ACTA MATHEMATICA VIETNAMICA

Vol. 15 No 2 (1990)

METHOD OF « CLEFT-OVERSTEP » BY PERPENDICULAR DIRECTION FOR SOLVING THE UNCONSTRAINED NONLINEAR OPTIMIZATION PROBLEM

NGUYEN VAN MANH and BUI MINH TRI

INTRODUCTION

We shall be concerned with the following unconstrained nonlinear optimization problems :

$$\text{minimize } J(u), \text{ subject to } u \in E^n,$$

where $J(u)$ is a nonlinear function and E^n is the n-dimensional Enclidean space.

In [1], an algorithm of « cleft-overstep » called the algorithm of quasi-bisector is proposed for solving this problem.

Based on the method of « cleft-overstep » we present in this paper a new method called the algorithm of « cleft-overstep » by perpendicular direction. Some examples are given in § 5.

I. DESCRIPTION OF THE ALGORITHM

We shall denote by $\|.\|$ the Enclidean norm and by $\langle ... \rangle$ the inner product in E^n. Suppose that J(u) is continuously differentiable and that the gradient mapping $J'(u)$ satisfies the Lipschitz condition with a constant. L. Let $s^k \in E^n$ denota the vector showing the changed direction of the function $J(u)$ on step $k+1$ ($k=0, 1, 2,...$). Taking $u^\circ \in E^n$ as the starting point, we construct a sequence of points $\{u^k\}$ by the method of « cleft-overstep » as follows :

$$u^k = u^{k-1} + \alpha_k S^{k-1}, K = 1. 2, 3... \tag{1}$$

$$S^o = - J'(u^o), \; S^k = -J'(u^k) + \beta_k S^{k-1}, \; k = 1, 2,... \tag{2}$$

$$\beta_k = \begin{cases} 0, \text{ if } k \in I_1 \\ \dfrac{\langle y, \mathcal{J}'(u^k)\rangle}{\langle y, S^{k-1}\rangle}, \text{ if } k \in I^2, \; y = S^{k-1} + \mathcal{J}'(u^k) \end{cases} \tag{3}$$

Where I_1 is the set of indices $k = 0, n, 2n, \ldots$ and of satisfying those endices k

$$\text{either } \frac{\| S^{k-1} \|}{\| \mathcal{J}'(u^k) \|} \leq \varepsilon_1 \tag{4}$$

$$\text{or } \; |\cos \varphi_{k-1}| = \frac{\langle \mathcal{J}'(u^k), S^{k-1}\rangle}{\| \mathcal{J}'(u^k) \| \cdot \| S^{k-1} \|} > \varepsilon_2, \; 1 > \varepsilon_1 > \varepsilon_2 > 0;$$

$I_2 = \{0, 1, 2, \ldots\} \setminus I_1$;

α_k, $(k = 1, 2, \ldots)$ is the step length, satisfying the following « cleft-overstep » condition :

$$J(u^{k-1} + \alpha_k S^{k-1}) \leqslant \mathcal{J}(u^{k-1} + \varepsilon S^{k-1}), \; 0 < \varepsilon < 1$$

$$\left.\frac{\partial \mathcal{J}(u^{k-1} + \alpha S^{k-1})}{\partial \alpha}\right|_{\alpha = \alpha_k} > 0. \tag{5}$$

2. THE BASIC LEMMAS

LEMMA 1. *Suppose that $J'(u^k) \not\equiv 0$ for all k. If S^k is a moving direction constructed using (1) - (5), then there exist $A_1 > 0$ and $A_2 > A_1$ such that :*

$$A_1 \| \mathcal{J}'(u^k) \|^2 \leq \| S^k \| \leqslant A_2 \| \mathcal{J}'(u^k) \|^2. \tag{6}$$

LEMMA 2. *Under the conditions as of Lemma 1, S^k is always a descreasing direction of $J(u)$, and the following inequality is satisfied for some $A_3 > 0$*

$$-\langle \mathcal{J}'(u^k), S^k \rangle \geqslant A_3 \| \mathcal{J}'(u^k) \|^2 \geqslant 0. \tag{7}$$

