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ON THE MIXED BOUNDARY VALUE PROBLEM FOR NON-LINEAR HYPERBOLIC EQUATION IN DOMAINS WITH CORNER POINTS

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§1. INTRODUCTION

The mixed boundary value problem for non-linear hyperbolic equation in domains with « enough smooth » boundary was studied by J.L.Lions [4]. In this paper, we consider the same problem but in domains with corner points.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega = \bigcup_{i=1}^{m} \Gamma_i$, where Γ_i, $i = 1, 2, \ldots, m$, are $(n-1)$- dimensional smooth manifolds, such that each Γ_i intersects Γ_{i-1} and Γ_{i+1} along $(n-2)$-dimensional smooth manifolds γ_{i-1} and γ_{i+1}.

We shall consider only the case $m = 2$ since the results can be easily generalized to the case $m > 2$ due to their local character.

Suppose that at a point $P \in \gamma = \Gamma_1 \cap \Gamma_2$, Γ_1 interesects Γ_2 with the angle $\Upsilon(P)$.

We denote by Q_T the cylinder $Q_T = \Omega \times]0, T[$, $0 < T < +\infty$, and by $S_T = \partial\Omega \times]0, T[$ its lateral surface.

We consider the problem:

$$\mathcal{L}u = u_{tt} - Lu + |u|^{\rho} u = f \tag{1.1}$$

$$u(x, 0) = \varphi(x) \tag{1.2}$$

$$u_t(x, 0) = \Psi(x) \tag{1.3}$$

$$u \mid S_T = 0 \tag{1.4}$$

where : $\rho > 0$,

where $\rho>0$

$$Lu = \sum_{i,\, j=1}^{n} (a_{ij}\, u_{xi})\, x_j + \sum_{i=1}^{n} a_i\, u_{xi} + au,$$

$$a_{ij} = a_{ji},\ a_{ij},\ a_i\ \text{, and } a \in C^{\infty}(\overline{Q}_T)$$

$$\nu\, |\, \xi\, |^2 \leq \sum_{i,\, j=1}^{n} a_{ij}\, \xi_i\, \xi_j \leq \nu^{-1}\, |\, \xi\, |^2, \qquad (1.1)'$$

$$\forall \xi \in \mathbb{R}^n\ ,\ |\, \xi\, | \neq 0.$$

This problem is different from the problem in [4] in that $\partial\pi$ has corner points and we do not require $L = L^*$ (see [4], Section 1. 9).

The main results of this paper are presented in §3 under the assumption that $n = 2$. The smoothness of solution with respect to the time-variable t is given in Theorem 3. 1, and the behaviour of solution in a neighbourhood of the edge is given in Theorem 3. 2.

We shall introduce some spaces of functions :

$W^K(G)$: the space of functions U defined in a domain $G \subset \mathbb{R}^n$ such that

$$\|\, U\, \|^2_{W^K(G)} = \int_G \sum_{s=o}^{K} \left|\, \frac{\partial^s u}{\partial x^s}\, \right|^2 dx < +\infty \ ;$$

$\overset{\circ}{W}{}^K(G)$: the closure of $C_o^{\infty}(G)$ in $W^K(G)$.

$\overset{\circ}{W}{}^K_{\alpha}(\pi)$: the space of functions U such that

$$\|\, U\, \|^2_{\overset{\circ}{W}{}^K_{\alpha}(\pi)} = \sum_{s=o}^{K} \int_{\pi} r^{\alpha - 2K + 2s} \left|\, \frac{\partial^s U}{\partial x^s}\, \right|^2 dx < +\infty,$$

where $r = \mathrm{dist}\,(x,\, \gamma)$.

$W^1_{2,\,o}(Q_T)$: the closure in $W^1(Q_T)$ of the set of the infinitely smooth functions $U(x,\, t)$ such that $U = 0$ in a neighbourhood of S_T .

$\widehat{W}{}^1_{2,\,o}(Q_T) = \left\{ U(x,\, t) \in W^1_{2,\,o}(Q_T),\ U(x,\, T) = 0 \right\}$: a subspace of $W^1_{2,\,o}(Q_T)$.

