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LINEAR PURSUIT GAMES WITH MIXED DYNAMICS

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INTRODUCTION

In this paper we study pursuit games in which the movement of a player is discribed by a linear differential equation and the movement of his opponent is discribed by a linear discrete equation. Our aim is to obtain some sufficient conditions for the game to be completed after a finite interval of time. The paper is divided into three sections. In the first two sections we consider games with geometrical constraints on controls and games with integral-geometrical constraints on controls in the last section.

I. GAMES WITH GEOMETRICAL CONSTRAINTS: CASE I.

Let the motion of the phase-vector $x \in R^n$ of the pursuer be discribed by the following differential equation

$$\dot{x} = Ax + Bu, \ x(0) = x_0 , \tag{1.1}$$

and the motion of the phase-vector $y \in R^m$ of the evader-by the following difference equation

$$y(k+1) = Cy(k) + Dv(k) \ , \ k = 0,1,...$$
$$y(0) = y_0 \ ,$$

where matrices A, B, C, D, have the degrees $n \times n$, $n \times p$, $n \times m$ and $m \times q$, respectively.

Suppose that P and Q are convex compact sets of R^p and R^q respectively. Denote by R the set of real numbers. A control $u(t)$, $t \in R$, of the pursuer is a measurable function which satisfies the constraint

$$u(t) \in P \ , \text{ for every } t \in R.$$

A control $v(k)$, $k = 0,1,...$ of the evader is a sequence of vectors satisfying the constraints

$$v(k) \in Q \ , \ \text{for } k = 0,1,...$$

In what follows such $u(.)$ and $v(.)$ will be called admissible controls.

For $r \leqslant \min(m, n)$ we consider R^r as a subspace of R^m (or R^n). Denote by π the orthogonal projection from R^m (or R^n) onto R^r. Let there be given a terminal subset $M \subset R^r$. Let $\alpha(k)$ be a strictly monotone function which maps the set of natural numbers into the set of nonnegative real numbers and satisfies the condition $\alpha(0) = 0$. For every natural number K we have

$$0 = \alpha(0) < \alpha(1) < \ldots \leqslant \alpha(K) = T.$$

Hence, for each $t \in [0, \alpha(K))]$ there exists a unique value $k, 0 \leqslant k < K$ such that $\alpha(k) \leqslant t < \alpha(k+1)$. Let there be given sets $N(k)$, $k = 0,1,\ldots$, satisfying $N(k) \subset \{0,1,\ldots, k\}$. We shall say that the game (1.1), (1.2) [starting from the position (x_0, y_0) is completed after the time $(\alpha(K), K)$, if for any admissible control $\{v(0),\ldots, v(K)\}$ of the evader there is an admissible control $u(\cdot)$ of the pursuer defined on $[0, \alpha(K))$ such that

$$\pi y(K) - \pi x(\alpha(K)) \in M.$$

Our hypotheses concerning the information of the game are as follows. To construct a control u at a moment $t \in [\alpha(k), \alpha(k+1))$ the pursuer can use the structure of the game, i.e. the systems (1.1), (1.2), and the controls of the evader at each moment $s \in N(k)$. Let us define the following sets

$$\Delta_1(K) = \{0 \leqslant k \leqslant K-1 : N(k) \neq \emptyset\}; \ \Delta_2(K) = \{0,1,\ldots, K-1\} \setminus \Delta_1(K);$$

$$\Delta_3(K) = \bigcup_{k \in \Delta_1(K)} N(k); \ \Delta_4(K) = \{0,1,\ldots, K-1\} \setminus \Delta_3(K);$$

$$H_1(K) = \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} DQ, \tag{1.3}$$

$$G_1(K) = \sum_{k \in \Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)A} BP\, dt. \tag{1.4}$$

If $\Delta_1(k) \neq \emptyset$, then we can order the sets $\Delta_1(K)$ and $\Delta_3(K)$ such that

$$\Delta_1(K) = \{s_1, s_2, \ldots, s_{|\Delta_1(K)|}\}; \ \Delta_3(K) = \{r_1, r_2, \ldots, r_{|\Delta_3(K)|}\};$$

$$s_1 < s_2 < \ldots < s_{|\Delta_1(K)|}; \ r_1 < r_2 < \ldots < r_{|\Delta_3(K)|};$$

where $|\Delta_i(K)|$, stands for the number of elements of $\Delta_i(K)$, $i = 1, 3$.