LEMMA 3. *If $J(u)$ is continuously differentiable and bounded from below and if the set $M(u^0) = \{u \mid J(u) \leqslant J(u^0)\}$ is bounded then the « cleft-overstep » condition (5) is always satisfied in $M(u^0)$.*

LEMMA 4. *If $J(u)$ is continuously differentiable and bounded from below then there exists $\alpha > 0$ such that :*

$$\frac{\partial \mathcal{J}(u^k + \alpha S^{k-1})}{\partial \alpha} \geqslant 0,$$

$$\mathcal{J}(u^{k-1} + \alpha S^{k-1}) \leqslant \mathcal{J}(u^{k-1} + \varepsilon S^{k-1}), \tag{8}$$

where ε is the positive number chosen by (5).

3. CONVERGENCE AND THE RATE OF CONVERGENCE

THEOREM 1. *Suppose that $J(u)$ is continuously differentiable and is bounded from below, and that $J(u)$ satisfies the Lipschitz condition with a constant L, then for every starting point $u^0 \in E_n'$, the sequence $\{u^k\}$, constructed by the above rule (1) — (5) has cluster points which are critical points of $J(u)$.*

Furthermore, if $J(u)$ is a convex function and if the set $M(u^0) = \{u|J(u) \leqslant J(u^0)\}$ is bounded, then $\{u^k\}$ is a minimizing sequence for $J(u)$. In the case when the minimum point is unique the sequence $\{u^k\}$ converges to this point.

THEOREM 2. *Suppose that $J(u)$ is a convex function, continuously differentiable in E^n and that $J'(u)$ satisfies the Lipschitz condition with a constant L. If $\{u^k\}$ is the sequence constructed by the rule (1) — (5), then*

$$0 < \mathcal{J}(u^k) - \mathcal{J}^* \leqslant \frac{D^2}{A_4} \cdot \frac{1}{k}, \quad A_4 = \frac{a\, \varepsilon\, A_3}{a+1}, \quad a > 0 \tag{9}$$

where $D = \sup_{u,\, v \in M(u^0)} \| u - v \|$ is the diameter of the set $M(u^0)$, and $J^* = \inf J(u) = J(u^*)$.

Moreove, if $J(u)$ is a strongly convex function in E^n then

$$0 < \mathcal{J}(u^k) - \mathcal{J}^* \leqslant (\mathcal{J}(u^0) - \mathcal{J}^*)\, q^k \tag{10}$$

$$\| u^k - u^* \|^2 \leqslant \frac{2}{\mathcal{X}} (\mathcal{J}(u^0) - \mathcal{J}^*)\, q^k, \tag{11}$$

$$q = 1 - A_4 \mu; \quad 0 < q < 1; \quad \mathcal{X}, \mu = \text{constant} > 0$$

4.1 *Proof of Lemma 1.*

By (1) — (5), if $k \in I_1$ or $S^{k-1} \equiv 0$ then $S^k \equiv - J'(u^k)$ and, hence,

$$\| S^k \|^2 = \| J'(u^k) \|^2, \ A_1 = A_2 = 1.$$

The inequality; (6) becomes equality.

If $k \in I_2$ and $S^{k-1} \not\equiv 0$ then we have:

$$\beta_k = \frac{\langle y, \mathcal{J}'(u^k)\rangle}{\langle y, S^{k-1}\rangle} = \frac{\langle \mathcal{J}'(u^k) + S^{k-1}, \mathcal{J}(u^k)\rangle}{\langle \mathcal{J}'(u^k) + S^{k-1}, S^{k-1}\rangle} =$$

$$= \frac{\| \mathcal{J}'(u^k) \|^2 + \langle \mathcal{J}'(u^k), S^{k-1}\rangle}{\| S^{k-1} \|^2 + \langle \mathcal{J}'(u^k), S^{k-1}\rangle} =$$

$$= \frac{\| \mathcal{J}'(u^k) \| / \| S^{k-1} \| + \langle \mathcal{J}'(u^k), S^{k-1}\rangle / (\| \mathcal{J}'(u^k) \| \cdot \| S^{k-1} \|)}{\| S^{k-1} \| / \| \mathcal{J}'(u^k) \| + \langle \mathcal{J}'(u^k), S^{k-1}\rangle / (\| \mathcal{J}'(u^k) \| \cdot \| S^{k-1} \|)}$$

$$= \frac{1/\gamma + \cos\varphi_{k-1}}{\gamma + \cos\varphi_{k-1}}, \ \textit{where} \ \gamma = \frac{\| S^{k-1} \|}{\| \mathcal{J}'(u^k) \|}$$