$W(Q_T) = \{\, U(x,\, t) \in L^{\infty}(o,\, T;\ \overset{\circ}{W}{}^1(\pi) \cap L^p(\pi)),\ p \geqslant 1,\ U_t \in L^{\infty}(0,\, T;\ L^2(\pi))\, \}$,

where

$$\|\, U\, \|_{W(Q_T)} = \|\, U\, \|_{L^{\infty}(0,\, T;\ \overset{\circ}{W}{}^1(\Omega) \cap L^p(\Omega))} + \|\, U_t\, \|_{L^{\infty}(0,\, T:\ L^2(\Omega))}.$$

We easily see that, if $u \in W(Q_T)$ then $u \in W^1_{2,0}(Q_T)$ and

$$\| u \|_{W^1_{2,0}(Q_T)} \leqslant C \| u \|_{W(Q_T)}$$

DEFINITION 1.1 A function $u(x, t) \in W(Q_T)$, $(P = \rho + 2)$, is called a weak solution of Problem (1.1) — (1.4), if $u(x, 0) = \mathcal{S}(x)$ and $u(x, t)$ satisfies the following integral identity on Q_T

$$(\mathcal{L}u - f, \eta) = \iint_{Q_T} \Big(- u_t \eta_t + \sum_{i,j=1}^{n} a_{ij} u_{x_i} \eta_{x_j} - \sum_{i=1}^{n} a_i u_{xi} \eta - au\eta +$$

$$+ |u|^\rho u\eta - f\eta \Big) dxdt - \int_\Omega \psi(x)\eta(x,0)dx = 0, \ \forall \eta \in \widehat{W}^1_{2,0}(Q_T) \cap L^P(Q_T). \qquad (1.5)$$

§2. EXISTENCE AND UNIQUENESS OF SOLUTION

THEOREM 2.1 *If* $f \in L^2(Q_T)$, $\mathcal{S}(x) \in \dot{W}^1(\Omega) \cap L^P(\Omega)$, $(P = \rho + 2)$, $\psi(\Omega) \in L^2(\pi)$, *then there exists a weak solution* $u(x, t) \in W(Q_T)$ *of Problem (1.1) – (1.4).*

Proof. Arguing as is the proof of Theorem 1.1 in [4], we can show the existence of a function $u(x,t) \in W(Q_T)$ satisfying $u(x,0) = \mathcal{S}(x)$. Moreover, we have the integral identity (1.5) for $\eta \in \mathcal{M}_m$, where

$\mathcal{M}_m = \{\eta \mid \eta = \sum_{K=1}^{m} d_K(t)\, w_K,\ d_K \in W^1([0, T]),\ d_K(T) = 0\}$, ($\{w_K\}$ being the base in [4].

But $\overline{\bigcup_{m=1}^{\infty} \mathcal{M}_m} = \widehat{W}^1_{2,0}(Q_T) \cap L^P(Q_T)$, the function $u(x, t)$ satisfies the integral identity (1.5) for all $\eta \in \widehat{W}^1_{2,0}(Q_T) \cap L^P(Q_T)$.

LEMMA 2.1. *Assume that* $u, v \in W(Q_T)$ *and* ρ is a *positive number such that* $\rho \leqslant \frac{2}{n-2}$ if $n > 2$. *Then the operator*

$$G(\eta) = \iint_{Q_\tau} (|u|^\rho u - |v|^\rho v)\, \eta\, dx\, dt$$

satisfies the inequality

$$|G(\eta)| \leqslant C(\| \eta \|^2_{W^1_{2,0}(Q_\tau)} + \| w \|^2_{L^2(Q_\tau)}),$$

$$\forall \eta \in W^1_{2,0}(Q_\tau),\ w = u - v,\ \tau \in]0,T],\ Q_\tau = \Omega \times]0,\tau[,$$

where C does not depend on η.

