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ON COINCIDENCE THEOREMS FOR SET–VALUED MAPPING AND VARIATIONAL INEQUALITIES

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0. Introduction

Variational inequalities and coincidence theorems (or fixed point theorems in special cases) are closely related. The theme has attracted much attention. The purpose of this paper is give a slight extension of a theorem of Browder on coincidence ([4]) and derive some new results about variational inequalities in infinite dimensional spaces. Theorem 1.3 in the first section extends a result of Browder ([4]) on open inverse images set–valued mappings to locally selectionable set–valued mappings defined on nonempty convex (non necessarily compact) sets. The main results are given in Section 2. Theorem 2.1 is a version of Ky Fan's minimax lemma, Theorem 2.3 deals with a coercivity condition on the algebraic boundary.

1. Coincidence theorem

Let C, D be nonempty compact convex subsets of two topological spaces X and Y, respectively. Let $T : C \to 2^D$ and $S : D \to 2^C$ be set–valued mappings. By a coincidence of T and S we mean a point $(x, y) \in C \times D$ such that $y \in T(x)$ and $x \in S(y)$ ([4]).

We recall without proof the following :

THEOREM 1.1 ([4]). *Let X and Y be Hausdorff locally convex spaces. Let C and D be nonempty compact convex subsets of X and Y, respectively. Let $S : C \to 2^D$ and $T : D \to 2^C$ be nonempty closed convex–valued mappings. Assume that S and T are upper semicontinuous. Then S and T have a coincidence.*

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THEOREM 1.2 ([4]). *Let X and Y be Hausdorff locally convex spaces. Let C and D be nonempty compact convex subsets of X and Y, respectively. Let $S : C \to 2^D$ and $T : D \to 2^C$ be nonempty convex-valued mappings. Assume that T is upper semicontinuous with closed values and $S^{-1}(y)$ is open in C for any $y \in D$. Then S and T have a coincidence.*

The aim of this section is to show that Theorem 1.2 is still valid under a weaker condition imposed on S.

Let X and Y be topological spaces. By a locally selectionable mapping we mean a set-valued mapping $S : X \to 2^Y$ with the following property : for any $x_0 \in X$ with $S(x_0) \neq \emptyset$ and $y_0 \in S(x_0)$ there exists an open neighborhood $\mathcal{U}_{x_0}$ of x_0 and a continuous point-valued mapping $s : \mathcal{U}_{x_0} \to Y$ such that $s(x_0) = y_0$ and $s(x) \in S(x)$ for all $x \in \mathcal{U}_{x_0}$ ([2]).

THEOREM 1.3. *Let X and Y be Hausdorff locally convex space. Let U and V be nonempty convex subsets of X and Y, respectively. Let $C \subset U$ and $D \subset V$ be nonempty compact convex subsets. Let $S : U \to 2^V$ and $T : D \to 2^C$ be nonempty convex-valued mappings. Assume that T is closed valued, upper semicontinuous and the mapping R defined by $R(x) = S(x) \cap D$ is locally selectionable with nonempty values. Then S and T have a coincidence.*

LEMMA 1.4. *Let X, Y be Hausdorff locally convex spaces. Let C and D be nonempty compact convex subsets of X and Y, respectively. Let $R : C \to 2^D$ be a nonempty convex-valued mapping. If R is locally selectionable, then R has a continuous selection.*

PROOF: We associate with any $x \in C$ a point $y \in R(x)$ and a local continuous selection $r_x : U_x \to D$, where U_x is an open neighborhood of x. This means that $r_x(x) = y$ and $r_x(t) \in R(t)$ for all $t \in U_x$. Since $\{U_x\}_{x \in C}$ is an open covering of the compact set C, C has a finite subcovering

$\{U_{x_i}\}_{i=1,\dots,n}$. Let $\{\alpha_i\}_{i=1,\dots,n}$ be a continuous partition of unity associated with this subcovering. Then the point-valued mapping r defined by

$$r(x) = \sum_{i=1}^{n} \alpha_i(x) r_{x_i}(x)$$

is a continuous selection of R. Indeed, $r_{x_i}(x) \in R(x)$ for $\alpha_i(x) \neq 0$, and from the fact that $R(x)$ is convex conclude that $r(x) \in R(x)$.