At the same time

$$\| S^k \|^2 = \| -\mathcal{J}'(u^k) + \beta_K S^{k-1} \|^2 = \| \mathcal{J}'(u^k) \|^2 + \beta_k^2 \| S^{k-1} \|^2 - 2\beta_k \langle \mathcal{J}'(u^k),$$

$$S^{k-1}\rangle = \| \mathcal{J}'(u^k) \|^2 \left[1 + \beta_k^2 \frac{\| S^{k-1} \|^2}{\| \mathcal{J}'(u^k) \|^2} - 2\beta_k \frac{\langle \mathcal{J}'(u^k), S^{k-1}\rangle}{\| \mathcal{J}'(u^k) \|^2} \right] =$$

$$= \| \mathcal{J}'(u^k) \|^2 \left[1 + \beta_k^2 \gamma^2 - 2\beta_k \gamma \cos \varphi_{k-1} \right].$$

We consider the expression:

$$A = 1 + \beta_k^2 \gamma^2 - 2 \beta_k \gamma \cos \varphi_{k-1},$$

Denoting $x = \beta_k \gamma$ we have

$$0 < x = \frac{1/\gamma + \cos\varphi_{k-1}}{\gamma + \cos\varphi_{k-1}} \cdot \gamma = \frac{1/\gamma + \cos\varphi_{k-1}}{1 + (\cos\varphi_{k-1})/\gamma} < \frac{1}{\varepsilon_1} + \varepsilon_2.$$

(Taking into account the «cleft-overstep» conditio . (5), at the point u^k the direction S^{k-1} is an increasing direction of the objective function $J(u)$. Therefore $\cos\varphi_{k-1} > 0$).

It follows that

$$A = 1 + x^2 - 2x\cos\varphi_{k-1} \leqslant 1 + x^2 \leqslant 1 + (1/\varepsilon_1 + \varepsilon_2)^2 = A_2$$

On the other hand, $1 + x - 2x\cos\varphi_{k-1}$, as a function of x, reaches its minimum value at a point such that

$$A'_x = 0 \Leftrightarrow 2x - 2\cos\varphi_{k-1} = 0 \Rightarrow x^* = \cos\varphi_{k-1}$$

Therefore

$$A > 1 + \cos^2\varphi_{k-1} - 2\cos^2\varphi_{k-1} = 1 - \cos^2\varphi_{k-1} \geqslant 1 - \varepsilon_2^2 = A_1$$

We have thus proved that

$$(1 - \varepsilon_2^2)\, \| \mathcal{J}'(u^K) \|^2 < \| S^K \|^2 < [1 + (1/\varepsilon_1 + \varepsilon_2)^2] \cdot \| \mathcal{J}'(u^k) \|^2,$$

where

$$A_2 = 1 + (1/\varepsilon_1 + \varepsilon_2)^2 > A_1 = 1 - \varepsilon_2^2 > 0.$$

The proof of Lemma 1 is complete:

4.2 Proof of Lemma 2.

For the sake of simplicity we denote $a = J'(u^k)$, $b = S^{k-1}$. If $k \in I_1$ or $S_{k-1} \equiv 0$ then $S^k = - J'(u^k)$. Lemma 2 is trivial