Proof. We shall prove the lemma for $n \geqslant 3$. The same argument can be used to show the validation of the lemme in the case $n = 2$.

Since $f(x) = \rho x^{\rho+1} - (\rho + 1) a x^{\rho} + a^{\rho+1}$, $(a > 0)$ is a nonnegative function on $[a, +\infty[$, we have

$$|v|(|u|^{\rho} - |v|^{\rho}) \leqslant \rho |u|^{\rho}(|u| - |v|) \leqslant \rho |u|^{\rho} |w| \quad \text{for} \quad |u| \geqslant |v|.$$

From this, it follows that

$$|(|u|^{\rho} u - |v|^{\rho} v)| \leqslant \rho |u|^{\rho} |w| + |v|^{\rho} |w|,$$

or $$|(|u|^{\rho} u - |v|^{\rho} v)| \leqslant \rho |v|^{\rho} |w| + |u|^{\rho} |w|,$$

and therefore

$$|(|u|^{\rho} u - |v|^{\rho} v)| \leqslant (\rho + 1)(|u|^{\rho} + |v|^{\rho})|w| \tag{2.1}$$

From (2.1) we have

$$\int_{\Omega} |(|u|^{\rho} u - |v|^{\rho} v)\eta| dx \leqslant (\rho + 1) \int_{\Omega} (|u|^{\rho} + |v|^{\rho})|w||\eta| dx.$$

Applying Holder's inequality for $\frac{1}{n} + \frac{1}{q} = \frac{1}{2}$ and Sobolev's imbedding theorem $\overset{\circ}{W}{}^{1}(\Omega) \to L^{q}(\Omega)$, we obtain

$$\frac{1}{\rho+1} \int_{\Omega} |(|u|^{\rho} u - |v|^{\rho} v)\eta| dx \leqslant (\|u\|^{\rho}_{\overset{\circ}{W}{}^{1}(\Omega)} + \|v\|^{\rho}_{\overset{\circ}{W}{}^{1}(\Omega)}) \times$$
$$\times (\|\eta\|^{2}_{\overset{\circ}{W}{}^{1}(\Omega)} + \|\omega\|^{2}_{L^{2}(\Omega)} \leqslant (\|u\|^{\rho}_{W(Q_T)} + \|v\|^{\rho}_{W(Q_T)}) \times$$
$$\times (\|\eta\|^{2}_{\overset{\circ}{W}{}^{1}(\Omega)} + \|\omega\|^{2}_{L^{2}(\Omega)})$$

for almost all $t \in [0, T]$, and therefore

$$|G(\eta)| \leqslant C(\|\eta\|^{2}_{W^{1}_{2,0}(Q_{\tau})} + \|\omega\|^{2}_{L^{2}(Q_{\tau})}),$$

where C does **not** depend **on** η.

THEOREM 2.2 *Assume that ρ is as in Lemma 3.1. Then the weak solution of Problem (1.1)—(1.4) is unique.*

Proof. Let $u, v \in W(Q_T)$ be two weak solutions of Problem (1.1) – (1.4). We shall prove that $u = v$ almost every where on Q_T.

Indeed, put $$\eta(x,t) = \begin{cases} 0 & , b \leqslant t \leqslant T, \\ \int_b^t \omega(x,\tau) d\tau, & 0 \leqslant t \leqslant b, \end{cases}$$

where $b \in]0, T[$, $\omega = u - v$.

We easily see that $\eta \in \widehat{W}{}^{1}_{2,0}(Q_T)$.

From the inequality

$$[\iint_{Q_T} (\int_0^T |\omega(x,t)| dt)^{P} dx dt]^{1/P} \leqslant \int_0^T (\iint_{Q_T} |\omega(x,t)|^{P} dx dt)^{1/P} dt, \; P = \rho + 2,$$

it follows that $\eta \in L^P(Q_T)$.

Thus, we have $\eta \in \widehat{W}^1_{2,0}(Q_T) \cap L^P(Q_T)$. (2.2)

Using the same argument of [5], Chapter IV, Theorem 3.1 and taking account of Lemma 2.1, and condition (2.2), we obtain $u = v$ almost everywhere on Q_T, as desired.

