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SOME RESULTS ON QF-n RINGS (n = 2,3)

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1. Introduction.

A ring R is called a quasi - Frobenius ring (briefly, QF - ring) if R is right artinian and right self - injective. For details about these rings see e.g. Faith [4, Chapter 24]. Generalizations of the class of QF - rings as $QF-1$, $QF-2$, $QF-3$ rings have been investigated by several authors, see Thrall [15], Morita [10], Harada [8], Tachikawa [14],... . By [15] a $QF-2$ algebra is $QF-3$. In [5, Theorem 4.1] Fuller showed that a right and left artinian ring is $QF-3$ if it is $QF-2$. Following this line we shall prove that a right or left artinian ring is $QF-3$ if it is $QF-2$ (Theorem 3.3).

Concerning QF - rings, Kupisch [9] proved that a right and left artinian ring R is a QF - ring if and only if R is a $QF-2$ ring and $Soc(R_R) = Soc(_RR)$. We shall prove that a right artinian ring R is QF if and only if R is $QF-2$ and $Soc(R_R) \hookrightarrow Soc(_RR)$ (Theorem 3.5).

Finally, we give a characterization of QF - rings in term of the "extending property" on finitely generated free modules (Theorem 3.6).

2. Definitions and Notations.

We assume throughout that all rings are associative with identity and all modules are unitary. For a module M we denote by $E(M), J(M), Z(M)$ and $Soc(M)$ the injective hull, the Jacobson radical, the singular submodule and the socle of M, respectively. We write M_R (resp. $_RM$) to indicate that M is a right (resp. left) R - module. For a submodule N of M (denoted

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by $N \hookrightarrow M$), $N \hookrightarrow_e M$ (resp. $N \overset{\oplus}{\hookrightarrow} M$) means that N is an essential submodule (resp. direct summand of M). For two modules A, B the symbol $A \overset{\hookrightarrow}{\sim} B$ means that A is isomorphic to a submodule of B.

A module M is said to be an extending module (or CS module in [2]) if for every submodule A of M there exists a submodule A^* of M such that $A \hookrightarrow_e A^* \overset{\oplus}{\hookrightarrow} M$.

A module M is called a small module if M is small in $E(M)$, i.e. for any proper submodule H of $E(M)$, $H + M \neq E(M)$. If M is not small, M is called non - small. Let e be an idempotent of R, then e is called nonsmall if eR_R is a nonsmall module. Dually, M is called a cosmall module if for any projective module P and any epimorphism $f : P \longrightarrow M$, $ker(f)$ is essential in P, i.e. for each non - zero submodule H of P, $ker(f) \cap H \neq 0$. If M is not cosmall, M is called a non-cosmall module (see e.g. [7], [13]). A ring R is called a right co - H - ring if every non - co - small right R - module contains a non - zero projective direct summand and R has ACC on right annihilator ideals.

Let R be a ring. R is said to be right $QF-2$ if R is a direct sum of uniform right ideals. R is defined to be right $QF-3$ if R contains a faithful injective right ideal. R is called a right $QF-3^0$ ring if R has a minimal faithful right R - module. R is call a right $QF-3'$ ring if the injective hull $E(R_R)$ of R_R is torsionless in the sense of Bass [1]. R is called a right $QF-3^+$ ring if the injective hull $E(R_R)$ of R_R is projective. Left $QF-n$ rings ($n = 2, 3, 3^0, 3', 3^+$) are defined similarly.

The following results are used in the study of this paper.

LEMMA 2.1. (TACHIKAWA [14, PROPOSITION 3.1), *If R is a left perfect ring, then the following conditions are equivalent :*

(1) *R is right $QF-3'$*

(2) $\quad R$ *is right* $QF-3^0$

(3) $\quad R$ *is right* $QF-3$

LEMMA 2.2. (TACHIKAWA [14, PROPOSITION 3.3]. *Let* R *be a right and left perfect ring. If* R *is left and right* $QF-3'$, *then* R *is left and right* $QF-3^+$.

LEMMA 2.3. (MORITA [10, THEOREM 1.1]]). *Let* R *be a right or left artinian ring. Then* R *is right* $QF-3$ *if and only if* R *is left* $QF-3$.

LEMMA 2.4. (FULLER [6, THEOREM 3.1]). *Let* R *be a right or left artinian ring, and let* e *and* f *be primitive idempotents of* R. *If* (eR, Rf) *is an injective pair (i.e.* $Soc(eR_R) \cong \frac{fR}{fJ}$ and $Soc({}_RRf) \cong \frac{Re}{Je}$, where $J = J(R)$, *then* eR_R *and* ${}_RRf$ *are injective.*

LEMMA 2.5. (RAYAR [13]). *Let* R *be a right artinian ring, and* M *be a right* R *- module. Then* M *is a small module if and only if* $M.Soc({}_RR) = 0$.