In the other cases when $k \in I_2$ and $S^{k-1} \not\equiv 0$ we have

$$-\langle \mathcal{J}'(u^k), S^k \rangle = \langle -\mathcal{J}'(u^k), -\mathcal{J}'(u^k) + \beta_k S^{k-1} \rangle =$$

$$= \langle -a, -a + \frac{\langle a, a+b \rangle}{\langle b, a+b \rangle} \cdot b \rangle = \| a \|^2 - \frac{\| a \|^2 + \langle a, b \rangle}{\| b \|^2 + \langle a, b \rangle} \cdot \langle a, b \rangle =$$

$$= \frac{\| a \|^2 \| b \|^2 + \langle a, b \rangle \| a \|^2 - \| a \|^2 \langle a, b \rangle - \langle a, b \rangle^2}{\| b \|^2 + \langle a, b \rangle} =$$

$$= \frac{\| a \|^2 \cdot \| b \|^2 - \langle a, b \rangle^2}{\| b \|^2 + \langle a, b \rangle} = \frac{1 - (\langle a, b \rangle / (\| a \| \cdot \| b \|))}{1 + \langle a, b \rangle / \| b \|^2} \cdot \| a \|^2 =$$

$$= \frac{1 - \cos^2\varphi_{k-1}}{1 + \frac{\langle a, b \rangle}{\|a\| \cdot \|b\|} \frac{\| a \|}{\| b \|}} \cdot \| a \|^2 = \frac{1 - \cos^2\varphi_{k-1}}{1 + \frac{\cos^2\varphi_{k-1}}{\gamma}} \| a \|^2 \geqslant$$

$$\geqslant \frac{1 - \varepsilon_2^2}{1 + \varepsilon_2/\varepsilon_1} \| a \|^2;$$

Hence

$$-\langle \mathcal{J}'(u^k), S^k \rangle \geqslant A_3 \| \mathcal{J}'(u^k) \| > 0,$$

where

$$A_3 = \frac{1-\varepsilon_2^2}{1+\varepsilon_1/\varepsilon_2}.$$

This result shows that S^k form a obtuse angle with $\mathcal{J}'(u^k)$ because their scalar product is a negative number. Therefore S^k is a discreasing direction for $J(u)$. This completes the proof of Lemma 2. The proofs of Lemma 3 and Lemma 4 are similar to those used in [1] and will therefore be omitted

4.3 Proof of Theorem 1

First we rewrite (5) as

$$\mathcal{J}(u^{k-1}) - \mathcal{J}(u^{k-1} + \alpha_k S^{k-1}) \geqslant \mathcal{J}(u^{k-1}) - \mathcal{J}(u^{k-1} + \varepsilon S^{k-1}),$$

where, ε is chosen according to the condition (5) so that

$$0 < \varepsilon \leqslant \frac{2A_3}{A_2 L(a+1)},$$

Using the same way as in the proof of Theorem 1 in [1] we have

$$\mathcal{J}(u^k) - \mathcal{J}(u^k + \varepsilon S^k) \geqslant -\varepsilon\langle S^k, \mathcal{J}'(u^k)\rangle \left[1 + \frac{\varepsilon L}{2} \cdot \frac{\|S^k\|^2}{\langle S^k, \mathcal{J}'(u^k)\rangle}\right] >$$

$$\geqslant -\varepsilon\langle S^k, \mathcal{J}'(u^k)\rangle \left[1 - \frac{\varepsilon L A_2}{2A_3}\right].$$

(by Lemma 1 and 2)

$$\geqslant \varepsilon A_3 \|\mathcal{J}'(u^k)\|^2 \left[1 - \frac{\varepsilon L A_2}{2A_3}\right] \qquad \text{(by Lemma 2)}$$

$$\geqslant \varepsilon A_3 \|\mathcal{J}'(u^k)\|^2 \left[1 - \frac{2A_3 L}{2A_2 L(a+1)} \cdot \frac{A_2}{A_3}\right] =$$

$$= \frac{\varepsilon a A_3}{a+1} \|\mathcal{J}'(u^k)\|^2 = A_4 \|\mathcal{J}'(u^k)\|^2 > 0,$$

where $A_4 = \dfrac{\varepsilon a A_3}{a+1} = \text{constant} > 0.$

Consequently, combining it with the condition of « cleft-overstep » (5) we have

$$\mathcal{J}(u^k) - \mathcal{J}(u^k + \alpha_{k+1} S^k) \geqslant \mathcal{J}(u^k) - \mathcal{J}(u^k + \varepsilon S^k) \geqslant$$