ASSUMPTION 1.1. K is a natural number such that

a/ $\Delta_1(K) \neq \emptyset$

b/ $M \overset{*}{-} H_1(K) \neq \emptyset$.

The definition of the geometrical difference $\overset{*}{-}$ and the notion of multivalued integral were given in /4/.

ASSUMPTION 1. 2. There exists a matrix of the degree $|\Delta_3(K)| \times |\Delta_1(K)|$

$$\varphi(K) = \begin{vmatrix} \gamma(s_1, r_1)\ \gamma(s_2, r_1) \ldots\ \gamma(s_{|\Delta_1(K)|}, r_1) \\ \gamma(s_1, r_2)\ \gamma(s_2, r_2) \ldots\ \gamma(s_{|\Delta_1(K)|}, r_2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \gamma(s_1, r_{|\Delta_3(K)|})\ \gamma(s_2, r_{|\Delta_3(K)|}) \ldots\ \gamma(s_{|\Delta_1(K)|}, r_{|\Delta_3(K)|}) \end{vmatrix}$$

with the following properties.

2a) $\gamma_K(s_i, r_j) = 0$ if $r_j \notin N(s_i)$

2b) $\sum_{i=1}^{|\Delta_1(K)|} \gamma_K(s_i, r_j) = 1$ for *all* $j = 1, 2, \ldots, |\Delta_3(K)|$

2c) $W(s_i) = \overline{P}(s_i \overset{*}{-} \sum_{r_j \in N(si)} \gamma_K(s_i, r_j)\, \pi C^{K-1-r_j} DQ \neq \varnothing$

for every $i = 1, 2, \ldots, |\Delta_1(K)|$, where

$$\overline{P}(s_i) = \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(T-t)A} BP\, dt\,.$$

ASSUMPTION 1.3. Suppose that

$$\pi C^K y_0 - \pi e^{\alpha(K)A} x_0 \in G_1(K) + \sum_{i=1}^{|\Delta_1(K)|} W(s_i) + (M \overset{*}{-} H_1(K)).$$

THEOREM 1. *If the assumptions 1.1—1.3 are satisfied, then the game (1.1) — (1.2) is completed after the time* $(\alpha(K), K)$.

Proof. It follows from Assumption 1.3 that there exist vectors $\widetilde{m} \in M \overset{*}{-} H_1(K)$; $g \in G_1(K)$, $\widetilde{\omega}(s_i) \in W(s_i)$, $i = 1, 2, \ldots, |\Delta_1(K)|$, such that

$$\pi C^K y_0 - \pi e^{\alpha(K)A} x_0 = \widetilde{m} + g + \sum_{i=1}^{|\Delta_1(K)|} \widetilde{\omega}(s_i) \tag{1.5}$$

This means that there are measurable functions $u_k(.)$, $u_k(t) \in P$ for all $t \in [\alpha(k), \alpha(k+1))$, $k \in \Delta_2(K)$, satisfying

$$g = \sum_{k \in \Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(\alpha(K)-t)A} Bu_k(t)\, dt\,. \tag{1.6}$$

Assume now that $\{v(0), \ldots, v(K)\}$ is an arbitrary admissible control of the evader. It is clear that there exists a vector $m \in M$ such that

$$\widetilde{m} = m - \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} Dv(k). \tag{1.7}$$

