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INTEGRAL REPRESENTATIONS OF SOME (p, q)-WAVE FUNCTIONS AND THEIR APPLICATION

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Some classes of (p, q)-wave functions have been studied in [1], [2]. In this paper we establish the integral representations for these functions with the characteristics $p = e^{\lambda x} y^{\pm k}$, $q = 0$ and present an application to boundary value problems.

1. INTEGRAL REPRESENTATIONS.

Let G be a simply connected region in the plane of complex variable $z = x + iy$, $f(z)$ a wave function in G, $J_\nu(z)$ the Bessel function of the first kind and ν-th order, k a positive constant, λ a non-zero real constant, C_1 and C_2 real constants. We now prove the following theorem.

THEOREM 1. *The function $F(z) = U(x, y) + iV(x, y)$ defined by the following formula*

$$e^{\frac{\lambda}{2}x} y^k [U(x, y) - C_1] - ie^{-\frac{\lambda}{2}x}[V(x, y) - C_2] =$$

$$= i\int_{-y}^{y} (y^2 - \gamma^2)^{\frac{k}{4}} J_{\frac{k}{2}}\left(\frac{\lambda}{2}\sqrt{y^2 - \gamma^2}\right)\overline{f(x, \gamma)}\, d\gamma +$$

$$+\int_{-y}^{y} (y + \gamma)(y^2 - \gamma^2)^{\frac{k}{4} - \frac{1}{2}} J_{\frac{k}{2} - 1}\left(\frac{\lambda}{2}\sqrt{y^2 - \gamma^2}\right) \overline{f(x, \gamma)} d\gamma, \qquad (1)$$

is an $e^{\lambda x}y^k$—wave function in G if G contains an entire line segment joining two arbitrary points in it with the same abscissa and if one of the following conditions is satisfied: a) G lies in the upper half-plane and the boundary of G contains a segment L of the real axis such that $f(x, y)/_L = 0$; b) The region G is symmetric with respect to the real axis and $f(x, y)|_{\overline{G} \cap (y=0)} = 0$.

Proof. First, it is easy to verify that $F(z) = U(x, y) + iV(x, y)$ is an $e^{\lambda x}y^k$—wave function in G if and only if the function $E(x, y)$ defined by

$$E(x, y) = e^{\frac{\lambda}{2}x} y^{\frac{k}{2}} [U(x, y) - C_1] - ie^{-\frac{\lambda}{x}x} y^{-\frac{k}{2}} [V(x, y) - C_2] \tag{2}$$

satisfies the equation

$$\frac{\partial E}{\partial x} + i\frac{\partial \overline{E}}{\partial y} = \frac{\lambda}{2}\overline{E} + \frac{k}{2}y^{-1}\, iE,\ (x, y) \in G. \tag{3}$$

Hence, to prove the theorem it is enough to find two real functions $A(y,\gamma)$ and $B(y,\gamma)$ such that the function

$$E(x,y) = \int_{-y}^{y} [A(y,\gamma) + iB(y,\gamma)]\, \overline{f(x,\gamma)}\, d\gamma \tag{4}$$

where $f(x, -y) = \overline{f(x,y)}$, satisfies (3).

It is evident from (4) that the function $E(x,y)$ satisfies (3) if

$$\frac{\partial A}{\partial y} + \frac{\partial A}{\partial \gamma} = -\frac{\lambda}{2} B + \frac{k}{2y} A,$$

$$\frac{\partial B}{\partial y} - \frac{\partial B}{\partial \gamma} = \frac{\lambda}{2} A - \frac{k}{2y} B,$$

and

$$\int_{-y}^{y} \gamma A_1(y, \gamma)\, u(x, \gamma)\, d\gamma = 0,$$

$$\int_{-y}^{y} B(y, \gamma)\, v(x,\gamma)\, d\gamma = 0,$$

$$B(y, y) = 0,\ A(y, -y) = 0,$$

$$A_1(y, \gamma) = (y + \gamma)^{-1} A(y,\gamma).$$

From these equalities, by using the relations [3]

$$J'_l(z) = -J_{l+1}(z) + lz^{-1} J_l(z),$$

$$J'_l(z) = J_{l-1}(z) - lz^{-l} J_l(z),$$

we find that

$$A(y,\gamma)=y^{-\frac{k}{2}}(y+\gamma)(y^2-\gamma^2)^{\frac{k}{4}-\frac{1}{2}}J_{\frac{k}{2}-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right),$$

$$B(y,\gamma)=y^{-\frac{k}{2}}(y^2-\gamma^2)^{\frac{k}{4}}J_{\frac{k}{2}}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right). \tag{5}$$

Finally, combining (5), (4) and (2) we get (1). This completes the proof of Theorem 1.