$$\geqslant A_4 \|\mathcal{J}'(u^k)\|^2 > 0$$

It follows that the sequence of numbers $\{J(u^k\}_{k\to\infty}$ is monotonically decreas. Since $J(u)$ it bounded from below, it follows from the Bolzano-Weierstrass theorem that the sequence of $\{u^k\}$ is convergent. Moreover

$$\lim_{k\to\infty} (\mathcal{J}(u^k) - \mathcal{J}(u^{k+1})$$

At the same time

$$\lim_{k\to\infty} \| \mathcal{J}'(u^k)\| = 0.$$

It means that the sequence $\{u^k\}$ always converges to a critical point.

By the hypothesis that $J(u)$ is a convex and continuous function, the set $M(u^0) = \{u \mid J(u) \leqslant J(u^0)\}$ is bounded. In a similar way as for the proof of Theorem 2 in [1] we obtain a minimizing sequence for J (u) and if J(u) has a unique minimum point then $\lim_{k\to\infty} u^k = u^*$

4.4 Proof of theorem 2.

Using the results in the proof of Theorem [1] we have

$$\mathcal{J}(u^k) - \mathcal{J}(u^{k+1}) \geqslant A_4 \| \mathcal{J}'(u^k)\|^2$$

where

$$A_4 = \frac{\varepsilon a A_3}{a+1} = \text{constant} > 0$$

We denote $a_k = \mathcal{J}(u^k) - \mathcal{J}^*$. If $J(u)$ is convex then

$$a^k = \mathcal{J}(u^k) - \mathcal{J}^* \leqslant \langle \mathcal{J}'(u^k), u^k - u^*\rangle \leqslant \| \mathcal{J}'(u^k)\| . \| u^k - u^*\| < D\|\mathcal{J}'(u^k)\|$$

where
$$D = \sup_{u,\, v \in M(u^0)} \| u - v \|$$

It follows that

$$a_k = a_{k+1} \geqslant \frac{A_4}{D^2} \cdot a_k^2$$

and by Lemma 2 of §1, Chapter 2 in [2] (page 65), we obtain

$$a_k < \frac{k_0+1}{A_4} . D^2 . \frac{1}{k} = \frac{D^2}{A_4} . \frac{1}{k},$$ for k = 0. If J(u) is a strongly convex function then by Theorem 6 of §1 Chapter 2 in [3] (page 60) we have :

$$a_k - a_{k+1} \geqslant A_4 \|\mathcal{J}'(u^k)\|^2 \geqslant \mu A_4 a^k, \mu > 0.$$

Therefore

$$a_{k+1} \leqslant a_k(1 - A_4\mu) = a_k q,\ q = 1 - A_4\mu.$$

We thus obtain

$$a_{k+1} \leqslant a_0\, q^k,\ \mathcal{J}(u^{k+1}) - \mathcal{J}^* \leqslant (\mathcal{J}(u^0) - \mathcal{J}^*)\, q^{k+1}$$

$$\|u^{k+1} - u^*\|^2 \leqslant \frac{2}{\mathcal{X}}(\mathcal{J}(u^{k+1}) - \mathcal{J}^* \leqslant \frac{2}{\mathcal{X}}\cdot(\mathcal{J}(u^0) - \mathcal{J}^*)\, q^{k+1},\ \text{where } \mathcal{X} > 0.$$

The proof of Theorem 2 is complete.

5. EXPERIMENTAL COMPUTATION

The proposed algorithm was coded in FORTRAN and tested on a Minicomputer CM4 for 5 different examples.

Example 1. The Rozenbrock function [4, 5]

$$\mathcal{J}_1(u) = 100(u_2 - u_1^2)^2 + (1 - u_1)^2$$

The starting point $u^0 = (-1,2;\ 1)$, $\mathcal{J}_1(u^0) = 24{,}2$.
The minimum point $u^* = (1,0;\ 1;\ 1,0)$, $\mathcal{J}_1(u^*) = 0$.
The results of computation are given in Table 1.

