]. It must remarked that in [4] for avoiding a misprint, the definition of percuspidal subgroups must be precised as follows. A cuspidal subgroup P is called percuspidal iff ${}^\circ P/\Gamma \cap P$ is compact. Our exposition is much influenced by Harish-Chandra's [10], [11], Landlands' [13] and Arthur's [1] works. We would like to address them our deep thanks for having brought to our knowledge of their remarkable works concerning Eisenstein series and trace formulae. The author thanks Dr. Dinh The Luc for valuable comments

Contents :

1. $\mathcal{A}_V(G/\Gamma,\sigma,\chi)$ has finite spectrum

Recall that G is a reductive Lie group, K a maximal compact subgroup of G, Γ a fixed Langlands type discrete subgroup and (τ, V) some unitary Γ-module of any dimension. We do not restrict to the finite dimensional case because for the Langlands type discrete groups the irreducible Γ-modules are ingeneral infinite dimensional, for example, the restrictions of irreducible G-modules to Γ. Let $P = MAU$ be a cuspidal subgroup of G. We fix the standard normalization of left invariant Haar measure dx on G by

$$dx = e^{2\varrho(log a)}dkdmdadu,$$

if $x = kmau \in G = KMAU$. Since $G = K.exp\ \mathcal{P}$, $\mathcal{P} = LieP$, following Harish-Chandra [11] we can defined functions $||\cdot||_{\sigma_0}$ and Ξ on G by

$$||X|| := ||k.exp\ X||_{\sigma_0||};\ \Xi(k.exp\ X) = \Xi(exp\ X)$$

Remark that in general $||\cdot||_{\sigma_0}$ and Ξ are independent of the choice of P, and we shall denote $||\cdot||_{\sigma_0}$ simply by $||\cdot||$. If $P = p_0 = M_0A_0U_0$ is a minimal parabolic subgroup, then

$$\Xi(x) = \int_K e^{-\varrho_{P_0}(H_{P_0}(xk)}dk \stackrel{\text{formally}}{=} \int_k (xk)^{-\varrho_0}dk.$$

Let (σ, V) be a unitary representation of K on V such that every $\sigma(k)$ $(k \in K)$ commutes with any $\tau(\gamma)$ $(\gamma \in \Gamma)$, $\chi : \mathcal{Z} = \text{Cent } (U(\mathcal{G}_{\mathbf{C}})) \in \text{End}_K V$ a representation of the center of the universal enveloping algebra $U(\mathcal{G}_{\mathbf{C}})$ on V commuting with any $\sigma(k)$ $(k \in K)$. For this reason we shall use the right actions of Γ and $\mathcal{Z} = \text{Cent } U(\mathcal{G}_C)$. The representation σ enables us to define the representation σ_M of $K_M := K \cap^0 P$ in V by

$$\sigma_M(\pi_{P|M}(k)) := \sigma(k) \qquad (k \in K_M).$$

Recall that the Lie algebra $\mathcal{G} = \mathit{Lie}\ G$ acts on the smooth functions on G by the left regular representation λ and by the right regular anti-representation ϱ

$$(\lambda(X)f)(x) := \frac{d}{dt}\,|_{t=0}\ f(x\ exp\ X) := f(x; X),$$

$$(\varrho(X)f)(x) := \frac{d}{dt}\,|_{t=0}\ f(exp\ X.x) := f(X; x).$$

These actions of $\mathcal{G}$ can be extended to the corresponding action of $U(\mathcal{G}_{\mathbf{C}}) \times U(\mathcal{G}_{\mathbf{C}})$ on the smooth functions on G, $f \mapsto (g_1 \otimes g_2)f \qquad (g_1, g_2 \in U(\mathcal{G}_{\mathbf{C}}))$, $((g_1 \otimes g_2)f(x) := f(g_1 x; g_2)$.

