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ON THE CONVERGENCE OF THE HOSCHSCHILD-SERRE SPECTRAL SEQUENCE FOR THE CONTINUOUS COHOMOLOGY OF PARABOLIC DISCRETE SUBGROUPS

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0. Introduction

The Eilenberg-Maclane cohomology group $H^*(\Gamma, \rho, E)$ of a discrete sugroup Γ of a reductive algebraic $\mathbb{Q}$ -group G with coeffiecients in a finite dimentional Γ-module (ρ, E) is equivalent to the continuous de Rham cohomology of the corresponding Riemannian symmetric space

$$X/\Gamma = K \setminus G_{\mathbb{R}} /\Gamma$$

of a maximal compact subgroup K in $G_{\mathbb{R}}$ with coefficients in the induced Γ-module

$$E_p = E \underset{K}{\times} G/\Gamma$$

(see [12]). So the scope is focused in the spectral decomposition of the Γ-equivariant Laplacian Δ on the space of smooth section of E_ρ. If Γ is cocompact, spectrum of Δ is discrete and one also has reasonable Hodge theory (see [1], Chapter II and III). The theory becomes more complicated in the non-cocompact case. G. Harder [7] had studied the case when G is of parabolic rank 1, K.F. Lai [11] the case when $G = Sp(n)$, and J. Schwermer [13], [14] the case when $G = SL_3(Z)$ and $G = SL_n(Z)$. The main idea is based on the index theory of elliptic operators (the kernel and cokernel of which are finite-dimensional) and the Hodge decomposition.

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At the present time, the orbit method has attained a flourishing development in the multidimensional context and in the multidimensional quantization procedure (see D.N. Diep [2], [4], [6]). This developmet shows that the description of the dual of Lie groups leads to the infinite-dimensional unitary representation, the non-compact coadjoint orbit can be of finite covolume. One hopes therefore to develope the theory of continuous cohomology in the non-compact case with infinite-dimensional continuous Γ-module (ρ, E). Analysing the index theorem of elliptic operators, D.N. Diep suggested an idea of replacing the finiteness of the kernel and the cokernel by the finitely generated property, and the compactness of symmetric spaces by the conditions of compact support at square-integrability.

Following Diep's ideal, we develope the Hodge theory for the case of square-integrable differential forms with coefficients in a Hilbert fiber bundle E_ρ over non-compact symmetric space. Applying this theory, we shall prove the theorem on the convergence to a direct sum of the E_2-terms of the Hochschild-Serre spectral sequence for the square-integrable cohomology classes of parabolic discrete groups (Theorem 3.5). This result extends a result of Harder ([7], Theoremn 2.8).

I would like to express my deepest gratitude to Dr. Do Ngoc Diep for posing problem, for inspiring suggestions and encouragements.

1. The cohomology of discrete subgroup

Let G be a Lie group, K a maximal compact subgroup, Γ a discrete subgroup without torsion of G. Then $X = K \setminus G$ is homeomorphic to an Euclidean space, and X/Γ is a $K(\Gamma, 1)$-space and the projection

$$G/\Gamma \xrightarrow{p_K} X/\Gamma$$

is a principal K-bundle.

Let $\rho : G \rightarrow Aut\ (E)$ be a differentiable representation of G in a Hilbert space E with finite spectrum. The restrictions of ρ on K and Γ induce the actions of these groups on E. The fibration $E_\rho = E \underset{K}{\times} G/\Gamma$ is a vector bundle associated with the principal K-bundle p_K with the fiber E. Denote by ∇ the flat canonical connection on E_ρ (see [3], p. 92 for the definition of connection on the infinite-dimention vector bundle and [7], p. 150 for the construction of this connection). Now, we consider the cohomology $H^*(\Gamma, \rho, E)$ of the group Γ with coefficients in a Γ-module E. It is well-known that $H^*(\Gamma, \rho, E) \simeq H(X/\Gamma, E_\rho)$, where the right hand side is the cohomology of the Rham complex $\{\Omega(X/\Gamma, E_\rho); d_\nabla\}$ of the differential forms of X/Γ with values in the bundle E_ρ, and the differential d_∇ is defined as follows