The condition $\widetilde{\omega}(s_i) \in W(s_i)$ implies the existence of vectors $\omega(s_i) \in \overline{P}(s_i)$ sati sfying

$$\widetilde{\omega}(s_i) = \omega(s_i) - \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \, \pi C^{K-1-r_j} Dv(r_j). \tag{1.8}$$

Since $\omega(s_i) \in \overline{P}(s_i)$, then according to the definition of the sets $\overline{P}(s_i)$ we can find measurable functions $u_{s_i}(\cdot)$, $u_{s_i}(t) \in P$ for every $t \in [\alpha(s_i), \alpha(s_i+1))$, suchthat

$$\omega(s_i) = \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(T-t)A} B u_{s_i}(t) \, dt. \tag{1.9}$$

Now, at each moment $t \in [0, \alpha(K))$ we define an admissible control of the pursuer by setting

$$u(t) = \begin{cases} u_k(t), \ t \in [\alpha(k), \alpha(k+1)), \ k \in \Delta_2(K) \\ u_{s_i}(t), \ t \in [\alpha(s_i), \alpha(s_i+1)), \ i = 1,2,\ldots, |\Delta_1(K)|. \end{cases}$$

Substituting the formulas (1.6), (1.7), (1.8) into (1.5) we get

$$\pi C^k y_0 - \pi e^{\alpha(K)A} x_0 = m - \sum_{k \in \Delta_{4(K)}} C^{K-1-k} Dv(k) +$$

$$+ \sum_{k \in \Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(\alpha(K)-t)A} Bu_k(t) \, dt +$$

$$+ \sum_{i=1}^{|\Delta_1(K)|} \omega(s_i) - \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \, \pi C^{K-1-r_j} Dv(r_j). \tag{1.10}$$

From (1.9) and (1.10) we obtain

$$\pi C^K y_0 - \pi e^{\alpha(K)A} x_0 = m - \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} Dv(k) +$$

$$+ \sum_{k \in \Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)A} Bu_k(t)\, dt + \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(T-t)A} Bu_{s_i}(t) dt$$

$$- \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \pi C^{K-1-r_j} Dv(r_j),$$

i.e.

$$m = \pi C^K y_0 - \pi e^{\alpha(K)A} x_0 + \sum_{k \in \Delta_4(K)} \pi (C)^{K-1-k} Dv(k) -$$

$$- \sum_{k \in \Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)A} Bu_k(t) dt - \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(T-t)A} Bu_{s_i}(t) dt +$$

$$+ \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \pi C^{K-1-r_j} Dv(r_j) = \pi C^K y_0 - \pi x(\alpha(K)) +$$

$$+ \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} Dv(k) + \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \pi C^{K-1-r_j} Dv(r_j). \tag{1.11}$$

From the properties of the matrix $\varphi(K)$ we have

$$\sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j)\, \pi C^{K-1-r_j} Dv(r_j) = \sum_{j=1}^{|\Delta_3(K)|} \pi C^{K-1-r_j} Dv(r_j). \tag{1.12}$$

Combining (1.11) and (1.12) yields

$$m = -\pi x(\alpha(K)) + \pi C^K y_0 + \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} Dv(k) +$$

$$+ \sum_{j=1}^{|\Delta_3(K)|} \pi C^{K-1-r_j} Dv(r_j) = \pi y(K) - \pi x(\alpha(K)).$$

This means that the game (1.1), (1.2) is completed after the time $(\alpha(K), K)$.

2. GAMES WITH GEOMETRICAL CONSTRAINTS: CASE 2

In this section we suppose that the motion of the vectors x and y is described by the following equations

$$\begin{cases} x(k+1) = Ax(k) + Bu(k),\ k = 0, 1, 2, \ldots \\ x(0) = x_o,\ x \in R^n \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{y} = Cy + Dv \\ y(o) = y_o,\ y \in R^m, \end{cases} \tag{2.2}$$

where the control u satifies the constraint $u \in P$ and the control v is a measurable function which satisfies the constraint $v \in Q$.
Such controls u and v are said to be admissible. As before, P and Q are compact convex subsets of R^p and R^q respectively. Other notations remain the same as in Section 1.