Let us consider the following functional spaces

$$C^\infty_V(G, \sigma) := \{f \in C^\infty_V(G); f(kx) = \sigma(k)f(x) \quad (k \in K,\ x \in G)\}$$

$$C_V(G/\Gamma, \sigma) := \{f \in C_V(G); f(kx\gamma) = \sigma(k)f(x)\tau(\gamma)\ (k \in K, x \in G, \gamma \in \Gamma)\}$$

$$L^2_V(G/\Gamma, \sigma) := \{f \in L^2_V(G); f(kx\gamma) = \sigma(k)f(x)\tau(\gamma)\ (k \in K, x \in G, \gamma \in \Gamma)\}$$

$$\mathcal{A}_V(G/\Gamma,\sigma) := \{f \in C_V(G/\Gamma,\sigma); \forall D \in U(\mathcal{G}_{\mathbf{C}}) \times U(\mathcal{G}_{\mathbf{C}}), \exists r \in \mathbb{R} \ ; \ \sup_{x\in G} |Df(x)|\Xi(x)^{-1}(1+\|x\|)^{-r} < \infty\}$$

$$\mathcal{A}_V(G/\Gamma,\sigma,\chi) := \{f \in \mathcal{A}_V(G/\Gamma,\sigma); \ (Zf)(x) = f(x)\chi(Z) \quad (Z \in \mathcal{Z}) \ \ f \text{ is } K-\text{finite and is of } \mathcal{Z}-\text{finite spectrum}\}$$

The last condition means that if $f \in \mathcal{A}_V(G/\Gamma,\sigma,\chi)$, then the set $\{f_k; f_k(x) := f(kx)\}$ generates a finite dimensional subspace and the set $\{Zf; \ z \in \mathcal{Z} = \text{Cent } U(\mathcal{G})_{\mathbf{C}}\}$ is a G-invariant subspace on which $\text{Ind}^G(\tau, V)$ has finite spectrum, i.e. every irreducible G-invariant closed subspace has a finite multiplicity.

Recall that the elements of $\mathcal{A}_V(G/\Gamma,\sigma,\chi)$ are called the <u>automorphic forms</u> of type (σ,χ). We set

$$\mathcal{A}_V^\infty(G/\Gamma,\sigma,\chi) := \mathcal{A}_V(G/\Gamma,\sigma,\chi) \cap C_V^\infty(G).$$

Let $f \in L^2_V(G/\Gamma)$. It is easy to see [10] that the following three conditions are equivalent :

(1) $P = MAU$, $\varphi \in C_c(G/U)$

$$\int_{G/U} \varphi(x)f(x)dx = 0,$$

(2) $f^P(x) := \int_{U/U\cap\Gamma} f(xu)du = 0, \quad \forall$ cuspidal P,

(3) $f^P(x) := \int_{U/U\cap\Gamma} f(xu)du = 0, \quad \forall$ percuspidal P.

If one of these equivalent conditions holds we say that f is a (V-valued) cusp form and denote $f \in {}^0L^2_V(G/\Gamma) = {}^0L^2(G/\Gamma)\hat{\otimes}_{\mathbf{C}} V$

THEOREM 1. *If* $\alpha \in C_c(G)$ *and* $\lambda = Ind_\Gamma^G(\tau, V)$ *then* ${}^0\lambda(\alpha) = \lambda(\alpha\ |_{{}^0L^2_V(G/\Gamma)}$ *is of finite spectrum.*

PROOF: From Gelfand-Shapiro's theorem, it is easy to see that ${}^0\lambda(\alpha)\ |_{{}^0L^2(G/\Gamma)}$ is compact. So it has only discrete spectrum, i.e. every eigenvalue has finite multiplicity. The theorem is proved. ∎

LEMMA 1. ${}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi) \hookrightarrow {}^0 L^2(G/\Gamma) \hat{\otimes} V$

PROOF: $f \in \mathcal{A}_V(G/\Gamma, \sigma, \chi)$, then it is analytic $\mathcal{Z}$-finite and K-finite (see Harish-Chandra [11]). Also following Harish-Chandra, there exists $\alpha \in C_c^\infty(G)$ such that $f = \alpha * f$. So we have

$$\begin{aligned} f(g_i x) &= (\alpha * f)(g_i x) = (g'\alpha * f)(x) \\ &= \int g'\alpha(y) f(y^{-1}x) dy. \end{aligned}$$

$$\begin{aligned} |f(g_i x)| &\leq C(1 + ||y^{-1}||||x||)^r \Xi(y^{-1}x) \int g'\alpha(y) dy \\ &\leq C'(1 + ||y^{-1}||)^r \Xi(y^{-1})(1 + ||x||)^r \Xi(x) \int g'\alpha(y) dy \\ &\leq C''(g) \Xi(x)(1 + ||x||)^r. \end{aligned}$$