LEMMA 2.6. (HARADA [7, THEOREM 1.3]). *Let* R *be a right perfect ring. Then* R *is right* $QF-3^+$ *if and only if* e_iR *is injective for every non - small primitive idempotent* e_i *of* R.

3. Results.

Let R be a right artinian, right and left $QF-2$ ring. Since R is right artinian, we can find orthogonal primitive idempotents $e_{11}, e_{12}, ..., e_{1j_1}, e_{21}, e_{22}, ..., e_{2j_2}, ..., e_{n1}, e_{n2}, ..., e_{nj_n}$ such that :

(a) $1 = \sum_{k=1}^{j_n} \sum_{i=1}^{n} e_{ik}$,

(b) $Soc(e_{ts}R_R) \cong Soc(e_{t's'}R_R)$ if and only if $t = t'$, and

(c) length $(e_{tk}R_R) \leq$ length $(e_{tk'}R_R)$ if $k < k'$.

LEMMA 3.1.. *For any* $i \in \{1, ..., n\}$, $(e_{ij_i}R)_R$ *is an injective module.*

PROOF: Let $e = e_{ij_i}$. We shall show that there exists idempotent f such that (e, f) is an injective pair, i.e.

$$Soc(eR) \cong \frac{fR}{fJ} , \; Soc(Rf) \cong \frac{Re}{Je} .$$

First, we prove that $Je\ell(J) = 0$, where $\ell(J)$ is the left anninilator of J in R. Assume on the contrary that $Je\ell(J) \neq 0$. Then $g.Je\ell(J) \neq 0$ for some idempotent $g \in \{e_{ik}\}$. It follows the existence of an element $y \in J$ such that $gyel(J) \neq 0$. Clearly $gyel(J) = Soc(gR)$. Consider an R - homomorphism f from eR to $gyeR$ defined by $f(er) = gyer$, $r \in R$. Since $f(el(J)) = gyel(J) = Soc(gR) \neq 0$ and since R is right QF - 2, eR is uniform. Therefore f is a monomorphism. Moreover, $f(eR) \underline{\hookrightarrow} gJ \underset{\neq}{\hookrightarrow} gR$. This show that $Soc(eR) \cong Soc(gR)$ and length$eR \underset{\neq}{<}$ length gR, a contradiction to the choice of e. Therefore we have $Jel(J) = 0$, as claimed.

Now, there is a primitive idempotent f in R such that $\frac{fR}{fJ} \cong Soc(eR)$. Hence $Soc(eR)f \neq 0$, i.e. $el(J)f \neq 0$. On the other hand, as we showed above, $el(J) \underline{\hookrightarrow} r(J)$, where $r(J)$ is the right annihilator of J in R. It follows that $er(J)f \neq 0$. Thus there exists $z \in r(J)$ such that $ezf \neq 0$ and $Rezf = Soc(Rf)$. Hence there exists R - homomorphism φ from Re to $Rezf$ with $Ker\varphi = Je$. Since R is left QF - 2, we have

$$Rezf \quad \cong \quad \frac{Re}{Je} , \quad \text{so} \quad Soc(Rf) \cong \frac{Re}{Je} .$$

Thus (e, f) is an injective pair. By Lemma 2.4, eR is injective.

LEMMA 3.2.. *For any* $i \in \{1, .., n\}$ *we have*

$$e_{ik}R \quad \overset{\hookrightarrow}{\sim} \quad e_{ij_i}R \, . \, .$$

In particular, R is right and left QF - 3 as well as right and left $QF-3^{+}$.

PROOF: Since for each $i \in \{1,...,n\}$, $e_{ij_i}R$ is injective by Lemma 3.1, it follows the existence of an R - homomorphism $g : e_{ik}R \longrightarrow e_{ij_i}R$ such that the diagram

$$\begin{array}{lcl} Soc(e_{ik}R) & \xrightarrow{\tau} & e_{ik}R \\ \varphi \downarrow & & \\ Soc(e_{ij_k}R) & g & \\ \tau' \downarrow & & \\ e_{ij_i}R \swarrow & & \end{array}$$

is commutative, where τ, τ' are the inclusion maps and φ is an isomorphism taken from (b). Since R is right $QF-2$, so $e_{ik}R$ is uniform, therefore $\tau' \varphi(Soc(e_{ik}R)) \neq 0$. It follows that g is monomorphic, i.c. $e_{ik}R$ is embedded into $e_{ij_i}R$.

Note that by (a), (b), (c) and Lemma 3.1, it is easy to see that $\sum_{i=1}^{n} e_{ij_i}R$ is a non -zero faithful injective right R - module, i.e. R is right $QF-3$, and by the first assertion of Lemma 3.2, $E(R_R)$ is projective, i.e. R is right $QF-3^{+}$. Then by Lemma 2.3, R is right and left $QF-3$ as well as right and left $QF-3^{+}$.