Example 2. The Wood function [4, 5]

$$\mathcal{J}_2(u) = 100(u_2 - u_1^2)^2 + (1 - u_1)^2 + 90(u_4 - u_3^2)^2 + (1 - u_3)^2 +$$

$$+ 10{,}1[(1 - u_2)^2 + (1 - u_4)^2] + 19{,}8(1 - u_2)(1 - u_4).$$

The staring point $u^0 = (-3;\ -1;\ -3;\ -1)$, $\mathcal{J}_2(u^0) = 19192$.
The minimum point $u^* = (1;\ 1;\ 1;\ 1)$, $\mathcal{J}_2(u^*) = 0$.
The results of computation are given in Table 2.

Example 3. The Powell function [4, 5]

$$\mathcal{J}_3(u) = (u_1 - 10u_2)^2 + 5(u_3 - u_4)^2 + (u_2 - 2u_3)^4 + 10(u_1 - u_2)^4$$

The starting point $u^0 = (3;\ -1;\ 0;\ 1)$, $\mathcal{J}_3(u^0) = 215$.
The minimum point $u^* = (0;\ 0;\ 0;\ 0)$, $\mathcal{J}_3(u^*) = 0$.
The results of computation are given in Table 3.

Example 4. The Poliak function [4, 5]

$$\mathcal{J}_4(u) = \sum_{i=1}^{10} (e^{-0,2i} + 2e^{-0,4i} - u_1 e^{-0,2iu_2} - u_3 e^{-0,2iu_4})^2$$

The starting point $u^0 = (0,5; 0; 2,5; 3)$; $\mathcal{J}_4(u^0) = 0,544$.

The minimum point $u^* = (1; 1; 2; 2;)$, $\mathcal{J}_4(u^*) = 0$.

The results of computation are given in Table 4

Example 5. The Nurminski function [3]

$$\mathcal{J}_5(u) = \max \left\{ |u_2|, \frac{u_1^2}{(1+u_2^2)^2} \right\}.$$

The starting point $u^0 = (7; 7)$, $\mathcal{J}_5(u^0) = 7$.

The minimum point $u^* = (0; 0)$, $\mathcal{J}_5(u^*) = 0$.

The results of computation are given in table 5.

In Tables 1—5 we denote by N the times of computing the objective function $J(u)$ and by $\Delta J = J(u)_N - J(u^*)$ the corresponding different between $J(u)_N$ and its optimal value $J(u^*)$;

In therse tables we also give the comparision of the efficiency of the algorithm of « cleft-overstep » by perpendicular direction (A. D. O. S. P.) with three known algorithms:

— the steppest descent algorithm (S. C. A.)

— the conjugate gradient algorithm (C. G. A.)

— the transformed polytope algorithm (T. P. A.)

In our algorithm (A. C. O. S. P.) we choose $\varepsilon = 0$; $\varepsilon_1 = 0,05$; $\varepsilon_2 = 0,95$; $a = 1$; $\gamma_1 = 10^{-2}$; $\gamma = 0,618$.

The process of finding the « cleft-overstep » length is as follows: we denote

$$g_k(\alpha) = \mathcal{J}(u^{k-1} + \alpha S^{k-1}).$$

Step 0. Pose $\alpha_a = 0$, $\alpha_b = \alpha_{k-1}$, $k = 1,2, \ldots$ $\alpha_0 = 0,5$.

Step 1. Calculate the value $g_k(\alpha_a)$ and $g_k(\alpha_b)$.

a) If $g_k(\alpha_b) < g_k(\alpha_a)$ then put $\alpha_a := \alpha_b$, $\alpha_b = 2\alpha_b$ and go back to step 1.

b) If $g_k(\alpha_b) \leqslant g_k(\alpha_a)$ the go to step 2.

Step 2.

a) If $g_k(\alpha_b) \leqslant g_k(\varepsilon)$ then put $\alpha_k := \alpha_b$ and stop the process of finding the « cleft-overstep » length.

b) If $g_k(\alpha_b) > g_k(\varepsilon)$ then go to step 3.