In other words, f has tempered growth. ∎

LEMMA 2. (A MODIFIED GODEMENT'S LEMMA). *Let* X *be a locally compact space,* $d\mu$ *a (positive) probability measure on it,* $\mathcal{H} \subset L^2_V(X, d\mu)$ *a closed subspace. If every function* f *of* $\mathcal{H}$ *is essentially bounded, then* $\mathcal{H}$ *is finitely generated, i.e.* $\mathcal{H} \subset \mathbb{C}^n \hat{\otimes} V$ *for some* n.

PROOF: We have $||f||_2 \leq ||f||_\infty$. So the identity map Id : $(\mathcal{H}, ||\cdot||_\infty) \longrightarrow (\mathcal{H}, ||\cdot||_2)$ is continuous. In virtue of the closed graph theorem, this map is a homeomorphism, i.e.

$$||f||_\infty \leq c||f||_2 \qquad (\forall f \in L^2_V(X, d\mu)).$$

If $f_1, \cdots, f_n$ is an orthonormal system, then so is $|f_1|, \cdots, |f_n|$ in $L^2(X, d\mu)$. Thus for almost all $x \in X$,

$$| \sum_{j=1}^{n} a_j |f_j(x)| \, | \leq c \left(\sum_{1 \leq j \leq n} |a_j|^2 \right)^{1/2}, \qquad (*)$$

$$a = (a_1, \cdots, a_n) \in \mathbb{C}^n.$$

$*$ holds for almost all $x \in X$ and a in some countable dense subset $D \subset \mathbb{C}^n$. For a fixed $x \in X$, $(*)$ holds for whole $\mathbb{C}^n$.

Taking $a_j = |f_j(x)|$, we have

$$\sum_{j=1}^{n} |f_j(x)|^2 \leq c^2$$

Hence,

$$n \leq c^2.$$

So $\mathcal{H} \subset \mathbb{C}^n \ \hat{\otimes} \ V = \underbrace{V \oplus \cdots \oplus V}_{n \text{times}}$. ∎

LEMMA 3. *In the space* ${}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi)$ *the induced representation* $\mathrm{Ind}^G(\tau, V)$ *has finite spectrum.*

PROOF: We take $X = G/\Gamma$, $d\mu = dx$, $\mathcal{H} = {}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi)$. Following Lemma 2, we only need to prove that $\mathcal{H}$ is closed. Indeed, if $\varphi_n \in \mathcal{H}$ and $\varphi_n \to \varphi$ in the distribution sense, then $Z.\varphi_n \to Z.\varphi \quad (\forall Z \in \mathcal{Z})$. Thus $Z.\varphi = \varphi.\chi(Z)$, $\varphi \in {}^0L^2_V(G/\Gamma, \sigma, \chi) = {}^0L^2(G/\Gamma, \sigma, \chi)\hat{\otimes}V$. Following Harish-Chandra, we also have $\varphi = \alpha * \varphi$, $\alpha \in C_c(G)$. So we conclude that $\varphi \in {}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi)$. ∎

In fact, we have ${}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi) = {}^0L^2_V(G/\Gamma, \sigma, \chi)$. With any unitary equivalence class $< \sigma >$ of representations of K, we define also the "scalar" automorphic forms spaces $\mathcal{A}(G/\Gamma, <\sigma>, \chi)$, which is generated by the functions $x \mapsto <v, f(x)> \quad (v \in V,\ f \in \mathcal{A}_V(G/\Gamma, \sigma, \chi))$.

THEOREM 2. *dim*$_{\mathbf{C}}$ $(G/\Gamma, \sigma, \chi) < \infty$ *and the representation* $\mathrm{Ind}^G(\tau, V)$ *of* G *on* $\mathcal{A}_V(G/\Gamma, \sigma, \chi)$ *has finite spectrum.*

PROOF: The first assertion $\dim_{\mathbf{C}} \mathcal{A}(G/\Gamma, \sigma, \chi) < \infty$ is proved in Langlands [13] and Harish-Chandra [10].