Let G be defined as in Theorem 1 and suppose that a wave function $g(z)$ satisfies the following conditions 1°/ $g(z)\,|_L=0$; 2°/ $g(z)$ is continued from G to G^*, where G^* is the region symmetric to G with respect to the real axis, and for $z\in G^*$

$$g(z)=\overline{-g(\bar{z})}.$$

Then, we have

THEOREM 2. *The function $F(z)=U(x,y)+iV(x,y)$ given by the formula*

$$e^{\frac{\lambda}{2}x}[U(x,y)-C_1]-ie^{-\frac{\lambda}{2}x}y^k[V(x,y)-C_2]=$$

$$=i\int_{-y}^{y}(y^2-\gamma^2)^{\frac{k}{4}}J_{\frac{k}{2}}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)g(x,\gamma)d\gamma-$$

$$-\int_{-y}^{y}(y+\gamma)(y^2-\gamma^2)^{\frac{k}{4}-\frac{1}{2}}J_{\frac{k}{2}-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)\overline{g(x,\gamma)}d\gamma, \tag{6}$$

is an $e^{\lambda x}y^{-k}$ —wave function in G.

The proof is analogous to that of Theorem 1

2. INVERSION FORMULA

Let G be defined as in the previous section. Suppose that G is an unbounded region and k is an even number.

THÉOREM 3. *The wave function in (1) $f(z)=u(x,y)+iv(x,y)$ can be expressed in terms of the $e^{\lambda x}y^k$ — wave function $F(z)$ as follows*

$$u(x,\ y) = 2^{4l-3}\, y e^{\frac{\lambda}{2}x} \lambda^{2-2l} \times$$

$$\times \frac{\partial^{2l}}{(\partial y^2)^{2l}} \int_0^y (y^2 - \gamma^2)^{\frac{l}{2}-\frac{1}{2}} I_{l-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)\gamma^2 [U(x,\ \gamma) - C_1]\, d\gamma, \qquad (7)$$

$$v\ (x,y) = \lambda^{2-2l}\, 2^{4l-3} \frac{\partial^{2l}}{(\partial y^2)^{2l}} \int_0^y (y^2-\gamma^2)^{\frac{l}{2}-\frac{1}{2}} I_{l-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) \times$$

$$\times \left[e^{-\frac{\lambda}{2}x} (V(x,\ \gamma) - C_2) + M\ (x,\ \gamma)\right] \gamma d\gamma, \qquad (8)$$

where

$$M(x,y) = 2\int_0^y (y^2-\gamma^2)^{\frac{k}{4}} J_{\frac{k}{2}}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u(x,\ \gamma) d\gamma,$$

$$C_1 = U(x,\ 0),\ C_2 = V(x,\ 0),$$

$$\frac{\partial^j V(x,y)}{\partial y^j} = 0\ (y^{2l-j})\ (y \to 0;\ l = \frac{k}{2} = 1, 2, \ldots;\ j = 1, 2, \ldots, l).$$

Proof. It follows from (1) that

$$e^{\frac{\lambda}{2}x} y^{k-1} [U(x,y) - C_1] =$$

$$= 2\int_0^y (y^2-\gamma^2)^{\frac{k}{4}-\frac{1}{2}} J_{\frac{k}{2}-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u\ (x,\ \gamma) d\gamma, \qquad (9)$$

$$e^{-\frac{\lambda}{2}x} [V(x,y) - C_2] =$$

$$= -2\int_0^y (y^2-\gamma^2)^{\frac{k}{4}} J_{\frac{k}{2}}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u\ (x,\ \gamma) d\gamma +$$

$$+ 2\int_0^y (y^2-\gamma^2)^{\frac{k}{4}-\frac{1}{2}} J_{\frac{k}{2}-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) v(x,\ \gamma)\gamma d\gamma, \qquad (10)$$

To obtain the inverse formula for the integral representations (9) and (10) we shall apply the Laplace—Carson transform:

$$\mathscr{C}\,[g(\gamma)] \equiv \tilde{g}(t) = t\int_0^\infty e^{-t\gamma}\, g(\gamma)\, d\gamma,$$

where t is a complex number.