PROOF OF THEOREM 1.3: Observe that $R : C \to 2^D$ is nonempty convex– valued, locally selectionable. By Lemma 1.4, R has a continuous selection $r : C \to D$. The set–valued mapping $p : C \to 2^C$ defined by

$$p(x) = T(r(x))$$

is upper semicontinuous with nonempty convex values.

Applying Theorem 1.1 for p and the identity mapping I of C we obtain a coincidence (x_0, y_0) of p and I, i.e. $y_0 = I(x_0) = x_0$ and $x_0 \in p(y_0) = T(r(y_0))$. This implies $x_0 \in T(r(x_0))$. Since $r(x_0) \in R(x_0)$, we have a coincidence $(x_0, r(x_0))$ of R and T which is also a coincidence of S and T.

2. Variational inequalities.

Let U be a subset of a topological vector space. We denote the algebraic interior and the algebraic boundary of U by

$$Core\ U = \{x \in U \ ,\ \forall y \in U \ ,\ \exists \delta > 0 : x + ty \in U \ ,\ |t| \leq \delta\} \ ,$$

$$Bdry\ U = \{x \in U \ ,\ x \notin Core\ U\} \ .$$

We begin with a version of Ky Fan's minimax lemma.

THEOREM 2.1. *Let X be a Hausdorff locally convex space, C a nonempty compact convex set in X, and φ a real–valued function on $C \times C$. Suppose* *a) $\varphi(x,x) \leq 0$ for all $x \in C$,*

b) the set–valued mapping T defined by

$$T(x) = \{y \in C / \varphi(x,y) > 0\}$$

is locally selectionable and has convex values.

Then there exists a vector $\bar{x}$ in C such that

$$\varphi(\bar{x},y) \leq 0 \ for \ all \ y \in C \ .$$

PROOF: Suppose the contrary, i.e. for any $x \in C$ we can find a vector $y \in C$ such that $\varphi(x,y) > 0$. Then $T(x)$ is nonempty. So $T : C \to 2^C$ is a locally selectionable nonempty convex–valued mapping. Applying Theorem 1.3 for T and the identity mapping I of C (with $U = U_1 = C_1 = C$) we obtain a coincidence (x^*, y^*) of T and I. This means that $x^* \in T(y^*)$ and $y^* = I(x^*) = x^*$. Hence $\varphi(x^*, x^*) > 0$, which leads to a contradiction.

REMARK: If the function φ satisfies one of the following conditions :

b') The set $\{y \in C \ , \ \varphi(x,y) > 0\}$ is convex for all $x \in C$ and the set $\{x \in C \ , \ \varphi(x,y) > 0\}$ is open for all $y \in C$.

b") The function $\varphi(x,.)$ is concave or every fixed $x \in C$ and $\varphi(.,y)$ is lower semicontinuous for every fixed $y \in C$; then condition b) of Theorem 2.1 is satisfied. Indeed, we have $b") \Rightarrow b') \Rightarrow b)$, i.e. b) is the weakest condition. Thus we have :

COROLLARY 2.2. (KY FAN'S MINIMAX LEMMA). *Let X be a Hausdorff locally space, C be a nonempty compact convex subset of X and φ a real–valued function on $C \times C$. Suppose*

i) $\varphi(x,x) \leq 0$ for all $x \in C$.

ii) *the function* $\varphi(x,.)$ *is concave for* $x \in C$ *and* $\varphi(.,y)$ *is lower semicontinuous for all* $y \in C$.

Then there exists a vector $\bar{x}$ *in* C *such that*

$$\varphi(\bar{x}, y) \le 0 \quad \textit{for all } y \in C .$$

THEOREM 2.3. *Let* U *be a nonempty convex subset of a Hausdorff locally convex space* X *and* φ *a real–valued function on* $U \times U$. *Suppose that*

a) $\varphi(x,x) = 0$ *for all* $x \in U$.

b) *There exists a nonempty convex subset* E *of* X *such that* $U \cap E$ *is nonempty compact and for every* $x \in U \cap Bdry\ E$ *we can find a vector* $y \in U \cap E$ *such that* $\varphi(x,y) > 0$.

c) *The function* $\varphi(x,.)$ *is concave for any* $x \in U$, *and for any* $y \in U \cap E$ *the set* $\{x \in U \mid \varphi(x,y) > 0\}$ *is open.*

Then there exists a vector $\bar{x} \in U \cap E$ *for which* $\varphi(\bar{x}, y) \le 0$ *for all* $y \in U$.