The proof of the 2^{nd} assertion is similar to the proof of the same assertion in finite dimensional case (see Harish-Chandra [10]), (use induction on rank G).

If rank $G = 0$, we have $\mathcal{A}_V(G/\Gamma, \sigma, \chi) = {}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi)$ and it has finite spectrum by Lemma 3.

Consider the case rank $G = \ell \geq 1$. Let $P = MAU$ be an own cuspidal subgroup, $f \in \mathcal{A}_V(G/\Gamma, \sigma, \chi)$. Recall that for $\mathcal{Z}_1 = \text{Cent }(U(\mathcal{M}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})) \simeq \text{Cent }(U(\mathcal{M}_{\mathbf{C}}) \otimes U(\mathfrak{a}_{\mathbf{C}}))$ there exists a canonical injective homomorphism

$$\mu : \mathcal{Z} = \text{Cent } U(\mathcal{G}) \hookrightarrow \mathcal{Z}_1 = \text{Cent } U(\mathcal{M}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})$$

such that $\mathcal{Z}_1$ is a free $\mathcal{Z}$-module of finite rank. Let us denote by ϱ_P the half-sum of positive roots of $P = MAU$

$$\varrho_P(X) = \frac{+1}{2} tr_{\mathcal{U}}(adX)$$

and put $X' = X + \varrho_P(X)$ for $X \in \mathcal{M}_1 = \mathcal{M} + \mathfrak{a}$. So $X \mapsto X'$ can be extended to homomorphism from $U(\mathcal{M}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})$ into itself. It is well-known that for $z \in \mathcal{Z},\ Z - \mu(Z)' \in U(\mathcal{G}_{\mathbf{C}})\mathcal{U}$ and

$$\mu(Z)' = d_P^{-1} \circ \mu(Z) \circ d_P$$

in the sense of differential operators on M_1, where

$$d_P(m) := |\det\ (Adm)|^{1/2} \qquad \text{and}$$

$$d_P(m)(a) = e^{\varrho_P(\ell n a)} \qquad (m \in M,\ a \in A).$$

We define now $\pi_P : \mathcal{A}_V(G/\Gamma, \sigma, \chi) \longrightarrow \mathcal{A}(M/\Gamma_M, \sigma_M \otimes 1, \chi_M)$ by the map $f \mapsto \varphi_f\ |_M$,

$$\varphi_f := \sum_{i=1}^{r} (\zeta_i \pi_P f) \otimes \zeta_i^{**},$$

where $\zeta_1, ..., \zeta_r \in \mathcal{Z}_1$ such that $\zeta_1^*, ..., \zeta_r^*$ form a basis for $\mathcal{Z}_1^* = \mathcal{Z}_1 / Ker\ \chi$ and $\zeta_1^{**}, ..., \zeta_r^{**}$ a basis for $\mathcal{Z}_1^{**}$,

$$\sum_{n=1}^{r} (\zeta \zeta_i^*) \otimes \zeta_i^{**}) = \sum_{i=1}^{r} \zeta_i^* \otimes (\zeta_i^{**} \zeta),$$

$$\sum (\zeta \zeta_i \pi_P f) \otimes \zeta_i^{**}) = \sum \zeta_i \pi_P f \otimes (\zeta_i^{**} \zeta),$$

$$\chi_1 : \mathcal{Z}_1 \to \text{End}\ (V \otimes_{\mathbf{C}} \mathcal{Z}_1^{**}),$$

$$(v \otimes \zeta^{**}) \chi_1(\eta) = v \otimes \zeta^{**} \eta,$$

$$\zeta \varphi_f = \varphi_f \chi_1(\zeta) \qquad (\zeta \in \mathcal{Z}_1),$$

$$H \varphi_f = \varphi_f \chi_1(H) \qquad (H \in \mathfrak{a}),$$

and finally

$$\varphi_f\ |_M = 0 \Longleftrightarrow \pi_P f = 0.$$

So we have $\mathcal{A}_V(G/\Gamma, \sigma, \chi)/\mathcal{A}_P \hookrightarrow \mathcal{A}_{V \otimes \mathcal{Z}_1^{**}}(M/\Gamma_M, \sigma_M \otimes 1, \chi_M)$ where $\mathcal{A}_P :=$ Ker π_P. By the induction hypothesis, $\mathcal{A}_{V \otimes \mathcal{Z}_1^{**}}(M/\Gamma_M, \sigma_M \otimes 1, \chi_M)$ has finite spectrum, then so is $\mathcal{A}_V(G/\Gamma, \sigma, \chi)/\mathcal{A}_P$.