$$(1)\qquad d_\nabla\omega(\xi_1, ..., \xi_{p+1}) = \sum_{1\leq i\leq p+1} (-1)^i\omega(\xi_1, ..., \hat{\xi}_i, ..., \hat{\xi}_{p+1}) + \\ + \sum_{i\leq i<j\leq p+1} (-1)^{i+j}\omega([\xi_i, \xi_j], \xi_1, .., \hat{\xi}_i., \hat{\xi}_j, ., \xi_{p+1})$$

Let $\mathcal{G} = Lie\ G$, $\mathcal{K} = Lie\ K$ be the Lie algebras of the Lie groups G and K respectively. Put $\mathcal{P} = \mathcal{G}/\mathcal{K}$. The adjoint action of K on $\mathcal{G}$ induces the representation $AD_\mathcal{P}$ of K in $\mathcal{P}$. Then the tangent bundle TX/Γ of X/Γ is itself the vector bundle on X/Γ associated with the principal K-bundle p_K and with the representation $Ad_\mathcal{P}$ (see [7], p.131).

The representations $Ad_\mathcal{P}$ and ρ define the action $\nu(k) = \wedge^p Ad_\mathcal{P}(k) \otimes \rho(k)$ of K on $\wedge^p\mathcal{P}^* \otimes E$.

For any p-forms ω of $\Omega^p(X/\Gamma, E_\rho)$ we have a $\nu(k)$-invariant smooth function

$$(2)\qquad \varphi_\omega : G/\Gamma \longrightarrow \wedge^p\mathcal{P}^* \otimes E.$$

Denote by $C^\infty_K(G/\Gamma, \wedge^p\mathcal{P}^* \otimes E)$ the space of $\nu(k)$-invariant smooth functions as in (2). Then there is an isomorphism

$$(3) \qquad C^\infty_K(G/\Gamma, \wedge^p\mathcal{P}^* \otimes E) \simeq Hom_K(\wedge^p\mathcal{P}, C^\infty(G/\Gamma, E)) = C^p(\mathcal{G}, K; C^\infty(G/\Gamma, E)),$$

where the last space consists of the relative cohomology cocycles of the Lie algebra $\mathcal{G}$ modulo K with values in $C^\infty(G/\Gamma, E)$. We have the following lemma whose proof is straightforward.

LEMMA 1.1. *There exists an isomorphism*

$$(4) \qquad h : \Omega(X/\Gamma, E_\rho) \xrightarrow{\simeq} C^p(\mathcal{G}, K; C^\infty(G/\Gamma, E))$$

which commutes with the differential operators of the complexes. Hence h induces the isomorphism on the cohomology groups

$$h^* : H^p(X/\Gamma, E_\rho) \xrightarrow{\simeq} H^p(\mathcal{G}, K; C^\infty(G/\Gamma, E)).$$

2. The Hodge decomposition

In this section we assume that G is a reductive Lie group, K is a maximal compact subgroup of G, θ is the Cartan involution corresponding to K. Then

$$\mathcal{P} = \{\xi \in \mathcal{G} \mid \theta(\xi) = -\xi\},$$

and we have the Cartan decomposition

$$\mathcal{G} = \mathcal{K} \otimes \mathcal{P}.$$

Put

$$B_\theta(\xi, \eta) = -B(\xi, \theta(\eta)),$$

where B is the Killing form on $\mathcal{G}$

DEFINITION 2.1. *A differentiable G-module (ρ, E) is said to be K-admissible if there exists a scalar product $(.,.)$ on the Hilbert space E which satisfies the following conditions*

(i) $(\rho(k)v, \rho(k)w) = (v, w)$ *for all* $k \in K$,

(ii) $(\rho(\xi)v, w) = (v, \rho(\xi)w)$ *for all* $\xi \in \mathcal{P}$.