Step 3. Chek the inequality $|\alpha_b - \alpha_a| \leqslant \gamma_1$

a) If $|\alpha_b - \alpha_a| \leqslant \gamma_1$ then put $\alpha_k := \alpha_a$ and stop the process of finding the « cleft—overstep » length.

b) If $|\alpha_b - \alpha_a| > \gamma_1$ then go to step 4.

Step 4. Calculate $\alpha = \alpha + \gamma(\alpha_b - \alpha_a)$ and $g_k(\alpha)$

a) If $g_k(\alpha_a) < g_k(\alpha) \leqslant g_k(\varepsilon)$ set α_k : α and stop the process of finding the cleft-overstep » length.

b) If $g_k(\alpha) \leqslant g_k(\alpha_a)$ then put $\alpha_a := \alpha$ and go back to step 3.

c) If $g_k(\alpha) > g_k(\varepsilon)$ then put $\alpha_b := \alpha$ and go back to step 3.

Table 1.

A.C.O.S.P.		S. D. A.		C. G. A.		T. P. A.	
N	ΔJ	N	ΔJ	0	ΔJ	N	ΔJ
0	0.242×10^2	0	0.242×10^2	0	0.242×10^2	0	0.242×10^2
75	0.4×10^0	13140	0.4×10^{-2}	546	0.13×10^0	86	0.9×10^0
169	0.13×10^{-2}	26140	0.67×10^{-3}	981	0.18×10^{-9}	169	0.81×10^{-3}
262	0.33×10^{-6}	31782	0.33×10^{-3}			216	0.5×10^{-7}
279	0.54×10^{-7}						

Table 2.

A. C. O. S. P.		S. D. A.		C. G. A.		T. P. A.	
N	ΔJ	N	ΔJ	N	ΔJ	N	ΔJ
0	19192	0	19192	0	19192	0	19192
413	0.12×10^{-2}	842	0.79×10^1	120	35 . 1	252	0.16×10^1
810	0.15×10^{-3}					471	0.26×10^{-1}
1029	0.68×10^{-6}	From here the algorithms are not stabilized and do not converge.				58	0.11×10^{-5}
1462	0.15×10^{-6}						

Table 3.

A. C. O. P.		S. D. A.		C. G. A.		T. P. A	
N	ΔJ	N	ΔJ	N	ΔJ	N	ΔJ
0	0.215×10^2	0	0.215×10^2	0	0.215×10^2	0	0.215×10^2
300	0.56×10^{-3}	6326	0.84×10^{-3}	258	0.60×10^{-1}	83	0.41×10^{-2}
622	0.56×10^{-4}	12326	0.25×10^{-3}	545	0.14×10^{-2}	164	0.21×10^{-5}
955	0.16×10^{-5}	16466	0.15×10^{-3}	1024	0.23×10^{-4}	221	0.14×10^{-8}
1049	0.1×10^{-5}			1200	0.80×10^{-6}		

Table 4.

A.C.O.P		S.D.A.		C.G.A.		T.P.A	
N	ΔJ	N	ΔJ	N	ΔJ	N	ΔJ
0	0.514	0	0.544	0	0.514	0	0.544
1051	0.24×10^{-5}	844	0.18×10^{-3}	607	0.25×10^{-3}	118	0.29×10^{-3}
2062	0.10×10^{-6}	1340	0.77×10^{-5}	1218	0.34×10^{-4}	231	0.20×10^{-3}
2905	0.10×10^{-6}	3160	0.45×10^{-5}	1310	0.29×10^{-4}	318	0.20×10^{-3}
						The paralysis occurs	

Table 5.

A, C. O. P.		S. D. A.		C. G. A.		T. P. A.	
N	ΔJ	N	ΔJ	N	ΔJ	N	ΔJ
0	0.70×10^{1}	0	0.70×10^{1}	0	0.70×10^{1}	0	0.70×10^{1}
82	0.92×10^{-3}	541	0.20×10^{1}	The Algorithm not stabilized		90	0.28×10^{1}
173	0.15×10^{-5}	1061	0.20×10^{1}			198	0.28×10^{1}
210	0.59×10^{-6}	The paralysis occurs ;				The paralysis occurs.	
				225	0.16×10^{-1}		
				308	0.41×10^{-3}		

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Received March 25, 1989

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