Let K be a given natural number, $\alpha(k)$ be a given strictly monotone function which satisfies the condition: $\alpha(0) = 0$, $\alpha(k) > 0$ for every k.

The game (2.1), (2.2) is said to be completed after the time $(K; \alpha(K))$ if for each admissible control $v(.)$ there exists an admissible control $\{u(0), \ldots, u(K)\}$ such that

$$\pi y\ (\alpha(K) - \pi x(K) \in M.$$

To construct the control $u(k)$ at each moment k the pursuer can use the structure of the game and the information about the controls of the evader at any moment $s \in N_1(k)$, where

$$N_1(k) = \bigcup_{i \in N(k)} [\alpha(i), \alpha(i+1)).$$

Let us consider the following sets

$$H_2(K) = \sum_{k \in \Delta_4(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)C} DQ\,dt; \quad G_2(K) = \sum_{k \in \Delta_2(K)} \pi A^{K-1-k} BP.$$

ASSUMPTION 2.1. K is a natural number satisfying

a/ $\Delta_1(K) \neq \phi$

b/ $M \overset{*}{-} H_2(K) \neq \phi$.

ASSUMPTION 2.2. There exists a matrix $\varphi(K) = (\gamma_k(s_i, r_j))$ such that the following conditions hold

2a/ $\gamma_K(s_i, r_j) = 0$, if $r_j \notin N(s_i)$

2b/ $\sum_{i=1}^{|\Delta_1(K)|} \gamma_K(s_i, r_j) = 1$ for all $j = 1, 2, \ldots, |\Delta_3(K)|$.

2c/ $W(s_i) = \pi A^{K-1-s_i} BP \overset{*}{-} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j) \overline{Q}(r_j) \neq \emptyset$,

where $\overline{Q}(r_j) = \int_{\alpha(r_j)}^{\alpha(r_j+1)} \pi e^{(T-t)} C DQ\, dt.$

ASSUMPTION 2.3.

$$\pi e^{\alpha(K)C} y_0 - \pi A^K x_0 \in G_2(K) + (M \overset{*}{-} H_2(K)) + \sum_{i=1}^{|\Delta_1(K)|} W(s_i).$$

THEOREM 2. *Under Assumptions (2.1)—(2.3), the game (2.1), (2.2) is completed after the time $(K, \alpha(K))$.*

Proof. It follows from Assumption 2.3 that there exist vectors

$$\tilde{m} \in M \overset{*}{-} H_2(K);\ g \in G_2(K);\ \tilde{\omega}(s_i) \in W(s_i),\ i = 1, 2, \ldots, |\Delta_1(k)|$$

satisfying

$$\pi e^{\alpha(K)C} y_0 - \pi A^K x_0 = \tilde{m} + g + \sum_{i=1}^{|\Delta_1(k)|} \tilde{\omega}(s_i).$$

Hence, we can find controls $u(k) \in P$, $k \in \Delta_2(K)$, such that

$$g = \sum_{k \in \Delta_2(K)} \pi A^{K-1-k} Bu(k).$$

For an arbitrary admissible control $v(\cdot)$ of the evader, there is a vector $m \in M$ satisfying

$$\tilde{m} = m - \sum_{k \in \Delta_4(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)C} Dv(t) dt.$$

The condition $\tilde{\omega}(s_i) \in W(s_i)$, $i = 1,2,\ldots, |\Delta_1(K)|$ implies the existence of vectors $u(s_i) \in P$ such that

$$\tilde{\omega}(s_i) = \pi A^{K-1-s_i} Bu(s_i) - \sum_{r_j \in N(s_i)} \int_{\alpha(r_j)}^{\alpha(r_j+1)} \gamma_K(s_i, r_j)\, \pi e^{(T-t)C} \cdot Dv(t)dt.$$