The proof of the following lemma is similar to the proof of Proposition 3.1 in [12].

LEMMA 2.2. *There exists an admissible scalar product on the Hilbert space E.*

REMARK 2.3: It follows from Definition 2.1 that any $\mathcal{G}$-invariant subspace of E has an $\mathcal{G}$-invariant orthogonal complement. Hence the representation ρ is either irreducible or completely reducible.

From now on, we fix a K-admissible scalar product on E. As in §1 [7] we can define the operators :

$$\aleph : \Omega(X/\Gamma, E_\rho) \longrightarrow \Omega(X/\Gamma, E^*_\rho),$$

$$* : \Omega^p(X/\Gamma, E_\rho) \longrightarrow \Omega^{N-p}(X/\Gamma, E_\rho),$$

$$\omega \longrightarrow *\omega = e_N L\omega,$$

where $N = dim\ X/\Gamma$. The evalution map

$$tr : E\hat{\otimes}E^* \longrightarrow \mathbf{C}$$

induces the map

$$tr : \Omega^N(X/\Gamma, E_\rho\hat{\otimes}E^*_\rho) \longrightarrow \Omega^N(X/\Gamma, \mathbf{C}).$$

For each $\omega, \omega' \in \Omega^*(X/\Gamma, E_\rho)$ we put

$$(1) \qquad (\omega, \omega') = \begin{cases} \int\limits_{X/\Gamma} tr(\omega \wedge \overline{\aleph \circ *\omega'} & \text{if} \quad deg\omega = deg\omega', \\ 0 & \text{if} \quad deg\omega \neq deg\omega', \end{cases}$$

whenever the integral of the right hand side is defined. The formula (1) defined a scalar product on the subspace $\Omega^*_c(X/\Gamma, E_\rho)$ of the differential forms with compact support in $\Omega^*(X/\Gamma, E_\rho)$.

We consider now the space $C_c^\infty(G/\Gamma, E)$ of the smooth functions with compact support. The group G acts naturally on $C_c^\infty(G/\Gamma, E)$ by

$$(2) \qquad (\pi(g)f)(x) = \rho(g^{-1})f(gx),$$

where $x \in G/\Gamma$, $g \in G$ and $f \in C_c^\infty(G/\Gamma, E)$. This action induces an action of $\mathcal{G}$ on $C_c^\infty(G/\Gamma, E)$ as follows

$$(3) \qquad (\pi_\xi f)(x) = (\mathcal{L}_\xi f)(x) - \rho(\xi)f(x),$$

where $\mathcal{L}_\xi$ is the Lie derivation on G/Γ. As usual we consider the following scalar product on $C_c^\infty(G/\Gamma, E)$

$$(4) \qquad (f, g) = \int\limits_{G/\Gamma} (f(x), g(x))_E dx.$$

The completion of $C_c^\infty(G/\Gamma, E)$ with the scalar product (4) is the space $L^2(G/\Gamma, E)$ of the square-integrable functions of G/Γ with values in E. We can consider the complex $\{C^p(\mathcal{G}, K; L^2(G/\Gamma, E)), d_\pi\}$, where the differential operator d_π is defined as follows

$$(5) \qquad d_\pi\varphi(\xi_1, .., \xi_{p+1}) = \sum_{i \leq i \leq p+1} (-1)^i \pi_{\xi_i}\varphi(\xi_1, .., \hat{\xi}_i, .., \xi_{p+1})$$

for all $\xi_1, .., \xi_{p+1} \in \mathcal{P}$.