REMARK.: Clearly, the Condition c) in the statement of Theorem 2.3 can be replaced by the Condition b'') given above.

PROOF OF THEOREM 2.3: Let T be a set–valued mapping defined by $Tx = \{y \in U \mid \varphi(x,y) > 0\}$, $x \in U$.

Let $C = U \cap E$. Then C is a convex compact subset of U.

Suppose now that for any $x \in U$ there exists a vector $y \in U$ such that $\varphi(x,y) > 0$. By hypothesis c), the mapping R defined by $R(x) = T(x) \cap C$ is locally selectionable convex–valued.

We claim that R has nonempty–values, i.e. $T(x) \cap C \neq \emptyset$ for any $x \in C$.

Indeed, if $x \in U \cap Bdry\ E$, then by assumtion b), $T(x) \cap C \neq \emptyset$. If $x \in U \cap core\ E$, then there exists a number $\alpha > 0$ such that

$$x - \varepsilon x \in E \ , \text{ for } 0 \leq \varepsilon \leq \alpha \ .$$

Now, let $y \in U$ with $\varphi(x, y) > 0$ and β a positive number such that

$$x + \varepsilon y \in E \ , \quad \text{for } 0 < \varepsilon \leq \beta \ .$$

By the convexity of E we get

$$\frac{1}{2}(x - \varepsilon x) + \frac{1}{2}(x + \varepsilon y) \in E \quad \text{for} \quad 0 < \varepsilon \leq Min\ \{\alpha, \beta\} \ .$$

Hence

$$(1 - \frac{1}{2}\varepsilon)x + \frac{\varepsilon}{2}y \in E \ , \ \text{for } 0 < \varepsilon \leq Min\ \{\alpha, \beta\} \ .$$

This implies

$$x^0 = (1 - \gamma)x + \gamma y \in U \cap E = C \text{ for } \gamma > 0 \text{ small enough.}$$

On the other hand, since $\varphi(x, .)$ is concave and $\varphi(x, x) = 0$, it follows that

$$\varphi(x, x^0) = \varphi(x, (1 - \gamma)x + \gamma y) \geq (1 - \gamma)\varphi(x, x) + \gamma\varphi(x, y) > 0 \ .$$

So we have $x^0 \in T(x) \cap C$, i.e. $T(x) \cap C \neq \emptyset$.

Now, applying Theorem 1.3 for T and the identity mapping I of C, we obtain a coincidence (x^*, y^*) of T and I, i.e. $x^* \in T(y^*)$ and $y^* = I(x^*) = x^*$.

Therefore $\varphi(x^*, x^*) > 0$ and this contradicts the assumption a).

Corollary 2.4. *Let X be a Hausdorff locally convex space, U a nonempty convex set in X and $f : X \to]-\infty, +\infty]$ a lower semicontinuous convex function which is finite on U. Let $A : X \to X^*$ (dual of X) be a mapping such that $x \to < Ax, x >$ is lower semicontinuous on U, $K \subset U$*

a nonempty convex compact set. Suppose that for each $x \in Bdry\ K$ there exists an element $y \in K$ for which

$$< A(x), x - y > > f(y) - f(x) .$$

Then there exists an element $\bar{x} \in K$ such that

$$< A\bar{x}, \bar{x} - y > \leq f(y) - f(x) \text{ for all } y \in U .$$

PROOF: Let φ be the real–valued function on $U \times U$ defined by

$$\varphi(x, y) = < Ax, x - y > \ + f(x) - f(y) .$$

Apply Theorem 2.3, we can find an element $\bar{x} \in K$ such that

$$\varphi(\bar{x}, y) \leq 0 \text{ for all } y \in U .$$

i.e.

$$< A(\bar{x}, \bar{x} - y > \leq f(y) - f(x) \text{ for all } y \in U .$$

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