Define a strategy of the pursuer by setting

$$u(k) = \begin{cases} u_k, & k \in \Delta_2(k) \\ u(s_i), & i = 1,2,\ldots,|\Delta_1(K)| \end{cases}$$

We have

$$\pi e^{\alpha(K)C} y_o - \pi A^K x_o = m - \sum_{k \in \Delta_4(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)C} Dv(t)dt +$$

$$+ \sum_{k \in \Delta_2(K)} \pi A^{K-1-k} Bu(k) + \sum_{i=1}^{|\Delta_1(K)|} A^{K-1-s_i} Bu(s_i) -$$

$$- \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \int_{\alpha(r_j)}^{\alpha(r_j+1)} \gamma_K(s_i, r_j) \pi e^{(T-t)C} Dv(t)dt =$$

$$= m + \sum_{k=1}^{k} \pi A^{K-1-k} Bu(k) - \sum_{k \in \Delta_4(K)} \int_{\alpha(k+1)}^{\alpha(k+1)} \pi e^{(T-t)C} Dv(t)dt -$$

$$- \sum_{i=1}^{\Delta_1(K)} \sum_{r_j \in N(s_i)} \int_{\alpha(r_j)}^{\alpha(r_j+1)} \gamma_K(s_i, r_j) \pi e^{(T-t)C} Dv(t)dt.$$

From the properties of the matrix $\varphi(K)$ we can deduce

$$\sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \int_{\alpha(r_j)}^{\alpha(r_j+1)} \gamma_K(s_i, r_j)\, \pi e^{(T-t)C} Dv(t)dt =$$

$$= \sum_{j=1}^{|\Delta_3(K)|} \left(\sum_{i=1}^{|\Delta_1(K)|} \gamma_K(s_i, r_j) \right) \int_{\alpha(r_j)}^{\alpha(r_j+1)} \pi e^{(T-t)C} Dv(t)dt =$$

$$= \sum_{j=1}^{|\Delta_3(K)|} \int_{\alpha(rj)}^{\alpha(r_j+1} \pi e^{(T-t)C} Dv(t)dt$$

Consequently,

$$\pi e^{\alpha(KC)} y_0 - \pi A^K x_0 = m + \sum_{k=1}^{K} \pi A^{K-1-k} Bu(k) -$$

$$- \sum_{k \in \Delta_4(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(T-t)C} Dv(t)dt - \sum_{j=1}^{|\Delta_3(K)|} \int_{\alpha(r_j)}^{\alpha(r_j+1)} \pi e^{(T-t)C} Dv(t)dt =$$

$$= m + \sum_{k=1}^{K} \pi A^{K-1-k} Bu(k) - \int_0^{\alpha(K)} \pi e^{(T-t)C} Dv(t)dt.$$

This means that $\pi y(\alpha(K)) - \pi x(K) = m$. The proof is complete.

3. GAMES WITH INTEGRAL CONSTRAINTS

In this section we consider the game (1.1), (1.2) in which the control u satisfies the following integral constraint

$$\int_0^{+\infty} \|u(s)\|^2 \, ds \leqslant \rho^2, \tag{3.1}$$

and the control v satisfies the constraint

$$v \in Q. \tag{3.2}$$

ASSUMPTION 3.1. K is a natural number such that

1a/ $\Delta_1(K) \neq \phi$.

1b/ $M \mathbin{\underline{*}} H_1(K) \neq \phi$.

where $H_1(K) = \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} DQ.$

ASSUMPTION 3.2. There exist a matrix $\varphi(K)=(\gamma_k(s_i, r_j))$ and a map $F(s_i, r_j, t)$ satisfying

2a/ $\quad \gamma_K(s_i, r_j) = 0$, if $r_j \notin N(s_i)$,

2b/ $\quad \sum_{i=1}^{|\Delta_1(k)|} \gamma_k(s_i, r_j) = 1$ for all $j = 1, 2, \ldots, |\Delta_3(K)|$

2c/ $\quad \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-T)A} BF(s_i, r_j, t)\, v(r_j)\, dt = \gamma_k(s_i, r_j)\pi C^{k-1-r_j}.Dv(r_j),$

for every $i = 1, 2, \ldots, |\Delta_1(k)|$; $r_j \in N(s_i)$; $v(r_j) \in Q$.