The following lemma is obvious

Lemma 2.4. *The scalar product on $C_c^\infty(G/\Gamma, E)$ and the restriction $B_{\theta|\mathcal{P}}$ of the for B_θ on $\mathcal{P}$ define a scalar product on $C^p(\mathcal{G}, K; C_c^\infty(G/\Gamma, E))$ whose completion is $C^p(\mathcal{G}, K; L^2(G/\Gamma, E))$.*

At last, we define the operator δ_π on $C^p(\mathcal{G}, K; L^2(G/\Gamma, E))$ by

$$(d_\pi \varphi, \varphi') = (\varphi, \delta_\pi \varphi'), \tag{6}$$

and put

$$\Delta_\pi = d_\pi \delta_\pi + \delta_\pi d_\pi. \tag{7}$$

Now we return to consider $(\Omega_c(X/\Gamma, E_\rho))$. It is clear that $d_\nabla(\Omega_c^*(X/\Gamma, E_\rho)) \subset \Omega_c^*(X/\Gamma, E_\rho)$. Therefore $\{\Omega_c^*(X/\Gamma, E_\rho), d_\nabla\}$ becomes a subcomplex of the complex $\{\Omega^*(X/\Gamma, E), d_\nabla\}$. On $\Omega_c^*(X/\Gamma, E_\rho)$ we define the operator δ_∇ by

$$(\delta_\nabla \omega, \omega') = (\omega, d_\nabla \omega') \quad \text{for all} \quad \omega, \omega' \in \Omega_c^*(X/\Gamma, E_\rho),$$

and put $\Delta_\nabla = d_\nabla \delta_\nabla + \delta_\nabla d_\nabla$.

We denote by $\Omega_{L^2}^*(X/\Gamma, E_\rho)$ the completion of $\Omega_c^*(X/\Gamma, E_\rho)$ with respect to the scalar product (3), so $\Omega_{L^2}^*(X/\Gamma, E_\rho)$ is the space of the square-integrable sections of the bundle $\wedge^* T^* X/\Gamma \hat{\otimes} E_\rho$. We also use the same notations d_∇, δ_∇ and Δ_∇ for their extensions on $\Omega_{L^2}^*(X/\Gamma, E_\rho)$. The cohomology group of the complex $\{\Omega_{L^2}^*(X/\Gamma, E_\rho), d_\nabla\}$ is denoted by $H_{L^2}^*(X/\Gamma, E_\rho)$ and the kernel of the operator Δ_∇ by $\mathcal{H}_{L^2}(X/\Gamma, E_\rho)$.

Theorem 2.5. (The Hodge decomposition).

$$\Omega_{L^2}^*(X/\Gamma, E_\rho) = Ker\, \Delta_\nabla \oplus Im\ d_\nabla \oplus Im\ \delta_\nabla. \tag{8}$$

Consider the isomorphism h in Lemma 1.1.

LEMMA 2.6. *The restriction of the isomorphism h on $\Omega^*_{L^2}(X/\Gamma, E_\rho)$ is a bijective map preserving the scalar products. Therefore, it deduces the following isomorphism*

$$\tilde{h} : \Omega^p_{L^2}(X/\Gamma, E_\rho) \rightleftarrows C^p(\mathcal{G}, K, L^2(X/\Gamma, E)) \tag{9}$$

The proof of this lemma is straightforward.

COROLLARY 2.7. *The isomorphism $\tilde{h}$ commutes with the codifferential operators δ_∇ and δ_π. Therefore $\tilde{h}$ commutes the Laplacians Δ_∇ and Δ_π.*

Denote by $\mathcal{H}^*(\mathcal{G}, K; L^2(G/\Gamma, E))$ the kernel of the operator Δ_π. For any $\varphi \in C^p(\mathcal{G}, K; L^2(G/\Gamma, E))$ we put

$$\begin{aligned} d_\tau\varphi(\xi_1, ., \xi_{p+1}) &= \sum_{i \leq i \leq p+1} (-1)^{i+1} \mathcal{L}_{\xi_i}\varphi(\xi_1, ., \hat{\xi}_i, ., \xi_{p+1}) \\ d_\rho\varphi(\xi_1, ., \xi_{p+1}) &= \sum_{i \leq i \leq p+1} (-1)^{i+1} \rho(\xi_i\varphi(\xi_1, ., \hat{\xi}_i, ., \xi_{p+1}) \end{aligned} \tag{10}$$

for all $\xi_1, ., \xi_{p+1} \in \mathcal{P}$.

