ACTA MATHEMATICA VIETNAMICA
Volume 16, Number 1, 1991

DOUBLE AND MULTIPLE HALF-RANGE FOURIER SERIES OF MEIJER'S G-FUNCTION

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1. Introduction

The object of this paper is to evaluate four double integrals of Meijer's G-function and utilize them to obtain four double half-range Fourier series of the G-function. We further derive one multiple integral and one multiple half-range Fourier series of the G-function analogous to our one double integral and one double half-range Fourier series of the G-function respectively.

The subject of Fourier series of the generalized hypergeometric functions occupies an important place in the field of special functions. Certain Fourier series of the generalized hypergeometric function play an important role in the development of the theory of special functions and certain Fourier series of the generalized hypergeometric functions enable us to obtain general solutions of some boundary value problems.

The Fourier series of the generalized hypergeometric functions were given from time to time by various mathematicians, with certain restrictions in parameters. An adequate list of references would be quite lengthy. However the references given here together with sources indicated in these references provide a good converge of the subject.

We now mention in brief some interesting work on this subject. MacRobert [15, 16] established a *cosine* and a *sine* Fourier series of MacRobert's E-function. Kesarwani [14], Jain [13] and the author [2, 3, 6] obtained some Fourier series of the G-function. Parashar [19], Anandani [1], the author [4, 5, 7], Shah [21], Saxena [20] and Taxak [23] established some Fourier series

Received by the editors November 14, 1989

of Fox's H-function [11]. Almost all research papers on Fourier series of generalized hypergeometric functions have been discussed and listed in [17, 18, 22]. It is important to note that all the Fourier series mentioned above are half-range Fourier series. Therefore, it is evident that to establish full-range Fourier series of the generalized hypergeometric functions are either very difficult or impossible. It is also amazing that so far nobody has attempted to establish double and multiple Fourier series of the generalized hypergeometric functions. This paper appears to be an attempt in the field of double and multiple Fourier series of the generalized hypergeometric functions. The author has been motivated to contribute in this direction by the work of Carslaw and Jaeger [9, pp. 180–183].

On specialising the parameters the G-function may be reduced to a great many of the special functions appearing in applied mathematics [10, pp. 216–222], so that each of the formulae developed in this paper becomes a master or key formula from which a very large number of relations can be deduced for Bessel, Legendre, Whittaker function, their combinations and other related functions. Hence, the Fourier series given in this paper are of a very general character and may encompass several cases of interest.

The following formulae are required in the proofs :

The Meijer's G-function introduced and defined by Meijer will be represented as follows [10, p. 207, (1)] :

$$G_{p,q}^{u,v}\left(z\,\Big|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right) = \frac{1}{2\pi i}\int_L \xi(s) z^s ds, \tag{1.1}$$

where $\xi(s) = \dfrac{\prod_{j=1}^{u}\Gamma(b_j-s)\prod_{j=1}^{v}\Gamma(1-a_j+s)}{\prod_{j=u+1}^{q}\Gamma(1-b_j+s)\prod_{j=v+1}^{p}\Gamma(a_j-s)}$,
and L is a suitable Mellin-Barnes type contour.

The multiplication formula for the gamma-function [10, p. 4, (11)] :

$$\Gamma(mz) = (2\pi)^{1/2-1/2m} m^{mz-1/2} \prod_{i=1}^{m-1} \Gamma(z+i/m), \tag{1.2}$$

where m is a positive integer.

The following integrals [12, p. 372, (1) & (8)] :

$$\int_0^\pi (sinx)^{w-1} sinmx\ dx \tag{1.3}$$

$$= \frac{\pi sin\frac{m\pi}{2}\Gamma(w)}{2^{w-1}\Gamma(\frac{w+m+1}{2})\Gamma(\frac{w-m+1}{2})}, Rew > 0.$$

$$\int_0^\pi (sinx)^{w-1} cosmx\ dx \tag{1.4}$$

$$= \frac{\pi cos\frac{m\pi}{2}\Gamma(w)}{2^{w-1}\Gamma(\frac{w+m+1}{2})\Gamma(\frac{w-m+1}{2})}, Rew > 0.$$

In what follows for sake of brevity a_p stands for $a_1, \cdots, a_p$, d and h are positive integers, the symbol $\Delta(d, w)$ represents the set of parameters $\frac{w}{d}, \frac{w+1}{d}, \cdots, \frac{w+d-1}{d}$, and the expression $\Delta(d, \frac{1-w\pm m}{2})$ stands for $\Delta(d, \frac{1-w+m}{2})$, $\Delta(d, \frac{1-w-m}{2})$.