ASSUMPTION 3.3. There is an admissible control $u^*(t)$, $t \in [\alpha(k), \alpha(k+1))$, $k \in \Delta_2(K)$, such that

$$\sum_{k\in\Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \| u^*(t) \|^2 dt \leqslant \rho^2.$$

Define

$$\tilde{\rho}^2 = \rho^2 - \sum_{k\in\Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \| u^*(t) \|^2 dt,$$

$$\chi^2(K) = \sup_{\substack{v(r_j)\in Q \\ r_j\in\Delta_3(K)}} \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \Big\| \sum_{r_j\in N(s_i)} F(s_i, r_j, t)\, v(r_j) \Big\|^2 dt.$$

ASSUMPTION 3.4.

$$\chi(K) \leqslant \tilde{\rho}.$$

ASSUMPTION 3.5.

$$\pi C^K y_0 - \pi e^{\alpha(K)A} x_0 \in G(K) + (M \underline{*} H_1(K)) + \sum_{k\in\Delta_2(K)} \int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(\alpha(K)-t)A} Bu^*(t)dt.$$

where

$$G(K) = \left\{ \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} B\omega_{s_i}(t)dt : \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \| \omega_{s_i}(t) \|^2 dt \leqslant (\tilde{\rho} - \chi(K)) \right\}.$$

THEOREM 3. *If Assumptions 3.1 — 3.5 are satisfied, then the game (1.1), (1.2) with the constraints (3.1), (3.2) is completed after the time $(K, \alpha(K))$.*

Proof. According to Assumption 3.5 there exist vectors $g \in G(K)$, $\tilde{m} \in M \underline{*} H_1(K)$, such that

$$\pi C^K y_0 - \pi r^{\alpha(K)A} x_0 = g + \tilde{m} + \sum_{k \in \Delta_2(K)} \int_{\alpha(K)}^{\alpha(k+1} \pi e^{(\alpha(K)-t)A} B u^*(t) dt.$$

Hence, there are measurable funciions $\omega_{s_i}(t)$ defined on

$[\alpha(s_i), \alpha(s_{i+1}))$, $i = 1, \dots, |\Delta_1(K)|$, satisfying

$$\sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \| \omega_{s_i}(t) \|^2 dt \leqslant (\tilde{\rho} - \chi(K))^2,$$

$$g = \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} B \omega_{s_i}(t) dt. \tag{3.4}$$

Now suppose that $\{ v(0), v(1), \dots, v(K) \}$ is an arbitrary admissible control of the evader. Then, from the condition $\tilde{m} \in M \underline{*} H_1(K)$ it follows that there exists $m \in M$ with the property

$$\tilde{m} = m - \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} D v(k) \tag{3.5}$$

By substituting (3.4) and (3.5) into (3.3), we have

$$\pi C^K y_0 - \pi e^{\alpha(K)A} x_0 = \sum_{i=1}^{|\Delta_1(K)|} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} B \omega_{s_i}(t) dt + m -$$

$$- \sum_{k \in \Delta_4(K)} \pi C^{K-1-k} D v(k) + \sum_{k \in \Delta_2(K)} \int_{\alpha(K)}^{\alpha(k+1} \pi e^{(\alpha(K)-t)A} B u^*(t) dt. \tag{3.6}$$

Define a control $u(\cdot)$ of the pursuer as follows

$$u(t) = \begin{cases} u^*(t), \; t \in [\alpha(k), \alpha(k+1)), \; k \in \Delta_2(K) \\ \sum_{r_j \in N(s_i)} F(s_i, r_j, t) v(r_j) + \omega_{s_i}(t), \quad t \in [\alpha(s_i), \alpha(s_i+1)], \\ \qquad i = 1, \dots, |\Delta_1(K)| \end{cases}$$

