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q – ANALOGUES OF CERTAIN RECURRENCE RELATIONS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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1. INTRODUCTION AND PRELIMINARIES:

Recently, Saxena, Modi and Kalla [5] defined the basic analogue of the G-function in the following manner:

$$G^{m_1, n_1}_{A, B}\left[z; q \left| \begin{matrix} (b_j) \\ (a_j) \end{matrix} \right.\right] =$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod_{j=1}^{m_1} G(a_j - \rho) \prod_{j=1}^{n_1} G(1 - b_j + \rho) \pi z^\rho d\rho}{\prod_{j=m_1+1}^{B} G(1 - a_j + \rho) \prod_{j=n_1+1}^{A} G(b_j - \rho) G(1-\rho) \sin \pi\rho} \tag{1}$$

where $G(a) = \left\{ \prod_{n=0}^{\infty} (1-q^{a+n}) \right\}^{-1}$ and $0 \leqq m_1 \leq B$; $0 \leq n_1 \leqq A$; $1 \leq j \leqq B$; $1 \leq j \leq A$.

The contour 'C' is a line parallel to Re $(wz) = 0$ with indentations, if necessary, in such a manner that all the poles of $G(a_j - \rho)$; $1 \leqq j \leqq m_1$ are to the right and those of $G(1 - b_j + \rho)$; $1 \leqq j \leq n_1$ to the left of 'C'.

The integral converges if Re $[\rho \log (z) - \log \sin \pi\rho] < 0$ for large values of $|\rho|$ on the contour, that is, if $|\{\arg (z) - w_2 w_1^{-1} \log |z|\} < \pi$ where for $|q|<1$, $\log q = -w = -(w_1 + iw_2)$, w, w_1, w_2 are definite quantities and w_1, w_2 being real.

Let $|q| < 1$ and

$$(a;q)_\mu \equiv (q^a;q)_\mu = \frac{\prod_{j=0}^{\infty}(1-q^{a+j})}{\prod_{j=0}^{\infty}(1-q^{a+\mu+j})} \quad \ldots (2)$$

for arbitrary 'a' and 'μ', so that

$$(a;q)_n = \begin{cases} 1 & \text{if } n=0 \\ (1-q^a)(1-q^{a+1})\ldots(1-q^{a+n-1}); & \forall n \in \{1,2,3,\ldots\} \end{cases}$$

and

$$(a;q)_{-n} = \frac{(-1)^n q^{\frac{n}{2}(n+1)}}{q^{na}(1-a;q)_n} \quad \ldots (3)$$

Also let $\Pi(q^a) = \frac{1}{G(a)} = (a;q)_\infty = \prod_{n=0}^{\infty}(1-q^{a+n})$

and

$$\Pi\begin{bmatrix} q^{a_1},\ldots,q^{a_r}; \\ q^{b_1},\ldots,q^{b_s} \end{bmatrix} = \frac{\Pi(q^{a_1})\ldots\Pi(q^{a_r})}{\Pi(q^{b_1})\ldots\Pi(q^{b_r})}.$$

we then define the basic generalized hypergeometric function as

$${}_r\Phi_s\begin{bmatrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{bmatrix}; q, z\bigg] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n\ldots(a_r;q)_n\, z^n}{(q;q)_n\,(b_1;q)_n\ldots(b_s;q)_n}, \ldots \quad (4)$$

where $z = x(1-q)^{s+1-r}$. The series converges if $|z| < 1$ and $|q| < 1$. As $q \to 1$ in (4), the ${}_r\Phi_s$ reduces to the ordinary hypergeometric function ${}_rF_s(\cdot)$.

A detailed account of the G — function with scalar argument can be found in the monograph by Mathai and Saxena [4].

The object of this paper is to establish six recurrence relations for the basic analogue of the G — function. As special cases, we derive the recurrence relation for the basic generalized hypergeometric function ${}_r\Phi_s(\cdot)$.