If $P_1, \cdots, P_s$ are the representatives of all Γ-conjugation classes of percuspidal subgroups. Then $\bigcap_{1 \leq i \leq s} \mathcal{A}_{P_i} =^0 \mathcal{A}(G/\Gamma, \sigma, \chi)$ shich also has finite spectrum in view of Lemma 3.3. the theorem is proved. ∎

If P is a cuspidal subgroup, and $f \in \mathcal{A}_V(G/\Gamma, \sigma, \chi)$, then its cuspidal component f_P has tempered growth.

Langlands' Theorem 3

For every cuspidal subgroup (P, A) and for every $\varphi \in^0 \mathcal{A}_V(M/\Gamma_M, \sigma_M, \chi_M)$. If

$$\int_{M/\Gamma_M} (\varphi(m), f_P(ma))dm = 0 \qquad \text{for almost all} \quad a \in A$$

then $f = 0$.

PROOF: No change is needed in the Harish-Chandra's exposition for finite dimensional case by remarking that

$$^0L^2_V(G/\Gamma, \sigma) = \text{closure} \; (\sum_{\chi \in \mathrm{Char}\ \mathcal{Z}} {}^0\mathcal{A}_V(G/\Gamma, \sigma, \chi))$$

2. Definition and properties of Eisenstein series.

Let (P, A) be a cuspidal subgroup of G, $< \lambda, \mu >$ the scalar product on $\mathfrak{a}^*_{\mathbb{C}}$ associated with the Killing form B on $\mathcal{G}_{\mathbb{C}}$. For all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, there exists a unique element $H_\lambda \in \mathcal{G}_{\mathbb{C}}$ such that $\lambda(H) = B(H_\lambda, H)$. So we have $< \lambda, \mu > = B(H_\lambda, H_\mu)$. Let us denote by $\alpha_1, ..., \alpha_\ell$ the simple roots of $(\mathcal{P}, \mathfrak{a})$ and $\lambda_1, ..., \lambda_\ell$ the corresponding fundametal weights, $< \lambda_i, \alpha_j > = \delta_{ij}$,

$$\mathfrak{a}^+ := \{H \in \mathfrak{a};\ \alpha_i(H) \geq 0,\ 1 \leq i \leq \ell\},$$

$$^+\mathfrak{a} := \{H \in \mathfrak{a};\ \lambda_i(H) \geq 0,\ 1 \leq i \leq \ell\}.$$

Since the matrix $(< \alpha_i, \alpha_j >)^{-1}$ only has positive entries, $^+\mathfrak{a} \supset \mathfrak{a}^+$. We introduce the complex domain

$$(\mathfrak{a}^*)^+ := \{\lambda \in \mathfrak{a}^*;\ < -\lambda + \varrho, \alpha > \ > 0,\ \forall \alpha \in \Delta(P/A)\},$$

$$(\mathfrak{a}^*_{\mathbb{C}})^+ := \{\lambda \in \mathfrak{a}^*_{\mathbb{C}};\ Re\lambda \in (\mathfrak{a}^*)^+\}.$$

Let $P = MAU$ be a cuspidal subgroup of G, (σ, V) a finite spectrum representation of K, $\varphi \in L^2_V(M/\Gamma_M, \sigma_M)$ a σ_M-function on M. We extend φ to a σ-function $\varphi_\Lambda \in L^2_V(G,\sigma)$, $\Lambda \in \mathfrak{a}^*_{\mathbb{C}}$, on G by

$$\varphi_\Lambda(x) = \varphi(kmau) := \sigma(k)\varphi(m)e^{-(\Lambda+\varrho)(H_P(a))} \underset{\text{formally}}{=} \sigma(k)\varphi(m)a^{-\Lambda-\varrho}.$$