By virtue of the Minkowski inequality we have

$$\sqrt{\sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)}\| \omega_{s_i}(t)+\sum_{r_j\in N(s_i)} F(s_i, r_j, t)\, v(r_j)\|^2 dt} \leqslant$$

$$\leqslant \sqrt{\sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)}\| \omega_{s_i}(t)\|^2 dt} + \sqrt{\sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)}\sum_{r_j\in N(s_i)}\|F(s_i, r_j, t)\, v(r_j)\|^2 dt}$$

$$\leqslant \tilde{\rho} - \chi(K) + \chi(K).$$

Therefore,

$$\int_0^{\alpha(K)} \| u(t)\|^2 dt \leqslant \sum_{k\in\Delta_2(K)}\int_{\alpha(k)}^{\alpha(k+1)} \| u^*(t)\|^2 dt +$$

$$+\sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)} \| \sum_{r_j\in N(s_i)} F(s_i, r_j, t)\, v(r_j) + \omega_{s_j}(t)\|^2 dt \leqslant$$

$$\leqslant \rho^2 - \tilde{\rho}^2 + \tilde{\rho}^2 = \rho^2.$$

From this it follows that $u(t)$ is an admissible control. Using Cauchi's formula and (3.6) we obtain

$$\pi y(K) - \pi x(\alpha(K)) = \pi C^K y_0 - \pi e^{\alpha(K)A} x_0 + \sum_{k=0}^{K} \pi C^{K-1-k} Dv(k) -$$

$$-\int_0^{\alpha(K)} \pi e^{(\alpha(K)-t)A} Bu(t)dt = \pi C^K y_0 - \pi e^{\alpha(K)A} x_0 + \sum_{k\in\Delta_4(K)} \pi C^{K-1-k} Dv(k) +$$

$$+\sum_{k\in\Delta_3(K)} \pi C^{K-1-k} Dv(k) - \sum_{k\in\Delta_2(K)}\int_{\alpha(k)}^{\alpha(k+1)} \pi e^{(\alpha(K)-t)A} Bu^*(t)dt -$$

$$-\sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} Bw_{s_i}(t)dt - \sum_{i=1}^{|\Delta_1(K)|}\int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} B\cdot$$

$$\cdot \sum_{r_j\in N(s_i)} F(s_i, r_j, t) v(r_j) dt = m + \sum_{k\in\Delta_3(K)} \pi C^{K-1-k} Dv(k) -$$

$$-\sum_{i=1}^{|\Delta_1(K)|}\sum_{r_j\in N(s_i)}\int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} BF(s_i, r_j, t) v(r_j) dt. \tag{3.7}$$

It follows from Assumption 3.2 that

$$\sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} BF(s_j, r_j, t)v(r_i)dt =$$

$$= \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \gamma_K(s_i, r_j)\, \pi C^{K-1-r_j} Dv(r_j) = \sum_{j=1}^{|\Delta_3(K)|} \pi C^{K-1-r_j} Dv(r_j). \qquad (3.8)$$

Combining (3.7) and (3.8) yields

$$\pi y(K) - \pi x(\alpha(K)) = m + \sum_{k \in \Delta_3(K)} \pi C^{K-1-k} Dv(k) -$$

$$- \sum_{i=1}^{|\Delta_1(K)|} \sum_{r_j \in N(s_i)} \int_{\alpha(s_i)}^{\alpha(s_i+1)} \pi e^{(\alpha(K)-t)A} BF(s_i, r_j, t)v(r_j)dt =$$

$$= m + \sum_{k \in \Delta_3(K)} \pi C^{K-1-k} Dv(k) - \sum_{j=1}^{|\Delta_3(K)|} \pi C^{K-1-r_j} Dv(r_j) = m.$$

The proof is complete.

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