The following double orthogonality properties of *sine* and *cosine* function, which may be verified easily :

$$\int_0^\pi \int_0^\pi sinmx\ sinrx\ sinny\ sinty\ dxdy \tag{1.5}$$

$$= \begin{cases} \frac{\pi^2}{4}, & m = r,\ n = t \\ 0, & m \neq r \text{ or } n \neq t. \end{cases}$$

$$\int_0^\pi \int_0^\pi sinmx\ sinry\ cosny\ costy\ dxdy \tag{1.6}$$

$$= \begin{cases} \frac{\pi^2}{4}, & m = r,\ n = t, \\ 0, & m \neq r \text{ or } n \neq t, \\ \frac{\pi^2}{2}, & m = r,\ n = t = 0. \end{cases}$$

(1.7) $$\int_0^{\pi}\int_0^{\pi} cosmx\ cosrx\ sinny\ sinty\ dxdy$$

$$= \begin{cases} \frac{\pi^2}{4}, & m = r,\ n = t \\ 0, & m \neq r \text{ or } n \neq t \\ \frac{\pi^2}{2}, & m = r = 0,\ n = t. \end{cases}$$

(1.8) $$\int_0^{\pi}\int_0^{\pi} cosmx\ cosrx\ sinny\ sinty\ dxdy$$

$$= \begin{cases} \frac{\pi^2}{4}, & m = r,\ n = t \\ 0, & m \neq r \text{ or } n \neq t \\ \frac{\pi^2}{2}, & \text{either } m = r = 0,\ n = t \text{ or } m = r,\ n = t = 0 \\ \pi^2, & m = r = n = t = 0. \end{cases}$$

2. Double integrals

The double integrals to be evaluated are

(2.1) $$\int_0^{\pi}\int_0^{\pi} (sinx)^{\lambda-1}(siny)^{\mu-1} sinrx\ sinty\ g(x,y)dxdy$$

$$= \frac{\pi sin\frac{r\pi}{2} sin\frac{t\pi}{2}}{\sqrt{(dh)}}\psi(r,t);$$

(2.2) $$\int_0^{\pi}\int_0^{\pi} (sinx)^{\lambda-1}(siny)^{\mu-1} sinrx\ costy\ g(x,y)dxdy$$

$$= \frac{\pi sin\frac{r\pi}{2} cos\frac{t\pi}{2}}{\sqrt{(dh)}}\psi(r,t);$$

(2.3) $$\int_0^\pi \int_0^\pi (sinx)^{\lambda-1}(siny)^{\mu-1} cosrx\ sinty\ g(x,y)dxdy$$

$$= \frac{\pi cos\frac{r\pi}{2} sin\frac{t\pi}{2}}{\sqrt(dh)}\psi(r,t);$$

(2.4) $$\int_0^\pi \int_0^\pi (sinx)^{\lambda-1}(siny)^{\mu-1} cosrx\ costy\ g(x,y)dxdy$$

$$= \frac{\pi cos\frac{r\pi}{2} cos\frac{t\pi}{2}}{\sqrt(dh)}\psi(r,t)$$

where $2(u+v) > p+q,\ |argz| < (u+v-\frac{1}{2}p-\frac{1}{2}q)\pi,$

$$Re\ (\lambda + +2db_j) > 0,\ Re\ (\mu + 2hb_j) > 0,\ j = 1, \cdots, u;$$

and

$$g(x,y) = G^{uv}_{p,q}\left[z(sinx)^{2d}(siny)^{2h}\ \Big|^{a_p}_{b_q}\right]$$

$$\psi(r,t) = G^{u,v+2d+2h}_{p+2d+2h,\ q+2d+2h}\left[z\ \Big|^{\Delta(2d,1-\lambda),\Delta(2h,1-\mu),a_p}_{b_q,\Delta(d,\frac{1-\lambda\pm r}{2}),\Delta(h,\frac{1-\mu\pm t}{2})}\right]$$

PROOF: To establish the integral (2.1), expressing the G-function in the integrand as the Mellin-Barnes type integral (1.1) and interchanging the orders of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \zeta(s)z^s\left[\int_0^\pi (sinx)^{\lambda+2sd-1} sinrx\ dx \times \int_0^\pi (siny)^{\mu+2sh-1} sinty\ dy\right]ds.$$

Evaluating the inner-integrals with the help of (1.3) and using the multiplication formula for gamma-function (1.2), we get

$$\frac{\pi sin\frac{r\pi}{2} sin\frac{t\pi}{2}}{\sqrt(dh)}\frac{1}{2\pi i}\int_L \zeta(s)\frac{\prod_{i=0}^{2d-1}\Gamma(\frac{\lambda+i}{2d}+.s)}{\prod_{i=0}^{d-1}\Gamma(\frac{(\lambda+r+1)/2+i}{d}+s)}$$

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$$\times \frac{\prod_{i=0}^{2h-1}\Gamma(\frac{\mu+i}{2h}+s)z^s ds}{\prod_{i=0}^{d-1}\Gamma(\frac{(\lambda-r+1)/2}{d}+s)\prod_{i=0}^{h-1}\Gamma(\frac{(\mu+t+1)/2+i}{h})\prod_{i=0}^{h-1}\Gamma(\frac{(\mu-t+1)/2+i}{h}+s)}$$

On applying (1.1), the value of the integral (2.1) is obtained.