Remark that in the decomposition $x = kmau$, $a = a(x)$ is unique, $H_P(a) = log\ a(x)$, and $k = k(x)$ can be changed up to a factor $k' \in K_M$. If $x = (kk')(k'^{-1}m)au$ then $\varphi_\Lambda(x) = \sigma(kk')\varphi(k'^{-1}m)a^{-\Lambda-\varrho} = \sigma(k)\varphi(m)a^{-\vee-\varrho}$ $= \varphi_\Lambda(kmau)$. Thus the extended φ_Λ is just a function on G.

It is easy to show (see Langlands [13] and Harish-Chandra [10]) that for any compact $\Omega \subset G$ we can choose a constant c and an element $H_1 \in \mathfrak{a}$ such that

$$\sum_{\gamma\in\Gamma/\Gamma\cap P} \int_\Omega |\varphi_\Lambda(y^{-1}xy_0\gamma)|dy \le c||\varphi||_1\Xi(x)(1+||x||)^2 \times$$

$$\times \frac{e^{(-\Lambda'+\varrho)(H'(x))+(-\Lambda+\varrho)(H_1)}}{\prod_{\alpha\in\Delta(P|A)} |< -\Lambda+\varrho, \alpha >|} \qquad (**)$$

for all $y_0 \in \Omega$, $x \in \mathcal{S}_0$, the Siegel domain $H' : H_{P'|A'}$, $\Lambda \in (\mathfrak{a}^*)^+$ and $\Lambda' = \xi_\Lambda$, associated to P_0. where P_0 is the percuspidal subgroup such that $(P', A' :=^\xi (P, A) \succ (P_0, A_0)$.

Remark that vol $(M/\Gamma_M) < \infty$, then from the assumption $\varphi \in L^2_V(M/\Gamma_M, \sigma)$ one deduces that $\varphi \in L^1_V(M/\Gamma_M, \sigma)$, what is used here.

Definition. *Let $\varphi \in L^2_V(M/\Gamma_M, \sigma_M, \chi)$, $\Lambda \in (\mathfrak{a}^*_{\mathbb{C}})^+$, $x \in G$. Consider the series*

$$E(\Lambda : \varphi : x) \overset{\text{def}}{=} \sum_{\gamma\in\Gamma/\Gamma\cap P} \varphi_\Lambda)x\gamma)$$

$$\overset{\text{formally}}{=} \sum_{\gamma \in \Gamma/\Gamma\cap P} \sigma(k(x\gamma))\varphi(x\gamma)x\gamma)^{-\wedge-\varrho}.$$

It converges absolutely and uniformly on any compact $\Omega \times \omega \subset G \times (\mathfrak{a}_{\mathbb{C}}^*)^+$. Its sum is a function of class $C^\infty(G \times (\mathfrak{a}_{\mathbb{C}}^*)^+)$, holomorphic on $\wedge \in (\mathfrak{a}_{\mathbb{C}}^*)^+$, right Γ-invariant on the variable x and a σ-function on x, satisfying equations $ZE(\wedge : \varphi) = E(\wedge : \varphi)\chi(\mu_\wedge(Z))$ and finally is an automorphic form from $\mathcal{A}_V^\infty(G/\Gamma, \sigma, \chi)$ if P is percuspidal. It is called an Einsenstein series associated with P.

The properties indicated in this definition is easily deduced from the preceeding estimation (**).