2. RECURRENCE RELATIONS

The following recurrence relations will be proved here:

$$(1-q^{a_1})\, G^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(b_j)\\(a_j)\end{matrix}\right] - G^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(b_j)\\(a_1+1),\ (a_2),\ldots;\ (a_B)\end{matrix}\right]$$

$$= \frac{q^{a_1}}{z}\, G^{m_1,n_1}_{A,B}\left[z;q\,\middle|\,\begin{matrix}(b_j+1)\\(a_j+1)\end{matrix}\right], \quad \ldots (5)$$

where $0 \leq m_1 \leq B$; $0 \leq n_1 \leq A$; $1 \leq j \leq A$; $1 \leq j \leq B$.

$$(1 - q^{b_A - 1}) G^{m_1, n_1}_{A, B}\left[z : q \,\middle|\, \begin{matrix}(b_j)\\(a_j)\end{matrix}\right] - G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_1), \ldots, (b_{A-1}), (b_A - 1)\\(a_j)\end{matrix}\right]$$

$$= \frac{q^{b_A - 1}}{z} G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j + 1)\\(a_j + 1)\end{matrix}\right] \quad \ldots (6)$$

where $0 \leq m_1 \leq B$; $0 \leq n_1$ A; $1 \leq j \leq B$; $1 \leq j \leq A$.

$$(1 - q^{a_2})\, G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j)\\(a_1 + 1), (a_2), \ldots, (a_B)\end{matrix}\right]$$

$$- (1 - q^{a_1}) G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j\\(a_1), (a_2 + 1), (a_3), \ldots, (a_B)\end{matrix}\right]$$

$$= \frac{(q^{a_2} - q^{a_1})}{z} G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j + 1)\\(a_j + 1)\end{matrix}\right], \quad \ldots (7)$$

where $0 \leq m_1 \leq B$; $0 \leq n_1 \leq A$; $1 \leq j \leq B$; $1 \leq j \leq A$;

$$(1 - q^{b_{n_1 + 2} - 1})\, G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_1), \ldots, (b_{n_1}), (b_{n_1+1} - 1), (b_{n_1 + 2}), \ldots, (b_A)\\(a_j)\end{matrix}\right]$$

$$- (1 - q^{b_{n_1 + 1} - 1}) G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_1), \ldots, (b_{n_1}), (b_{n_1 + 1}), (b_{n_1 + 2} - 1), (b_{n_1 + 3}), \ldots (b_A)\\(a_j\end{matrix}\right]$$

$$= \frac{(q^{b_{n_1 + 1}} - q^{b_{n_1 + 2}})}{qz} G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j + 1)\\(a_j + 1)\end{matrix}\right], \quad \ldots (8)$$

where $0 \leq m_1 \leq B$; $0 \leq n_1 \leq A$; $1 \leq j \leq B$; $1 \leq j \leq A$.

$$(1 - q^{b_{n_1 + 2} - 1})\, G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_j)\\(a_j)\end{matrix}\right]$$

$$- G^{m_1, n_1}_{A, B}\left[z ; q \,\middle|\, \begin{matrix}(b_1, \ldots, (b_{n_1}), (b_{n_1 + 1}), (b_{n_1 + 2} - 1), (b_{n_1 + 3}), \ldots, (b_A)\\(a_j)\end{matrix}\right]$$

$$= \frac{q^{b_{n_1 + 2} - 1}}{z} G^{m_1, n_1}_{A, B}\left|z ; q \,\middle|\, \begin{matrix}(b_j + 1)\\(a_j + 1)\end{matrix}\right|; \quad \ldots (9)$$

where $0 \leq m_1 \leq B$: $0 \leq n_1 \leq A$; $1 \leq j \leq B$; $1 \leq j \leq A$.