§ 3. SMOOTHNESS OF SOLUTION

In this section we consider only the case $n = 2$. Then by Sobolev's imbedding theorem, the space $W(Q_T)$ has the following form:

$$W(Q_T) = \{u \in L^\infty(0, T; \mathring{W}^1(\Omega)),\ u_t \in L^\infty(0, T; L^2(\Omega))\}.$$

LEMMA 3.1. 1) Suppose that

$$i)\ (u)_t^{(i)} \in W(Q_T),\ (\forall i \leqslant l),$$

$$ii)\ \rho > l.$$

Then $(|u|^\rho)_t^{(i)} \in L^P(Q_\tau),\ (\forall i \leqslant l),\ (\forall \tau \in]0, T[),\ (\forall P > 1)$,

2) If, in addition,

$$\| (|u|)_t^{(i)} \|_{W(Q_T)} \leqslant C,\ (\forall i \leqslant l),$$

then $\| (|u|^\rho)_t^{(i)} \|_{L^P_{(Q_\tau)}} \leqslant C_1,\ (\forall i \leqslant l),\ (\forall p > 1),\ (\forall \tau \in]0, T[)$, *where* C_1 *depends on* C, ρ, l *and* Q_T.

Proof. We first observe that condition i) imples that

$$(|u|)_t^{(i)} \in W(Q_T) \text{ and } \| (|u|)_t^{(i)} \|_{W(Q_T)} = \| (u)_t^{(i)} \|_{W(Q_T)},\ (\forall i \leqslant l).$$

Now, we prove the lemma by induction.

For $l = 0$, the first assertion of the lemma is valid by Sobolev's imbedding theorem $W(Q_\tau) \to L^{P\rho}_{(Q_\tau)}$.

Assuming that this assertion holds for $l = l_0 - 1 \geq 0$, let us consider the case $l = l_0$.

We have

$$\| (|u|^\rho)_t^{(l_0)} \|_{L^P(Q_\tau)} = \| [\rho |u|)^{\rho-1} (|u|)_t]_t^{(l_0-1)} \|_{L^P(Q_\tau)} =$$

$$= \| \rho [|u|^{\rho-1} (|u|)_t^{(l_0)} + \sum_{r=1}^{l_0-1} C^r_{l_0-1} (|u|^{\rho-1})_t^{(r)} (|u|)_t^{(l_0-r)}] \|_{L^P(Q_\tau)} \leq$$

$$\leq \rho \{ \| |u|^{\rho-1} (|u|)_t^{(l_0)} \|_{L^P(Q_\tau)} + \sum_{r=1}^{l_0-1} C_{l_0-1}^r \| (|u|^{\rho-1})_t^{(r)} \cdot (|u|)_t^{(l_0-r)} \|_{L^P(Q_\tau)} \} \leq$$

$$\leq \rho \{ \| |u|^{\rho-1} \|_{L^{P'(\rho-1)}(Q_\tau)} \cdot \| (u)_t^{(l_0)} \|_{L^{q'}(Q_\tau)} +$$

$$+ \sum_{r=1}^{l_0-1} C_{l_0-1}^r \| (|u|^{\rho-1})_t^{(r)} \|_{L^{P'}(Q_\tau)} \cdot \| (|u|)_t^{(l_0-r)} \|_{L^{q'}(Q_\tau)}$$

for $\frac{1}{P'} + \frac{1}{q'} = \frac{1}{P}$, $P > 1$, $P'(\rho - 1) > 1$.

Applying Sobolev's imbedding theorem $W(Q_\tau) \subsetneq L^P(Q_\tau)$, $(\forall P \geqslant 1)$ and using the induction hypothesis, we have

$$(|u|^\rho)_t^{(i)} \in L^P(Q_\tau),\ (\forall i \leqslant l).$$

The second assertion of the lemma follows from the first assertion and the observation at the beginning of the proof.