From Lemma 3.2 we can state the following theorem :

THEOREM 3.3.. *Let R be a right or left artinian ring. Then R is $QF-3$ if R is $QF-2$.*

Now we are going to characterize QF - rings.

LEMMA 3.4.. *Let R be a right artinian ring and $E(R_R)$ be projective. If $Soc(R_R) \hookrightarrow Soc({}_RR)$, then R is right self - injective, i.e. R is a QF - ring.*

PROOF: Let e be any primitive idempotent of R. Consider the right R - module eR. Since $0 \neq Soc(eR) = eR$ and $Soc(R_R) \hookrightarrow eR.Soc({}_RR)$, we have $eR.Soc({}_RR) \neq 0$. Since R is right artinian, it follows that eR_R is a non - small module, by Lemma 2.5. But $E(R_R)$ is projective, then by Lemma 2.6, eR_R is injective. Hence R is right self - injective, i.e. R is a QF - ring.

THEOREM 3.5.. *For a right artinian ring R the following conditions are equivalent :*

1) R is a QF - ring.

2) a) R is a $QF-2$ ring,
b) $Soc(R_R) \hookrightarrow Soc({}_RR)$.

PROOF.: 1) $\Rightarrow$ 2) is clear

2) $\Rightarrow$ 1). Assume 2). By Lemma 3.2, R is $QF-3^+$, i.e. $E(R_R)$ and $E({}_RR)$ are both projective. The result now follows from Lemma 3.4.

THEOREM 3.6.. *For a right artinian ring R the following conditions are equivalent :*

1) R is a QF - ring.
2) a) $R_R \oplus R_R$ or ${}_RR \oplus {}_RR$ is an extending module,
b) $Soc(R_R) \hookrightarrow Soc({}_RR)$.

PROOF: It is clear that every injective module is an extending module. By a result of Faith - Walker (see [4, Theorem 24, 20]), every projection module over a QF - ring is injective. Hence 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 1). Assume that $R_R \oplus R_R$ is an extending module and b). Then by [12, Corollary 3'], $E(R_R)$ is projective. By Lemma 3.4, R satisfies. Now

assume that ${}_RR \oplus {}_RR$ is an extending module and $Soc(R_R) \hookrightarrow Soc({}_RR)$. By [12, Corollary 3'], $E({}_RR)$ is projective. From this and [3, Corollary 3.4] it follows that $E(R_R)$ is projective. Hence R is a QF - ring by Lemma 3.4.

REMARK: It was shown in [12, Theorem 5] that if a right artinian ring R satisfies a) of Theorem 3.6, then R is a right co - H - ring. In [11, Theorem 5.5] Oshiro gave an example of a (right and left) artinian and (right and left) co - H - ring R but R is not a QF - ring. This together with [12, Theorem 5] shows that the condition b) of Theorem 3.6 can not be obmitted.

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REFERENCES

1. H.Bass, *Finitistic dimension and a homological generalization of semiprimary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–468.
2. A.W. Chatters and C.R. Hajanavis, *Rings in which every complement right ideal is a direct summand*, Quart. J. Oxford **(2)28** (1977), 61–80.
3. Dinh Van Huynh and Phan Dan, *Some characterizations of right co - H - rings*, Preprint No.88.21 (1988), Institute of Mathematics, Hanoi.
4. C. Faith, *Algebra II : Ring theory*, Springer - Verlag.
5. K.R. Fuller, *Structure of QF - 3 rings*, Trans. Amer. Math. Soc. **134** (1968), 343–354.
6. K.R. Fuller, *On indecomposable injective over artinian rings*, Pacific J. Math. **29** (1969), 115–135.
7. M. Harada, *Non–small modules and non–cosmall modules*, Proc. of the 1978, Antw. Conf. Mercel. Dekker, 669–689.
8. M. Harada, *On one–sided $QF-2$ rings I, II*, Osaka J. Math. **17** (1980), 421–431, 433–438.
9. H. Kupisch, *Beiträge zur Theorie nichthalbeifacher Ringe mit Minimalbedingung*, J. Rein Angew Math. **201** (1959), 100–112.
10. K. Morita, *Duality in $QF-3$ rings*, Math. Z. **108** (1969), 237–252.
11. K. Oshiro, *Lifting modules, extending modules and their applications to QF–rings*, Hokkaido Math. J. **13** (1984), 310–338.
12. Phan Dan, *Right perfect rings with the extending property on finitely generated free modules*, Osaka J. Math. **26** (1989), 265–273.

13. M. Rayar, "Small and cosmall modules. Ph.D. Dissertation," Indiana Univ., 1971.
14. H. Tachikawa, *On left QF − 3 rings*, Pacific J. Math. **32** (1970), 255–268.
15. R.M. Thrall, *Some generalizations of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. **64** (1948), 173–183..

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