It is clear that $d_\pi = d_\tau - d_\rho$. We use δ_τ and δ_ρ to denote the conjugate operators of d_τ and d_ρ with respect to the scalar product on $C^*(\mathcal{G}, K; L^2(G/\Gamma, E))$ and put

$$\begin{aligned} \Delta_\tau &= d_\tau\delta_\tau + \delta_\tau d_\tau, \\ \Delta_\rho &= d_\rho\delta_\rho + \delta_\rho d_\rho \end{aligned} \tag{11}$$

LEMMA 2.8. *Let C be the Casimir operator $\mathcal{G}$. Then*

(i) $\Delta_\pi = \Delta_\tau - \Delta_\rho = -C + \rho(C)$ (the Kuga's lemma)

(12)

(ii) $Ker\ \Delta_\pi = Ker\ \Delta_\tau \cap Ker\ \Delta_\rho$,

The proof is obvious.

We remark that the form B_0 and the scalar product defined in (4) on $L^2(G/\Gamma, E)$ define a scalar product on the space $D^* = \wedge^p \mathcal{G}^* \hat{\otimes} L^2(G/\Gamma, E)) \simeq Hom\,(\wedge^p \mathcal{G}, L^2(G/\Gamma, E))$, and the space $C^* = C^*(\mathcal{G}, K; L^2(G/\Gamma, E))$ becomes the Hilbert subspace of the K-invariant elements of D^* annihilated on $\mathcal{K}$. Moreover $d_\tau = d_{|C^*}$, hence $\delta_\tau = \delta_{|C^*}$, $\Delta = \Delta_{|C^*}$. On the other hand, the space D^* can be identified with the space $\Omega^* L^2(G/\Gamma) \otimes E$ where $\Omega^*_{L^2}(G/\Gamma)$ is the space of the square - integrable forms on G/Γ with values in $\mathbb{C}$. So we have the decompositions

$$\begin{aligned} d &= d_0 \otimes 1_E, \\ \delta &= \delta_0 \otimes 1_E, \\ \Delta &= \Delta_0 \otimes 1_E, \end{aligned} \tag{13}$$

where d_0, δ_0, Δ_0 are the operators on $\Omega^*_{L^2}(G/\Gamma)$. For the operators d_0, δ_0, Δ_0, there is the Hodge-Kodaira's decomposition (see de Rham [15] Theorem 2.4).

$$\Omega^*_{L^2}(G/\Gamma) = Ker\ \Delta_0 \oplus Im\ d_0 \oplus Im\ \delta_0. \tag{14}$$

Hence we get

$$\begin{aligned} D^* &= Ker\ \Delta \oplus Im\ d \oplus Im\ \delta \\ &= (Ker\ \Delta_0 \oplus E) \oplus (Im\ d_0 \oplus E) \oplus Im\ \delta_0 \oplus E), \end{aligned} \tag{15}$$

and this deduces the decomposition

$$\begin{aligned} C^* = C^* \cap D^* &= (C^* \cap Ker\ \Delta) \oplus (C^* \cap Im\ d) \oplus (C^* \cap Im\ \delta) \\ &= Ker\ \Delta_\tau \oplus Im\ d_\tau \oplus Im\ \delta_\tau. \end{aligned} \tag{16}$$

