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THE APPROXIMATE SOLUTION OF THE FIRST KIND OPERATOR EQUATION IN LOCALLY CONVEX SPACES BY DISCREPANCY METHOD

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1. Introduction.

We consider the first kind operator equation

$$(1) \qquad Ax = y_0 \ , \ x \in X \ , \ y_0 \in R(A) \subset Y \ ,$$

where X is a locally convex space, Y is a separated locally convex space, and A is a linear operator from X into Y which is weakly continuous and such that there exists the inverse operator A^{-1} defined on the range $R(A)$ of A.

Given a filter base of y_0 consisting of closed convex subsets $\{V_\delta\}$, we want to establish approximate solutions x_δ of equation (1) such that

$$x_\delta \longrightarrow x_0 \ ,$$

where x_0 is an exact solution of (1), i.e. $x_0 = A^{-1}y_0$.

For given X, Y, A, y_0, we associate each filter base $\{V_\delta\}$ with the sequence of the approximate solutions $\{x_\delta\}$ and denote the above problem by $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$. If the approximate solutions x_δ are established by a discrepancy method and $x_\delta \to x_0$, then we say that this discrepancy method stabilizes the problem.

In this paper we give conditions under which a discrepancy method stabilizes any problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$.

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REMARK: : The case where the space X satisfies condition (A), Y is a metrizable locally convex space, and the operator A is linear continuous has been examined in [2].

2. Preliminaries

Assume that the topology of X is given by a family of seminorms $\{p_\gamma\}_{\gamma\in\Gamma}$. We shall write $X \equiv (X, p_\gamma, \Gamma)$.

Let us first recall some definitions and theorems which will be needed in the next section.

DEFINITION 1. *A filter $\mathcal{F}$ in X is called a Cauchy filter if for any neighbourhood U of the origin of X there exists an element $A \in \mathcal{F}$ such that*

$$x - y \in U \quad \textit{for all} \quad x, y \in A\ .$$

If each Cauchy filter in X is convergent, then X is called complete.

LEMMA 1 (SEE [4]). *Each locally convex space $X \equiv (X, p_\gamma, \Gamma)$ has the following properties :*

a) X is isomorphic to a subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$, where each X_γ is a normed space defined by $X_\gamma = X/p_\gamma^{-1}(0)$.

b) The isomorphism can be defined by mapping

$$V(x) = (V_\gamma(x))\ ,\ x \in X\ ,$$

where each V_γ is a canonical mapping from X onto X_γ.

c) $pr_\gamma \circ V = V_\gamma$ for all $\gamma \in \Gamma$,

where pr_γ is a canonical projection from $\prod_{\gamma\in\Gamma} X_\gamma$ on to X_γ.

LEMMA 2 (SEE [4]). *The quotient space X/M is separated if and only if M is a closed subspace in X.*

LEMMA 3 (SEE [4,5]). *The product of weak topologies is a weak topology.*

LEMMA 4 (SEE [4,5]). *Suppose that X is a product of locally convex spaces X_γ. Then X is complete if and only if each X_γ is complete.*

DEFINITION 2. *We say that the locally convex space $X \equiv (X, p_\gamma, \Gamma)$ satisfies the condition (A') if :*

a) The mapping V in Lemma 1 is an isomorphism from X onto $\prod_{\gamma\in\Gamma} X_\gamma$.

b) X is weak complete.

c) For each net $\{x_\omega\} \subset X$ with $x_\omega \xrightarrow{w} x$ and $p_\gamma(x_\omega) \longrightarrow p_\gamma(x)$, $\forall\gamma \in \Gamma$, we have $x_\omega \longrightarrow x$.

d) If $p_\gamma(x_1 + x_2) = p_\gamma(x_1) + p_\gamma(x_2)$ for $\gamma \in \Gamma$, then there exists a number λ_γ such that $p_\gamma(x_1 - \lambda_\gamma x_2) = 0$.

REMARK:

1. It is clear that if $X = \prod_{\gamma\in\Gamma} X_\gamma$, where each X_γ is a Hilbert space, then the space X satisfies condition (A').

2. Any space satisfying condition (A') is separated.

COROLLARY 5 (CF. [1]). *If X is a locally convex space satisfying condition (A'), then each $X_\gamma = X/p_\gamma^{-1}(0)$ is an E–space.*

PROOF: It follows from Lemmas 1, 3, 4.