$$(1-q^{a_2})G^{m_1,n_1}_{A,B}\left[z;q\left|\begin{matrix}(b_j)\\(a_j)\end{matrix}\right.\right]$$

$$-(1-q^{a_1-1})G^{m_1,n_1}_{A,B}\left[z;q\left|\begin{matrix}(b_j)\\(a_1-1),(a_2+1),(a_3,\ldots,(a_B)\end{matrix}\right.\right]$$

$$=\frac{q^{a_2}}{z}(1-q^{a_1-a_2-1})G^{m_1,n_1}_{A,B}\left[z;q\left|\begin{matrix}(b_j+1)\\(a_1),(a_2+1),\ldots,(a_B+1)\end{matrix}\right.\right] \quad (10)$$

where $0\leq m_1\leq B;\ 0\leq n_1\leq A;\ 1\leq j\leq B;\ 1\leq j\leq A.$

Proof of(5) : To prove (5), we **substitute** the value of the G_q—function from (1). So the L.H.S. of (5) becomes

$$\frac{(1-q^{a_1})}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G(a_j-x)\prod_{j=1}^{n_1}G(1-b_j+x)\pi z^x dx}{\prod_{j=m_1+1}^{B}G(1-a_j+x)\prod_{j=n_1+1}^{A}G(b_j-x)G(1-x)\sin\pi x}$$

$$-\frac{1}{2\pi i}\int_C\frac{G(a_1+1-x)\,G(a_2-x)\ldots G(a_{m_1}-x)\prod_{j=1}^{n_1}G(1-b_j+x)\pi z_x}{\prod_{j=m_1+1}^{B}G(1-a_j+x)\prod_{j=n_1+1}^{A}G(B_j-x)\,G(1-x)\sin\pi x}dx$$

which on simplification gives

$$-\frac{q^{a_1}}{2\pi i}\int_C\frac{\prod_{j=1}^{m_1}G(a_j-x)\prod_{j=1}^{n_1}G(1-b_j+x)\pi z^x(1-q^{-x})}{\prod_{j=m_1+1}^{B}G(1-a_j+x)\prod_{j=n_1+1}^{A}G(b_j-x)G(1-x)\sin\pi x}dx.$$

Putting $x=\xi-1$ we get the R. H. S. of (5).

In a similar manner, the remaining recurrence relations can be established.

3. APPLICATIONS

(i) If we take $m_1=B=r;\ n_1=0;\ A=s$ and replace z by $-\dfrac{1}{y(1-q)^{s+1-r}}$ in (5) and use the relation:

$$G^{r,0}_{s,r}\left[\frac{-1}{z};q\,\middle|\,\begin{matrix}(b_j)\\(a_j)\end{matrix}\right]=\Pi\begin{bmatrix}b_1 & & b_s\\ q,q & ,\ldots, & q\ ;\\ q^{a_1} & ,\ldots, & q^{a_r}\ ;\end{bmatrix}{}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right]\quad\ldots(11)$$

where $z=\mathcal{G}\,(1-q)^{s+1-r}$, *we then obtain*

$$ {}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right]-{}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,xq\right]$$

$$=\frac{x\,(1-q^{a_1})\ldots(1-q^{a_r})}{(1-q^{b_1})\ldots(1-q^{b_s})}\,{}_r\Phi_s\left[\begin{matrix}a_1+1,\ldots,a_r+1;\\ b_1+1,\ldots,b_s+1;\end{matrix}\,q,x\right]\quad(12)$$

where $x=y\,(1-q)^{s+1-r}$

which for $r=1$ and $s=0$ give the known result, Slater [6].

(ii) On the other hand if we take $m_1=B=r$; $n_1=0$; $A=s$ and use the relation

$$G^{r,0}_{s,r}\left[z;q\,\middle|\,\begin{matrix}(b_j)\\(a_j)\end{matrix}\right]=E_q\,(r;a_t:s;b_p;z)\quad(13)$$

in (5), (6) and (10) then we get the known recurrence relations for the E_q—function due to Agarwal, N. [1].

(iii) Next if we take $m_1=B=2$ and $n_1=A=0$ in (5) and (10) then the recurrence relations for the E_q — function given earlier by Agarwal, R. P. [2] are obtained.

(iv) Finally, it is quite interesting to note that (5), (6) and (7) are the q — analogues of the known recurrence relations, Mac Robert [3; 345; (i), (ii), (iii)] for ordinary hypergeometric function ${}_rF_s(\cdot)$.

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