LEMMA 3.2 *1) Suppose that*

i) $(u)_t^{(i)} \in W(Q_\tau)$, $(\forall i \leqslant l)$,

ii) $\rho > 0$ *and* $\rho > l - 1$.

Then $(|u|^\rho u)_t^{(i)} \in L^2(Q_\tau)$, $(\forall i \leqslant l)$, $(\forall \tau \in]0, T])$.

2) Moreover if

$$\| (u)_t^{(i)} \|_{W(Q_\tau)} \leqslant C,\ (\forall i \leqslant l-1),\ (l \geqq 1),$$

then $\| (|u|^\rho u)_t^{(l)} \|_{L^2(Q_\tau)} \leqslant C_1 + C_1 \| (u)_t^{(l)} \|_{W^1_{2,0}(Q_\tau)}$;

$(\forall \tau \in]0, T])$, *where* C_1 *depends on* C, ρ, l *and* Q_T.

Proof. Since $\big| |u|^\rho u \big| = |u|^{\rho+1}$, the first assertion of Lemma 3. 2 is a direct consequence of Lemma 3. 1 and it suffices to prove the secon donce.

Reasonning as in the proof of Lemma 3. 1, we can show that

$$\| (|u|^{\rho+1})_t^{(l)} \|_{L^2(Q_\tau)} \leqslant (\rho + 1) \{ \| |u|^\rho (|u|)_t^{(l)} \|_{L^2(Q_\tau)} +$$

$$+ \sum_{r=1}^{l-1} C_{l-1}^r \| (u|^\rho)_t^{(r)} \|_{L^4(Q_\tau)} \| (|u|)_t^{(l-r)} \|_{L^4(Q_\tau)} \}$$

for l>1

Furthermore, we have

$$\| |u|^\rho (|u|)_t^{(l)} \|^2_{L^2(\Omega)} \leqslant (\| u \|^\rho_{L^{4\rho}(\Omega)} \cdot \| (u)_t^{(l)} \|_{L^4(\Omega)})^2 \leqslant$$

$$\leqslant \|u\|^{2\rho}_{\mathring{W}^1(\Omega)} \cdot \|(u)^{(l)}_t\|_{\mathring{W}^1(\Omega)} \leqslant$$
$$\leqslant \|u\|^{2\rho}_{L^\infty(0,T;\, \mathring{W}^1(\Omega))} \cdot \|(u)^{(l)}_t\|^2_{\mathring{W}^1(\Omega)} \tag{3.1}$$

for almost all $t \in [0, T]$.

To complete the proof it remains to apply Lemma 3. 1 and inequality 3.1.

LEMMA 3.3. If the conditions 1), 2) of Lemma 3.2 are satisfied, then

$$\Big| \iint\limits_{Q_\tau} (|u|^\rho u)^{(l)}_t (u)^{(l+1)}_t \, dxdt \Big| \leqslant C_2 + C_2 \|(u)^{(l)}_t\|_{W^1_{2,0}(Q_\tau)}, \ (\forall \tau \in]0, T[),$$

where C_2 depends on C_1 in Lemma 3. 2.

Proof. We have

$$\Big| \iint\limits_{Q_\tau} (|u|^\rho u)^{(l)}_t (u)^{(l+1)}_t \, dxdt \Big| \leqslant \|(|u|^\rho u)^{(l)}_t\|^2_{L^2(Q_\tau)} + \|(u)^{(l+1)}_t\|^2_{L^2(Q_\tau)} \leqslant$$

$$\leqslant \|(|u|^\rho u)^{(l)}_t\|^2_{L^2(Q_\tau)} + \|(u)^{(l)}_t\|^2_{W^1_{2,0}(Q_\tau)}.$$

Therefore, applying Lemma 3. 2, we obtain the second conclusion of the lemma.

Remark 1. Lemma 3.1, 3.2 and 3.3 remain valid if the conditions on ρ are replaced by the condition $\rho \in \mathbb{N}, \rho \geqq 1$.