Setting

$$A(\gamma) = e^{\frac{\lambda}{2}x}\, \gamma^{\frac{k-1}{2}}\, [U(x, \sqrt{\gamma}) - C_1],$$

$$a(\gamma) = \gamma^{-\frac{1}{2}}\, u(x, \sqrt{\gamma}). \tag{11}$$

and using the formula [4]

$$\mathscr{C}\,[\gamma^{\frac{\alpha}{2}} J_\alpha (2\sqrt{\mu\gamma})] = \mu^{\frac{\alpha}{2}}\, t^{-\alpha}\, e^{-\frac{\mu}{t}} \quad (Re\alpha > -1)$$

we get from (9)

$$\tilde{A}(t) = 2^{2-k}\, \lambda^{\frac{k}{2}-1}\, t^{1-\frac{k}{2}}\, e^{-\lambda^2 4^{-2} t^{-1}}\, \tilde{a}(t),$$

or

$$\tilde{a}(t) = 4^{k-2}\, \lambda^{2-k}\, t^k \times$$

$$\times\, t^{-1}[2^{2-k}\lambda^{\frac{k}{2}-1}\, t^{1-\frac{k}{2}}\, e^{\frac{\lambda^2}{4^2 t}}\, \tilde{A}(t)].$$

Hence, taking into account the relation

$$\mathscr{C}\,[\gamma^{\frac{\alpha}{2}} I_\alpha (2\sqrt{\mu\gamma})] = \mu^{\frac{\alpha}{2}}\, t^{-\alpha}\, e^{\frac{\mu}{t}} \quad (Re\alpha > -1),$$

we obtain

$$a(\gamma) = 4^{2l-2}\,\lambda^{2-2l}\, \frac{d^{2l}}{d\gamma^{2l}} \int_0^\gamma (\gamma-\beta)^{\frac{l}{2}-\frac{1}{2}}\, I_{l-1}\left(\frac{\lambda}{2}\sqrt{\gamma-\beta}\right) A(\beta)\, d\beta. \tag{12}$$

$$l = \frac{k}{2} = 1, 2, 3, \ldots$$

Finally, combining (12) and (11) we get (7). In the same manner, from (10) we derive (8).

Note that when k is an arbitrary positive constant the theorem is proved similarly.

In the same way as we did for Theorem 3 we can obtain the inverse formula for the integral representation (6).

3. APPLICATION

Using the results obtained in the preceding sections we can find explicitly solutions of some boundary value problems for $e^{\lambda y} y^k$ — wave functions.

Problem 1. Let G be the first orthant: $\{(x, y) : x > 0,\ y > 0\}$. Find an $e^{\lambda x} y^k$ — wave function $F(z) = U(x, y) + iV(x, y)$ in G *such that*

$$U(0, y) = D(y) \text{ for } 0 \leqslant y < \infty, \tag{13}$$

where $D(y) \in C^{2l+1}$ $(y \geqslant 0)$, $l = \frac{k}{2} = 1, 2, 3, \ldots$

We shall find the solution $F(z)$ in the form (9) and (10) such that the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(z)$ are real wave functions in G, i. e.

$$\frac{\partial^2 u(x, y)}{\partial y^2} = \frac{\partial^2 u(x, y)}{\partial x^2}, \tag{14}$$

$$\frac{\partial^2 v(x, y)}{\partial y^2} = \frac{\partial^2 v(x, y)}{\partial x^2}, \tag{15}$$

and

$$u(x, 0) + iv(x, 0) = 0, \tag{16}$$

$$C_1 = U(0), C_2 = V(x, 0)\ (0 \leqslant x < \infty). \tag{17}$$

Taking into account the boundary condition (13) and using the inverse formula (7) we obtain

$$u(0, y) = d(y) =$$

$$= 2^{4l-3} \lambda^{2-2l}\, y \times$$

$$\times \frac{\partial^{2l}}{(\partial y^2)^{2l}} \int_0^y (y^2 - \gamma^2)^{\frac{l}{2} - \frac{1}{2}} I_{l-1}\left(\frac{\lambda}{2}\sqrt{y^2 - \gamma^2}\right) \gamma^{2l} [D(\gamma) - C_l]\, d\gamma, \tag{18}$$

$$\left.\frac{\partial v(x, y)}{\partial x}\right|_{x=0} = d'(y)\ (0 \leqslant y < \infty), \tag{19}$$

where $I_l(z)$ is the modified Bessel function of the first kind and l - th order.

Now, it follows from (16) that

$$\left.\frac{\partial u(x, y)}{\partial y}\right|_{y=0} = 0, \tag{20}$$

$$\left.\frac{\partial v(x, y)}{\partial y}\right|_{y=0} = 0,\ (0 \leqslant x < \infty). \tag{21}$$

We have from (13), (17) and (18)

$$d(y) \in C^2\ (y \geqslant 0),\ d(0) = 0 \tag{22}$$

We can verify directly that under the condition (22) the unique solution of the problem (14), (16), (18) and (20) is given by

$$u(x,y) = \begin{cases} 0 & for\ 0 \leqslant y \leqslant x < \infty, \\ d(y-x) & for\ \infty > y \geqslant x \geqslant 0. \end{cases} \tag{23}$$

Similarly, the problem (15), (16), (19) and (21) has the solution

$$v(x,y) = \begin{cases} 0 & for\quad 0 \leqslant y \leqslant x < \infty, \\ -d(y-x) & for\quad 0 \leqslant x \leqslant y < \infty. \end{cases} \tag{24}$$

Finally, combining (23), (24) and (9), (10) we get the desired solution of Problem 1.