3. Discrepancy method

DEFINITION 3. *Suppose that* $\Omega \subset \prod_{\gamma \in \Gamma} X_\gamma$, *where each* X_γ *is a normed space. An element* $x \in \Omega$ *is said to realize a minimum of* Ω *if it satisfies the following condition :*

$$(2) \qquad ||pr_\gamma(x)|| = \inf_{y \in \Omega} ||pr_\gamma(y)|| \quad \text{for all} \quad \gamma \in \Gamma .$$

Let us consider the problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$. Set $\Omega_\delta^\gamma = V_\gamma[A^{-1}(V_\delta)]$ and $\Omega_\delta = \prod_{\gamma \in \Gamma} \Omega_\delta^\gamma$. The approximate solution of the equation (1) depending on V_δ (established by a discrepancy method) is an element $x_\delta \in X$ such that $x_\delta = V^{-1}(y_\delta)$, where y_δ realizes a minimum of Ω_δ. If $x_\delta \longrightarrow x_0$, we say that the discrepancy method stabilizes the problem.

Now we discuss the existence of approximate solutions of the equation (1) by a discrepancy method.

THEOREM 6. *If* X *is a locally convex space satisfying condition (A'), then there exists a unique approximate solution of the equation (1) for each* V_δ *corresponding to the problem* $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$.

PROOF: For each $\gamma \in \Gamma$, put

$$\mu^\gamma = \inf_{y \in \Omega_\delta} ||pr_\gamma(y)|| .$$

Then there exists a sequence $\{y_k\} \subset \Omega_\delta$ such that

$$(3) \qquad ||y_k^\gamma|| \longrightarrow \mu^\gamma \quad \text{as} \quad k \longrightarrow \infty ,$$

where $y_k^\gamma = pr_\gamma(y_k)$.

Since X_γ is an E–space and by (3), we may assume without loss of generality that

$$y_k^\gamma \xrightarrow{w} y_\delta^\gamma \in X_\gamma \ . \tag{4}$$

We consider the element $y_\delta = (y_\delta^\gamma)$. We shall prove that y_δ realizes a minimum of Ω_δ. Indeed, if $y_\delta \notin \Omega_\delta$, then there exists $\gamma_0 \in \Gamma$ such that

$$y_\delta^{\gamma_0} \notin \Omega_\delta^{\gamma_0} \ . \tag{5}$$

Since the mapping $V_{\gamma_0} = pr_{\gamma_0} \circ V$ is linear continuous open from X onto X_{γ_0}, the set $\Omega_\delta^{\gamma_0} = V_{\gamma_0}[A^{-1}(V_\delta)]$ has at least one interior point, i.e. $int\,(\Omega_\delta^{\gamma_0}) \neq \emptyset$. By the Hahn– Banach theorem, there exists a linear continuous functional $f \in X_{\gamma_0}^*$ such that

$$f(y_\delta^{\gamma_0}) > 1 \quad \text{and} \quad f(y^{\gamma_0}) \leq 1 \quad \text{for all} \quad y^{\gamma_0} \in \Omega_\delta^{\gamma_0} \ . \tag{6}$$

This is a contradiction to (4). Thus

$$y_\delta \in \Omega_\delta \ . \tag{7}$$

From (7) it follows that

$$\mu^\gamma \leq ||y_\delta^\gamma|| \quad \text{for all} \quad \gamma \in \Gamma \ . \tag{8}$$

From (4) and by the Banach–Steinhaus theorem, it follows that

$$||y_\delta^\gamma|| \leq \underline{\lim}_{k\to\infty} ||y_k^\gamma|| = \mu^\gamma \ . \tag{9}$$

From (8) and (9), it follows that

$$||y_\delta^\gamma|| = \mu^\gamma \quad \text{for all} \quad \gamma \in \Gamma \ . \tag{10}$$

From (7) and (10), it follows that y_δ is a minimum of Ω_δ.

The uniqueness of the minimum of Ω_δ follows from the fact that Ω_δ is convex in $\prod_{\gamma\in\Gamma} X_\gamma$ and each X_γ is an E–space.

It follows that the approximate solution x_δ of (1) uniquely exists and is defined by

$$x_\delta = V^{-1}(y_\delta) .$$

DEFINITION 4. *We say that the weak continuous linear operator $A : X \to Y$, having A^{-1}, belongs to the set $\alpha[X, Y]$ if and only if a discrepancy method stabilizes every problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$.*

Our purpose is to find conditions for A to be in $\alpha[X, Y]$.