On applying the same procedure as above and using (1.3) and (1.4), the integral (2.2) is established.

Similarly the integral (2.3) is established with the help of (1.4) and (1.3).

The integral (2.4) is established similarly with the help of (1.4).

3. Double half-range Fourier series

The double half-range Fourier series to be established are

$$(3.1) \quad f(x,y) = \frac{4}{\pi\sqrt{(dh)}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} sin\frac{m\pi}{2} sin\frac{n\pi}{2}\psi(m,n) \times sinmx\ sinny;$$

$$(3.2) \quad f(x,y) = \frac{4}{\pi\sqrt{(dh)}}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} sin\frac{m\pi}{2} cos\frac{n\pi}{2}\psi(m,n) \times sinmx\ cosny;$$

$$(3.3) \quad f(x,y) = \frac{4}{\pi\sqrt{(dh)}}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} cos\frac{m\pi}{2} sin\frac{n\pi}{2}\psi(m,n) \times cosmx\ sinny;$$

$$(3.4) \quad f(x,y) = \frac{4}{\pi\sqrt{(dh)}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} cos\frac{m\pi}{2} cos\frac{n\pi}{2}\psi(m,n) \times cosmx\ cosny;$$

where $2(u+v) > p+q,\ |argz| < (u+v-\frac{1}{2}p-\frac{1}{2}q)\pi,$

$$Re\ (\lambda+2db_j) > 0,\ Re\ (\mu+2hb_j) > 0,\ j=1,\cdots,u;$$

and

$$f(x,y) = (sinx)^{\lambda-1}(siny)^{\mu-1}g(x,y);$$

provided $B_m, C, C_0,n, D_0,n, D_m,0$ are one-half and $D_0,0$ is one-quarter of the values of B_m,n, C_m,n and D_m,n with reference to (3.8), (3.10) and (3.12).

PROOF: To establish (3.1), let

$$f(x,y) = (sinx)^{\lambda-1}(siny)^{\mu-1}g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n} sinmx\ sinny. \tag{3.5}$$

Equation (3.5) is valid since $f(x,y)$ is continuous and of bounded variation in the open interval $(0,\pi)$.

Multiplying both sides of (3.5) by $sinrx\ sinty$ and integrating from 0 to π with respect to both x and y, we get.

$$\int_0^{\pi}\int_0^{\pi}(sinx)^{\lambda-1}(siny)^{\mu-1} sinrx\ sinty\ g(x,y)dxdy$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\int_0^{\pi}\int_0^{\pi} sinmx\ sinrx\ sinnx\ sinty\ dxdy.$$

Now using (2.1) and (1.5), we have

$$A_{r,t} = \frac{4}{\pi\sqrt{(dh)}}(sin\frac{r\pi}{2} sin\frac{t\pi}{2})\psi(r,t). \tag{3.6}$$

Substituting the value of $A_{m,n}$ from (3.6) in (3.5), the double half-range Fourier series (3.1) is established.

To prove (3.2), let us set

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{m,n} sinmx\ cosny. \tag{3.7}$$

Multiplying both sides of (3.7) by *sinrx costy* and integrating from 0 to π with respect to both x and y, and using (2.2) and (1.6), we obtain

$$B_{r,t} = \frac{4}{\pi\sqrt{(dh)}}(sin\frac{r\pi}{2}cos\frac{t\pi}{2})\psi(r,t), \tag{3.8}$$

except that $B_{r,0}$ is one-half of the above value.

From (3.7) and (3.8), the double half-range Fourier series (3.2) follows.

To establish (3.3), we set

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} C_{m,n} cosmx\ sinnx. \tag{3.9}$$

Multiplying both sides of (3.9) by *cosrxsinty* and integrating from 0 to π with respect to both x and y, and using (2.3) and (1.7), we get

$$C_{r,t} = \frac{4}{\pi\sqrt{(dh)}}(cos\frac{r\pi}{2}sin\frac{t\pi}{2})\psi(r,t), \tag{3.10}$$

except that C_0, t is one-half of the above value.