Now we return to consider the operators Δ_ρ. By Kuga's lemma (see Lemma 2.8.(i)), the action of Δ_ρ on C^* is the action of the Casimir operator $\rho(C)$ of the representation ρ. Since (ρ, E) is completely reducible and E is assummed to be of finite spectrum we can restrict to the case when (ρ, E) is irreducible. Then by Schur's lemma we get

$$\rho(C) = \lambda . Id_{C^*}. \tag{17}$$

If $\lambda = 0$, we have $\Delta_\pi = \Delta_\tau$. The decomposition (16) becomes

$$C^* = Ker\ \Delta_\pi \oplus Im\ d_\pi \oplus Im\ \delta_\pi. \tag{18}$$

If $\lambda \neq 0$ we have $\Delta_\rho = \{0\} = Ker\ \Delta_\pi$, therefore $Im\ \Delta_\pi = C^*$. So Theorem 2.5 is deduced from Corollary 2.5.

COROLLARY 2.9.

$$H^*_{L^2}(X/\Gamma, E) \simeq Ker\ \Delta_\nabla \simeq Ker\ \Delta_\pi \simeq H^* C^*$$

3. The cohomology of the arithmetical parabolic subgroup

In this section we consider a reductive algebraic $\underline{Q}$. Let Γ be an arithmetical subgroup without torsion of $\underline{G}$, $\underline{\rho} : \underline{G} \longrightarrow Aut\ (V)$ a rational representation of $\underline{G}$. From this we get a representation ρ of the Lie group G of the real points of $\underline{G}$ on the vector space $E_0 = V \otimes \mathbf{R}$.

Let us fix a maximal compact subgroup K of G, and choose an K-admissible scalar product on E_0. Put $E = E_0 \otimes C$. Let θ be the Cartan involution corresponding to K.

Let $\underline{P}$ be a parabolic subgroup of $\underline{G}$ and P the Lie group of its real points. Consider the Langlands decomposition $P = MAU$ of P (see [10] Chapter I §1). Put $P(1) = MU$, $K_M = \pi_{P|M}(K \cap P(1))$ where $\pi_{P|M} : P(1) \longrightarrow M$ is the projection. Then K_M is a maximal compact subgroup in M and the restriction of θ on M is again a Cartan involution (corresponding to K_M).

It is well-known that $\Gamma \cap \underline{P} = \Gamma \cap P(1)$, and we write Γ_P for $\Gamma \cap P$. We define $X(1) = x_0 P(1) \simeq K_M \setminus P(1)$, $X_M = X_M \setminus M$, where $x_0 = [K] \in K \setminus G = X$. Let Γ_M be the image of Γ_P in M. Then Γ_M is again an arithmetical subgroup. The projection $\pi_{P|M}$ induces the projection $X(1)/\Gamma_P \longrightarrow X_M/\Gamma_M$ which is easily seen to be a fibration with the fiber U/Γ_U, where $\Gamma_U = \Gamma_P \cap U$. Let us denote the Lie algebras of the Lie group K_M, $P(1)$, U, M by $\mathcal{K}_M$, $\mathcal{P}_1$, $\mathcal{U}$, $\mathcal{M}$, respectively. We have the Cartan decomposition corresponding to the involution θ_M :

$$\mathcal{M} = \mathcal{K}_M \oplus \mathcal{P}_M. \tag{1}$$

This decomposition deduces a decomposition of $\mathcal{P}_1$:

$$\mathcal{P}_1 = \mathcal{M} \oplus \mathcal{U} = \mathcal{K}_M \oplus \mathcal{P}_M \oplus \mathcal{U} \tag{2}$$

On the complex $C^* = C^*\ (\mathcal{P}_1,\ K_M,\ L^2(P(1)/\Gamma_P, E))$ we define the filter

$$F^pC^n = \{\varphi \in C^n \mid \varphi(\xi_1, .., \xi_n) = 0 \text{ whenever there are more than } n-p \text{ vectors } \xi_i \in \mathcal{U}\}$$

