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A REMARK ON LIMITS FOR GAMES WHICH BECOME FAIRER WITH TIME

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{A}_n)$ be an increasing sequence of subo-fields of $\mathcal{A}$. A sequence (X_n) in L^1_R, always assumed to be adapted to $(\mathcal{A}_n)$, is said to be a **mil** [3] or a game which becomes fairer with time [1], respectively if for every $\varepsilon > 0$ there exists p such that for all $n \geqq m \geqq p$, we have

$$P\left(\sup_{p \leq q \leq n} \| X_q(n) - X_q \| \geqq \varepsilon\right) \leqq \varepsilon,$$

or

$$P\left(\| X_m(n) - X_m \| \leq \varepsilon\right) \geqslant \varepsilon, \text{ respectively,}$$

Here $X_m(n)$ denotes the $\mathcal{A}_m$ — conditional expectation of X_n. Using the structure results of Talagrand [3], we have recently proved in ([2], Theorem 2.3) the following statement:

THEOREM 1. Let (X_n) be an L^1 — bounded real-valued game which becomes fairer with time. Then (X_n) converges in probability to some $X \in L^1_R$.

To prove the theorem we showed in [2] that for every subsequence (m_k) of N there exists a subsequence (n_k) of (m_k) such that the subsequence (X_{n_k}) is an L^1 — bounded mil which must converge a. s., by virtue of Theorem 4 [3]. However, there we did not mention that all these chosen mils (X_{n_k}) really converge a.s. to the same limit. Thus, the aim of this note is to fill this gap and to give a complete proof of the theorem.

2. PROOF OF THEOREM 1.

First, let (X_n) be a game which becomes fairer with time. Then by definition there exists an increasing subsequence (l_k) of N such that for all $h \geqq m \geqq l_k$ we have

$$P(\| X_m(h) - X_m \| \geq 2^{-k}) \leq 2^{-k}.$$

Now suppose that (X_n) is L^1-bounded. To prove Theorem 1 it is sufficient to show that if (m_k) is a subsequence of N then there exists a subsequence (n_k) of (m_k) such that both subsequences (X_{l_k}) and (X_{n_k}) are mils which converge a.s. and to the same limit. To see this let us consider an arbitrary subsequence (m_k) of N. Then one can construct a subsequence (n_k) of (m_k) such that, for every k, $n_k \geq l_k$. Now let (s_k) be the superimposed sequence of (l_k) with (n_k). Then for any $h, k \in N$ with $h \geqslant l_k$, the above inequality yields

$$P(\sup_{l_k \leq s_q \leq h} \| X_{s_q}(h) - X_{s_q} \| \geq 2^{-k+2}) \leq P(\sup_{l_k \leq s_q \leq h} \| X_{s_q}(h) - X_{s_q} \| \geq 2^{-k})$$

$$\leq P(\sup_{l_k \leq l_q \leq h} \| X_{l_q}(h) - X_{l_q} \| \geq 2^{-k}) + P(\sup_{l_k \leq n_q \leq h} \| X_{n_q}(h) - X_{n_q} \| \geq 2^{-k})$$

$$\leq \sum_{l_k \leq l_q \leq h} P(\| X_{l_q}(h) - X_{l_q} \| \geq 2^{-q}) + \sum_{l_k \leq n_q \leq h} P(\| X_{n_q}(h) - X_{n_q} \| \geq 2^{-q})$$

$$\leq \sum_{l_k \leq l_q \leq h} 2^{-q} + \sum_{l_k \leq n_q \leq h} 2^{-q}$$

$$< \sum_{q=k}^{\infty} 2^{-q} + \sum_{q=k}^{\infty} 2^{-q} = 2^{-k+2}.$$

Thus in particular, by taking the only h from each of the sequences (l_k), (n_k) and (s_k), we see that each of the sequences (X_{l_k}), (X_{n_k}) and (X_{s_k}) is itself an L^1-bounded mil in the sense of Talagrand [3]. Therefore by Theorem 4 of Talagrand [3], the subsequences (X_{l_k}), (X_{n_k}) and (X_{s_k}) converge a.s. and obviously to the same limit $X \in L^1_R$. This completes the proof of the theorem.

For further related results, see [2].

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