The double half-range Fourier series (3.3) is obtained from (3.9) and (3.10).

To establish (3.4), let

$$f(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{m,n} cosmx\ cosny. \tag{3.11}$$

Multiplying both sides of (3.11) by *cosrx costy* and integrating from 0 to π with respect to both x and y, and using (2.4) and (1.8), we have

$$D_{r,t} = \frac{4}{\pi\sqrt{(dh)}}(cos\frac{r\pi}{2}cos\frac{t\pi}{2})\psi(r,t), \tag{3.12}$$

except that D_0, t, $D_{r,0}$ are one-half and $D_{0,0}$ is one-quarter of the above value.

The double half-range Fourier series (3.4) follows from (3.11) and (3.12) immediately.

4. Multiple integrals

The following multiple integral analogous to (2.1) can be derived on following the procedure as given in Section 2 with the help of (1.3) :

$$\text{(4.1)} \qquad \int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}(sinx_1)^{\lambda_1-1}(sinx_2)^{\lambda_3-1}\cdots(sinx_n)^{\lambda_n-1}$$

$$\times sinr_1x_1\ sinr_2x_2\ sinr_3x_3\cdots sinr_nx_n$$

$$\times g(x_1,x_2,x_3,\cdots,x_n)dx_1dx_2dx_3\cdots dx_n$$

$$\frac{(\pi)^{n/2}sin\frac{r_1\pi}{2}sin\frac{r_2\pi}{2}sin\frac{r_3\pi}{2}\cdots sin\frac{r_n\pi}{2}}{\Gamma(d_1d_2d_3\cdots d_n)}\psi(r_1,r_2,r_3,\cdots,r_n),$$

where $2(u+v)>p+q,\ |argz|<(u+v-\frac{1}{2}p-\frac{1}{2}q)\pi,$

$$Re\ (\lambda_1+2d_1b_j)>0,\cdots,\ Re\ (\lambda+2d_nb_j)>0,\ j=1,\cdots,u;$$

and

$$g(x_1,x_2,x_3,\cdots,x_n)$$

$$=G_{p,q}^{u,v}\left[z(sinx_1)^{2d_1}(sinx_2)^{2d_2}(sinx_3)^{2d_3}\cdots(sinx_n)^{2d_n}\ \Big|_{b_q}^{a_p}\right];$$

and

$$\psi(r_1,r_2,r_3,\cdots,r_n)$$

$$=G_{p+2d_1+\cdots+2d_n,q+2d_1+\cdots+2d_n}^{u,v+2d_1+\cdots+2d_n}\left[z\ \Big|_{b_q,\Delta(d_1,1-\lambda_1\pm r_1),\cdots,\Delta(d_n,\frac{1-\lambda_n\pm r_n}{2})}^{\Delta(2d_1,1-\lambda_1),\cdots,\Delta(2d_n,1-\lambda_n),a_p;}\right]$$

The multiple integrals analogous to (2.2), (2.3) and (2.4) can also be derived similarly.

5. Multiple half-range Fourier series

The following multiple half-range Fourier series analogous to (3.1) can be derived on the following as given in Section 3, using the integral (4.1) and the multiple orthogonality property of sine functions analogous to (1.5) :

(5.1)
$$f(x_1, x_2, x_3, \cdots, x_n)$$
$$= \frac{2^n}{(\pi)^{n/2}\sqrt{(d_1 d_2 d_3 \cdots d_n)}} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \cdots \sum_{m_n=1}^{\infty}$$
$$sin\frac{m_1\pi}{2} sin\frac{m_2\pi}{2} sin\frac{m_3\pi}{2} \cdots sin\frac{m_n\pi}{2} \psi(m_1, m_2, m_3, \cdots, m_n)$$
$$\times\ sin m_1 x_1\ sin m_2 x_2\ sin m_3 x_3\ \cdots\ sin m_n x_n;$$

where $2(u+v) > p+q$, $|argz| < (u+v-\frac{1}{2}p-\frac{1}{2}q)\pi$,

$$Re\ (\lambda_1 + 2d_1 b_j) > 0, \cdots,\ Re\ (\lambda + 2d_n b_j) > 0,\ j = 1, \cdots, u;$$

and
$$f(x_1, x_2, x_3, \cdots, x_n)$$
$$= (sin x_1)^{\lambda_1 - 1}(sin x_2)^{\lambda_2 - 1}(sin x_3)^{\lambda_3 - 1} \cdots (sin x_n)^{\lambda_n - 1}$$
$$\times g(x_1, x_2, x_3, \cdots, x_n).$$

Similarly, the multiple half-range Fourier series analogous to (3.2), (3.3) and (3.4) can also be derived.

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