Consider the problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$. Put

$$U_\delta = \{x \in X : Ax \in V_\delta(y_0) - Ax_0\} .$$

By [4] there exists a topology (T) in X such that X together with (T) is a locally convex space. Moreover, the family $\{U_\delta\}$ becomes a filter base of neighbourhoods of the origin in X. We denote the space by $X_{(T)}$.

THEOREM 7. *The quotient space $X_{(T)/p_\gamma^{-1}(0)}$ is separated if and only if every set $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$.*

PROOF: Necessity : Suppose that $X_{(T)/p_\gamma^{-1}(0)}$ is separated. By Theorem 2, $p_\gamma^{-1}(0)$ is a closed subset in $X_{(T)}$. On the other hand, $p_\gamma^{-1}(0)$ is convex in $X_{(T)}$. Hence $p_\gamma^{-1}(0)$ is a weakly closed subset in $X_{(T)}$. It is not difficult to show that the operator $A^{-1} : R(A) \longrightarrow X_{(T)}$ is continuous. Hence it is weakly continuous (see [4]). It follows that $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$.

Sufficiency : Since the operator A is one to one, we have

$$A^{-1}[A(p_\gamma^{-1}(0))] = p_\gamma^{-1}(0) . \qquad (11)$$

It is evident that A is continuous in the topology (T). Hence it is weakly continuous.

From (11) and by the weak continuity of the operator A, it follows that $p_\gamma^{-1}(0)$ is weakly closed in $X_{(T)}$. Hence it is closed in $X_{(T)}$. By Theorem 2 it follows that $X_{(T)/p_\gamma^{-1}(0)}$ is separated.

The following lemmas will be used in the proof of Theorem 8.

LEMMA 1 (SEE [3]). *Suppose that X is a compact space and $\{E_\delta\}$ is a filter in X (or a filter base). If $\{E_\delta\}$ has a unique limit point x, then $\{E_\delta\}$ converges to x.*

LEMMA 2. *Suppose that the mapping $A : X \to Y$ has a closed graph and for a fixed point $y_0 \in Y$ there exists a unique element $x_0 \in X$ such that $Ax_0 = y_0$, where X is a compact space and Y is a topological space. If $\{V_\delta\}$ is a filter of neighbourhoods of y_0, then $\{E_\delta\}$ is a convergent filter base of x_0, where $E_\delta = A^{-1}(V_\delta)$.*

PROOF: It is clear that $\{E_\delta\}$ is a filter base in X. Since X is a compact space, $\{E_\delta\}$ has a limit point $\bar{x}$ (see [3]). We shall show that $\bar{x}$ coincides with x_0, hence by Lemma 1, $E_\delta \to x_0$. In fact, by definition there exists a filter $\{Q_\gamma\}$ in X such that it exceeds $\{E_\delta\}$ and converges to $\bar{x}$ (see [6]). Therefore $\{A(Q_\gamma)\}$ exceeds $\{V_\delta\}$. It follows that $\{A(Q_\gamma)\}$ also converges to y_0. We obtain

$$(12) \qquad Q_\gamma \to \bar{x} \quad \text{and} \quad A(Q_\gamma) \to y_0 \ .$$

Since the graph of A is closed, (12) implies

$$A\bar{x} = y_0 \ .$$

Thus $x_0 = \bar{x}$, and $E_\delta \to x_0$.

Now we are able to formulate and prove the main result.

THEOREM 8. *A weak continuous linear operator A from a locally convex space $X \equiv (X, p_\gamma, \Gamma)$ satisfying the condition (A') into a locally convex separated space Y and having A^{-1} belongs to $\alpha[X,Y]$, if and only if each $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$.*

PROOF: Necessity : Let A be a weak continuous linear operator from X into Y . Suppose that A^{-1} exists and $A \in \alpha[X,Y]$. We show that each $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$. In fact, if not, there exists $\gamma_0 \in \Gamma$ such that $A(p_\gamma^{-1}(0))$ is not weak closed in $R(A)$. We want to construct a problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$ so that a discrepancy method does not stabilize it. This, of course, contradicts $A \in \alpha[X,Y]$.