The spectral sequence associated with this filter concentrates in the first quadrant. Therefore, it converges to

$$H^*C^* = H^*(\mathcal{P}_1, K_M; L^2(P(1)/\Gamma_P, E)). \tag{3}$$

Furthermore, we have

LEMMA 3.1. *The $E_2^{p,q}$-terms of this spectral sequence are isomorphic to $H^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E)_\rho)$, where $\mathcal{H}^q(\mathcal{U}, E)$ is the space of the harmonic q-forms in the cohomology classes of the Lie algebra $\mathcal{U}$ with respects to Laplacian L which is defined by B. Kostant (see [10] and [7] §2).*

Consider the inclusion $\mathcal{H}^q(\mathcal{U}, E) \longrightarrow Hom\ (\wedge^q\mathcal{U}, E)$. It implies the following conclusions

$$\begin{aligned} Hom\ (\wedge^p\mathcal{P}_M, \mathcal{H}^q(\mathcal{U}, E)) &\longrightarrow Hom\ (\wedge^p\mathcal{P}_M,\ Hom\ (\wedge^q\mathcal{U}, E)) \\ &\longrightarrow Hom\ (\wedge^{p+q}(\mathcal{P}_M \oplus \mathcal{U}), E) \end{aligned} \tag{3'}$$

The argument in §1 shows that we can identify each forms $\omega \in \Omega^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E))$ with a square-integrable K_M-equivalent function

$$(4) \qquad h\omega = \varphi_\omega : M/\Gamma_M \longrightarrow Hom\,(\wedge^p \mathcal{P}_M, \mathcal{H}^q(\mathcal{U}, E)).$$

Composing (5) with the inclusions (4), we get the function

$$(5) \qquad \varphi_\omega : M/\Gamma_M \longrightarrow Hom\,(\wedge^{p+q}(\mathcal{P}_M \oplus \mathcal{U}), E) = Hom\,(\wedge^{p+q}\mathcal{P}_1, E)$$

For each $[mu] \in P(1)/\Gamma_P$, we put

$$(6) \qquad (hiq\omega)([mu]) = \varphi_\omega([m]).$$

Then $iq\omega \in \Omega^{p+q}_{L^2}(X(1)/\Gamma_P, E_\rho)$. The corresponding map $\omega \longrightarrow iq\omega$ defines a homomorphism

$$(7) \qquad iq : \Omega^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E)_\rho) \longrightarrow \Omega^{p+q}_{L^2}(X(1)/\Gamma_P, E_\rho)$$

The following lemma is obvious.

LEMMA 3.2. *Denote by d_M and d_P the differentials of the complexes $\Omega^*_{L^2}(X_M/\Gamma_M, \mathcal{H}^*(\mathcal{U}, E)_\rho)$ and $\Omega^*_{L^2}(X(1)/\Gamma_P, E_\rho)$ respectively. Then $iqd_M = d_p i_q$. Therefore i_q induces a homomorphism*

$$(8) \; i^* = \bigoplus_{p+q=n} i^*_q : \bigoplus_{p+q=n} H^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E)_\rho) \longrightarrow H^n_{L^2}(X(1)/\Gamma_P, E_\rho).$$

COROLLARY 3.3. *i^* is an epimorphism.*

This Corollary follows from Lemma 3.1, Lemma 3.2 and the assertion on the convergence of the above spectral sequence.