Recall that two cuspidal subgroups $P_i = M_i A_i U_i$, $i = 1,2$, are said to be associated if there exists $\gamma \in \Gamma$ such that $\gamma A_1 \gamma^{-1} = A_2$. We denote by $\mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ the group of all isomorphisms : $\mathfrak{a}_1 \to \mathfrak{a}_2$ (see Langlands [13, §2, p. 33]). If $\xi \subset \mathrm{Char}\, \mathcal{Z}$ is a finite set of characters then

$$^0L_V^2(G/\Gamma, \sigma, \xi) := \sum_{\chi \in \xi}^{\oplus} {}^0L_V^2(G/\Gamma, \sigma, \chi)$$

THEOREM 4. *For any finite subset $\xi \subset \mathrm{Char}\mathcal{Z}$ $s \in \mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ and $\wedge \in (\mathfrak{a}_{1,\mathbb{C}}^*)^+$, there exists an unique linear operator*

$$c(s : \wedge);\ {}^0L_V^2(M_1/\Gamma_{M_1}, \sigma_{M_1}; \xi) \longrightarrow {}^0 L_V^2(M_1/\Gamma_{M_2}, \sigma_{M_2}, {}^s\xi)$$

such that

1) it is a holomorphic function on $\wedge \in (\mathfrak{a}_{1,\mathbb{C}}^*)^+$ and interwines the M_i-module structures,

2) $(c(s, \wedge)\varphi)(x) = exp(+s \wedge + \varrho_{P_2}, H_{P_2}(x)) \int_{U_2 \cap wv_1/\Gamma \cap U_2 \cap wv_2} \varphi(xu\gamma)du$

3) $E^{P_2}(\wedge : \varphi : x) = \sum_{s \in \mathcal{W}(\mathfrak{a}_1, a_2)} (c(s : \wedge)\varphi)(x).exp((-s \wedge - \varrho_{P_2})(H_{P_2}(a(x))$

The proof of this theorem is the same as in finite dimensional V-valued case, and we omit it.

Now we finish the paper by stating the main theorem of the spectrum theory of Eisenstein series without proof, because it is completely similar to Langlands' theorem. Some of the statements are already proved. The others require an analytic continuation of Eisenstein series [13], [10], [1].

THEOREM 5. *(a) Suppose that* $\varphi \in^0 L^2_V(M/\Gamma_M, \sigma_M, \chi)$, *then :*

(1) $E(\Lambda : \varphi : x)$ *and* $c(s : \Lambda)\varphi$ *can be analytically continued as meromorphic functions on the whole* $\mathfrak{a}^*_{\mathbb{C}}$.

2) On $i\mathfrak{a}^*$, $E(\Lambda : \varphi : x)$ *is regular and* $c(s, \Lambda)$ *is unitary.*

For a K*-biinvariant function* $f \in C_c(G)^K$ *and* $t \in \mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$ *we have the following functional equations :*

(i) $E(\Lambda : Ind^G_P(\sigma \otimes \exp(\Lambda + \varrho_P)(f)\varphi : x) = \int f(y)E(\Lambda : \varphi)y)dy$,

(ii) $E(s\Lambda : c(s : \Lambda)\varphi : x) : E(\Lambda : \varphi : x)$,

(iii) $c(ts, \Lambda)\varphi = c(t, s\Lambda)c(s, \Lambda)\varphi$.

(b) Let $<P>_{ass}$ *be a class of associated cuspidal subgroups,*

$$\widehat{L}_{<P>_{ass}} := \{F = (F_{\tilde{P}})_{\tilde{P}\in<P>_{ass}}, F_P : i\mathfrak{a}_P \to L^2_{V,P}(M/\Gamma_M, \sigma_M, \chi)\}$$

(i) $\forall s \in \mathcal{W}(\mathfrak{a}_1, \mathfrak{a}_2)$, $F_{P_2}(s\Lambda) = c(s : \Lambda)F_{P_1}(\Lambda)$

(ii) $\|F\|^2 = \sum_{\tilde{P}\in<P>_{ass}} n(A)^{-1}(\frac{1}{2\pi i})^{\dim \tilde{A}} \int_{i\mathfrak{a}_{\tilde{P}}} \|F_{\tilde{P}}\|^2 d\Lambda < \infty$

where $n(A)$ *is the number of chambers in* $\mathfrak{a}$. *Then*
$\widehat{L}_{<P>_{ass}} \cong L^2_{V,<P>_{ass}}(G/\Gamma) \hookrightarrow L^2_V(G/\Gamma)$ *and*

$$L^2_V(G/\Gamma) = \bigoplus_{<P>_{ass}} L^2_{V,<P>_{ass}}(G/\Gamma)$$

The direct sume is taken over the set of all representations of associated classes of cuspidal subgroups.

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