Let $\{V_\delta(0)\}$ be a filter base of neighbourhoods of the origin in Y such that each $V_\delta(0)$ is convex closed in Y. Since Y is separated, $X_{(T)}$ is also separated. Since $A(p_\gamma^{-1}(0))$ is not weakly closed in $R(A)$, by Theorem 7 $X_{(T)}/p_{\gamma_0}^{-1}(0)$ is not separated. It follows that

$$\bigcap_\delta V_{\gamma_0}(U_\delta) \neq \{0\} \ , \tag{13}$$

where $U_\delta = A^{-1}(V_\delta(0))$, and $V(x) = (V_\gamma(x))$ is the isomorphism in Theorem 1.

On the other hand, we have $V_{\gamma_0} = pr_{\gamma_0} \circ V$. Therefore there exists an element $y^{\gamma_0} \neq 0$, $y^{\gamma_0} \in X_{\gamma_0}$ such that

$$y^{\gamma_0} \in \bigcap_\delta pr_{\gamma_0} \circ V(U_\delta)$$

We choose an element $(y^\gamma) \in \prod_{\gamma\in\Gamma} X_\gamma$ such that

$$pr_{\gamma_0}[(y^\gamma)] = -y^{\gamma_0} \ . \tag{14}$$

Let $x_0 = V^{-1}((y^\gamma))$, and consider the problem $\alpha[X, Y, A, x_0, \{V_\delta(0) + Ax_0\}]$. Since $A \in \alpha[X, Y]$, a discrepancy method stabilizes the problem. Therefore the sequence of the approximate solutions x_δ of (1) established by a discrepancy method converges to x_0

(15) $$x_\delta \longrightarrow x_0 \ .$$

On the other hand, we have

(16) $$\|pr_{\gamma_0} \circ V(x_\delta)\| = \inf_{x_{\gamma_0} \in V_{\gamma_0}(U_\delta) - y^{\gamma_0}} \|x_{\gamma_0}\| \leq \|y^{\gamma_0} - y^{\gamma_0}\| = 0 \ ,$$

(17) $$\|pr_{\gamma_0} \circ V(x_0)\| = \|y^{\gamma_0}\| > 0 \ .$$

From (16) and (17) it follows that $x_\delta \not\to x_0$. This contradicts (15). Thus $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$.

Sufficiency : Suppose that $A : X \to Y$ is weakly continuous linear such that A^{-1} exists and every $A(p_\gamma^{-1}(0))$ is weakly closed in $R(A)$. We show that $A \in \alpha[X, Y]$. We assume that $A \notin \alpha[X, Y]$. There exists a problem $\alpha[X, Y, A, x_0, \{V_\delta(y_0)\}]$ such that it is not stabilized by a discrepancy method. Since X is an (A')-space, we have $\gamma_0 \in \Gamma$ and a subsequence $\{x_{\delta_n}\}$ of the approximate solutions $\{x_\delta\}$ (established by a discrepancy method) such that

(18) $$\|pr_{\gamma_0} \circ V(x_{\delta_n}) - pr_{\gamma_0} \circ V(x_0)\| \geq \varepsilon > 0 \ ,$$

where ε is a fixed real number.

Let us consider an operator $M_{\gamma_0} : X_{\gamma_0} \to Y$ defined by

$$M_{\gamma_0} = A \circ V^{-1} \circ j_{\gamma_0} \ ,$$

where j_{γ_0} is an inclusion mapping from X_{γ_0} into $\prod_{\gamma \in \Gamma} X_\gamma$, and V is the isomorphism of Theorem 1.

It is clear that the operator M_{γ_0} is a weakly continuous linear one to one mapping from X_{γ_0} into Y.

In $R(M_{\gamma_0})$ we consider the following family of subsets

$$F_{\delta_n}^{\gamma_0} = M_{\gamma_0} \circ V_{\gamma_0}(U_{\delta_n}) ,$$

where $U_{\delta_n} = \{x \in X : Ax \in V_{\delta_n} - Ax_0\}$.

By [4] there exists a topology (T_{γ_0}) in $R(M_{\gamma_0})$ such that $R(M_{\gamma_0})$ together with (T_{γ_0}) becomes a locally convex space and $\{F_{\delta_n}^{\gamma_0}\}$ becomes a filter base of neighbourhoods of the origin in $R(M_{\gamma_0})$. We denote it by $R(M_{\gamma_0})_{(T_{\gamma_0})}$.