Now Lemma 2.4 in [7] show that the form $B_{\theta|_\mathcal{U}}$ is an K_M-admissible scalar product on $\mathcal{U}$. This scalar product and the K-admissible scalar product on E define an K-admissible scalar product on $\mathcal{H}^q(\mathcal{U}, E)$. Therefore, as in section §2, we can construct the Laplacian Δ_M on $\Omega^*_{L^2}(X_M/\Gamma_M, \mathcal{H}^*(\mathcal{U}, E)_\rho)$ and the Laplacian Δ_p on $\Omega^*_{L^2}(X/(1)/\Gamma_P, E_\rho)$. Put

$$\mathcal{H}^*_{L^2}(X(1)/\Gamma_P, E_\rho) = Ker\ \Delta_p,$$

$$\mathcal{H}^*_{L^2}(X_M/\Gamma_M, \mathcal{H}^*(\mathcal{U}, E)) = Ker\ \Delta_M.$$

Then by Corollary 2.9, we have

$$\begin{aligned} &\mathcal{H}^n_{L^2}(X(1)/\Gamma_P, E_\rho) \simeq H^n_{L^2}(X(1)/\Gamma_P, E_\rho), \\ (10)\quad &\mathcal{H}^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E_\rho)) \simeq H^p(X_M/\Gamma_M, H^q(U/\Gamma_P, E)_\rho). \end{aligned}$$

The proof of the following lemma is analogous to the one of Lemma 2.7 of [8].

LEMMA 3.4. *$i_q.\Delta_M = \Delta_P i_q$ and i_q induces a monomorphism*

$$i^* : \bigoplus_{p+q=n} \mathcal{H}^p_{L^2}(X_M/\Gamma_M, \mathcal{H}^q(\mathcal{U}, E)_\rho) \longrightarrow \mathcal{H}^n_{L^2}(X(1)/\Gamma_P, E_\rho))$$

Now combining the result of Corollary 3.3, Lemma 3.4 with the isomorphism (10) we get our main result.

THEOREM 3.5. *The homomorphism i^* defined in the Lemma 3.2 is an isomorphism.*

REFERENCES

1. A. Borel and N. Wallach, *Continuous cohomology, discrete subgroup and representations of reductive groups*, Annals of Math. Studies 4, Princeton University Press, 1980.
2. D.N. Diep, *Multidimensional quantization I*, Acta Math. Vietnamica **5** 2 (1980), 42–55.
3. D.N. Diep, *Multidimensional quantization II*, Acta Math. Vietnamica **7** 1 (1982), 87–93.
4. D.N. Diep, *Quelques aspects topologiques en analyses harmonique*, Acta Math. Vietnamica **8** 2 (1983), 35–131.
5. D.N. Diep, *On the Langlands type discrete group I : The Borel-Serre compactification*, Acta Math. Vietnamica **12** 1 (1987), 41–54.

6. D.N. Diep, *Multidimensional quantization and the generic representation*, Preprint series 29, Inst. Math. Hanoi, 1986.
7. G. Harder, *On the cohomology of discrete arithmetically defined group*, In : Proc. of the Int. Collog. : On Discrete subgroup of Lie groups and Applications to Moduli, Bombay (1973), Oxford University Press (1975), 129–160.
8. G. Harder, *On the cohomology of $SL_2(\theta)$*, In : Lie groups and their representaitons, Edited by I.M. Gelfand, London (1975), 139–150.
9. Harish-Chandra, *Automorphic forms on semisimple Lie group*, Springer Lecture Notes in Math. **62** (1968).
10. B. Kostant, *Lie algebra cohomology and generalized Borel-Weil theorem*, Annals of Math. **74** (1961), 329–387.
11. K.F. Lai, *On the cohomology of congruènce subgroup of sympletic group*, Nagoya Math. J. **85** (1982), 155–174.
12. Y. Masushima and S. Murakomi, *On the vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds*, Annals of Math. **78** (1963), 365–416.
13. J. Schwermer, *Sur la cohomologie des sous groupes de congruence de $SL_3(Z)$*, C.R. Acad. Sc. Paris **283** (1976), 817–820.
14. J. Schwermer, *Eisensteinreihen und die Kohomologie von Kongruenzuntergrupper von $SL_n(Z)$*, Monner Mathematische Schriften **99** (1977).
15. G. de Rham, *Variété différentiable*, Paris Hermann (1956).

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