Problem 2. Let G be the half-strip: $\{ (x,y) : 0 < x < h,\ 0 < y < \infty \}$. Find an $e^{\lambda x} y^{k}$ — wave function $F(z) = U(x,y) + iV(x,y)$ in G such that

$$\begin{aligned} &U(0,y) = D(y), \\ &U(h,y) = Q(y),\ for\ 0 \leqslant y < \infty, \end{aligned} \tag{25}$$

where $D(0) = Q(0)$; $D(y), Q(y) \in C^{2l+1}$ $(y \geqslant 0,\ l = \frac{k}{2} = 1, 2, 3, \dots$

we find the solution $F(z)$ in the form (9), (10) and (14) — (17) (for $0 < x < h$ $0 < y < \infty$). In view of (7), (16) and (25) we have

$$u(O,y) \equiv d(y) = 2^{4l-3}\ \lambda^{2-2l}\, y \times$$

$$\times \frac{\partial^{2l}}{(y^2)^2} \int_0^y \gamma^{2l}(y^2-\gamma^2)^{\frac{l}{2}-\frac{1}{2}}\, I_{l-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) [D(\gamma) - C_l]\, d\gamma,$$

$$u(h,y) \equiv q(y) = 2^{4l-3}\ \lambda^{2-2l}\ e^{\frac{\lambda}{2}h}\, y \times$$

$$\times \frac{\partial^{2l}}{(\partial y^2)^{2l}} \int_0^y \gamma^{2l}\, (y^2-y^2)^{\frac{l}{2}-\frac{1}{2}}\, I_{l-1}\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)[Q(\gamma)-C_1]\, d\gamma, \tag{26}$$

$$C_1 = D(O),$$

$$\left.\frac{\partial v(x,y)}{\partial x}\right|_{x=0} = d'(y),$$

$$\left.\frac{\partial v(x,y)}{\partial x}\right|_{x=h} = q'(y),\ \text{for } 0 \leqslant y < \infty. \tag{27}$$

It follows from (16) (for $0 \leqslant x \leqslant h$) that

$$u(x,0)\ 0,\ \left.\frac{\partial u(x,y)}{\partial y}\right|_{y=o} = 0, \tag{28}$$

$$v(x,O) = ,\ \left.\frac{\partial v(x,y)}{\partial y}\right|_{y=o} = 0,\ \text{for } 0 \leqslant x \leqslant h. \tag{29}$$

Using the relations (26) we have from (25)

$$d(y)\ ,\ q(y) \in C^2\ (y \geqslant 0)\ ,$$
$$d(0) = q\ (0) = 0. \tag{30}$$

It is known that under the condition (30) the solution of the problem (14) (26) and (28) is unique and is given in the form [5]

$$u(x,y) = \sum_{n=0}^{\infty} \widehat{d}\ (y - x - 2nh) - \sum_{n=1}^{\infty} \widehat{d}(y + x - 2nh) +$$
$$+ \sum_{n=0}^{\infty} \widehat{q}\ (y + x - (2n + 1)\ h) - \sum_{n=0}^{\infty} \widehat{q}\ (y - x - (2n + 1)\ h),$$
$$0 \leqslant x \leqslant h,\ 0 \leqslant y < \infty, \tag{31}$$

where

$$\widehat{d}\ (y) = \begin{cases} 0 & \text{for } y < 0, \\ d(y) & \text{for } y \geqslant 0, \end{cases} \qquad \widehat{q}(y) = \begin{cases} 0 & \text{for } y < 0, \\ q(y) & \text{for } y \geqslant 0. \end{cases}$$

Similarly, the problem (15), (27) and (29) has the solution

$$v(x, y) = - \sum_{n=0}^{\infty} [\widehat{d}(y - x - 2nh) + \widehat{d}\ (y + x - 2\ (n+1)\ h)] +$$
$$+ \sum_{n=0}^{\infty} [\ \widehat{q}(y + x - (2n+1)\ h) + \widehat{q}\ (y + x - (2n+1)\ h)],$$
$$0 \leqslant x \leqslant h,\ 0 \leqslant y < \infty. \tag{32}$$

Finally, putting (31), (32) into (9), (10) we obtain the explicit solution of Problem 2.

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