As a mapping from X_{γ_0} onto $R(M_{\gamma_0})_{(T_{\gamma_0})}$, M_{γ_0} is linear continuous. Hence it is also weakly continuous.

Put $h_0 = M_{\gamma_0} \circ V_{\gamma_0}(x_0)$ and $F_{\delta_n}^{\gamma_0}(h_0) = h_0 + F_{\delta_n}^{\gamma_0}$. It follows that $h_0 \in R(M_{\gamma_0})_{(T_{\gamma_0})}$. It is not difficult to show that $\{F_{\delta_n}^{\gamma_0}(h_0)\}$ is a filter base of neighbourhoods of h_0 in the space $R(M_{\gamma_0})_{(T_{\gamma_0})}$. Therefore it is also a filter base of weak neighbourhoods of h_0 in $R(M_{\gamma_0})_{(T_{\gamma_0})}$.

We consider the ball $B^{\gamma}(x_0)$ in X_{γ_0} defined by

$$B^{\gamma_0}(x_0) = \{x \in X_{\gamma_0} : ||x|| \leq ||V_{\gamma_0}(x_0)||\} .$$

Since X_{γ_0} is reflexive, $B^{\gamma_0}(x_0)$ is weakly compact in X_{γ_0}. Put $E_{\delta_n}^{\gamma_0} = B^{\gamma_0}(x_0) \cap M_{\gamma_0}^{-1}[F_{\delta_n}^{\gamma_0}(h_0)]$. It is clear that $V_{\gamma_0}(x_0) \in E_{\delta_n}^{\gamma_0}$. Since M_{γ_0} is weakly continuous and $\{F_{\delta_n}^{\gamma_0}(h_0)\}$ is a filter base of weak neighbourhoods of h_0, by Lemma 2 it follows that $\{E_{\delta_n}^{\gamma_0}\}$ is a weakly convergent filter base of $V_{\gamma_0}(x_0)$.

It is not difficult to show that

$$\Omega_{\delta_n}^{\gamma_0} \cap B^{\gamma_0}(x_0) \subset E_{\delta_n}^{\gamma_0} ,$$

where $\Omega_{\delta_n}^{\gamma_0} = V_{\gamma_0}[A^{-1}(V_{\delta_n})]$.

Since $pr_{\gamma_0} \circ V(x_{\delta_n}) \in \Omega_{\delta_n}^{\gamma_0} \cap B^{\gamma_0}(x_0)$, we have $pr_{\gamma_0} \circ V(x_{\delta_n}) \in E_{\delta_n}^{\gamma_0}$.

By the Banach–Steinhaus theorem, it follows that

$$(19) \qquad ||pr_{\gamma_0} \circ V(x_0)|| \leq \underline{lim}_k ||pr_{\gamma_0} \circ V(x_{\delta_{n_k}})|| \,.$$

On the other hand, we have

$$||pr_{\gamma_0} \circ V(x_{\delta_{n_k}})|| \leq ||pr_{\gamma_0} \circ V(x_0)|| \quad \text{for all} \quad k \,.$$

Hence

$$(20) \qquad \overline{lim}_k ||pr_{\gamma_0} \circ V(x_{\delta_{n_k}})|| \leq ||pr_{\gamma_0} \circ V(x_0)|| \,.$$

From (19) and (20) it follows that

$$\lim_k ||pr_{\gamma_0} \circ V(x_{\delta_{n_k}})|| = ||pr_{\gamma_0} \circ V(x_0)|| \,.$$

Since X_{γ_0} is an E-space, we have

$$\lim_k pr_{\gamma_0} \circ V(x_{\delta_{n_k}}) = pr_{\gamma_0} \circ V(x_0) \,.$$

This equality contradicts (18). Thus $x_\delta \to x_0$.

We end the paper by giving an example to show the existence of an operator $A \in \alpha[X, Y]$.

EXAMPLE: Let $X \equiv Y$ be the product of Hilbert spaces X_γ, $\gamma \in \Gamma$, and $A_\gamma : X_\gamma \to X_\gamma$ weak continuous linear one to one operators. The operator $A : X \to X$ is defined by

$$A[(x_\gamma)] = (A_\gamma(x_\gamma)) \,, \; (x_\gamma) \in \prod_{\gamma\in\Gamma} X_\gamma \,.$$

It is not difficult to show that $A \in \alpha[X, Y]$.

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