THEOREM 3.1 *Suppose that the following conditions are satisfied for* $l \geqq 1$

$$i) \ (f)^{(i)}_t \in L^2(Q_T), \ (\forall i \leqslant l)$$
$$and \ (f)^{(i)}_t (x, 0) = 0, \ (\forall i \leqslant l-1);$$
$$ii) \ u(x, 0) = u_t(x, 0) = 0;$$
$$iii) \ \rho \in \{1, 2, \ldots, l-1\} \cup]\, l-1, +\infty [$$

Then the weak solution $u(x, t)$ *of Problem (1.1)—(1.4) satisfies the inequality*

$$\sum_{i=0}^{l} \|(u)^{(i)}_t\|_{W(Q_T)} \leq C$$

where C *depends on* ρ, l, f *and* Q_T.

Proof. 1. For $l=1$. Due to Theorem 2.2, we see that the weak solution $u(x, t) \in \in W(Q_T)$ of Problem (1.1) — (1.4) may be found by Faedo-Galekin' s method.

We know that the approximate solution

$u_m = \sum\limits_{k=1}^{m} h_{km}(t)\, w_k(x)$, with $\{w_k\}$ being the base in [4], is computed from the system of non-linear differential equations

$$(u''_m(t), w_k) - (Lu_m(t), w_k) + (|u_m(t)|(t)|^\rho u_m(t), w_k) = (f(t), w_k) \tag{3.2}$$

with the initial conditions

$$u_m(0) = u'_m(0) = 0 \tag{3.3}$$

Differentiating both sides of equation (3.2) with respect to t and multiplying both sides of the just obtained equality by $h''_{km}(t)$, then summing up with respect to K, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\| u_{mtt} \|^2_{L^2(\Omega)} + \sum_{i,j=1}^{2} (a_{ij}\, u_{mtx_j}, u_{mtx_i})\right] = \frac{3}{2}\sum_{i,j=1}^{2} (a_{ijt} u_{mtxi}, u_{mtx_j}) +$$
$$+ \sum_{i,j=1}^{2} (a_{ijtt}\, u_{mtxj}, u_{mtxj}) - \frac{d}{dt}\left[\sum_{i,j=1}^{2} (a_{ijt} u_{mxi}, u_{mtxj})\right] +$$
$$+ \left(\sum_{i=1}^{2} a_i\, u_{mtxi} + a u_{mt}, u_{mtt}\right) + \left(\sum_{i=1}^{2} a_{it}\, u_{mxi} + a_t\, u_m, u_{mtt}\right) -$$
$$- \left((| u_m |^{\rho} u_m)_t, u_{mtt}\right) + (f_t, u_{mtt}). \tag{3.4}$$

Integrating both sides of equation (3.4) on $[0, \tau]$, for $\tau \in]0,T[$ and using Lemma 3.3, Remark 1, condition (1.1)', condition (3.3), we can show that

$$\| u_{mtt}(\tau) \|^2_{L^2(\Omega)} + \| u_{mtx}(\tau) \|^2_{L^2(\Omega)} \leq \gamma_1[1 + \int_0^{\tau}(\| u_{mtt} \|^2_{L^2(\Omega)} +$$
$$+ \| u_{mtx} \|^2_{L^2(\Omega)})dt] \text{ for almost all } \tau \in [0, T], \tag{3.5}$$

where γ_1 is a constant depending on ρ, f and Q_T.

Applying Gronwall-Bellman's inequality to (3.5), we obtain

$$\| u_{mt} \|_{W(Q_T)} \leq \gamma_2, (\forall m),$$

where γ_2 is a constant depending on ρ, f and Q_T. It follows that

$$\sum_{i=0}^{1} \| (u_m)_t^{(i)} \|_{W(Q_T)} \leq C, (\forall m),$$

where C depends on ρ, f and Q_T.

Passing to the limit as in the proof Theorem 1.1 in [4], we obtain the conclusion of the theorem for $l = 1$.

2) Assume that the theorem holds for $l = l_0 \geq 1$, we have to prove it for $l = l_0 + 1$. The proof can be made by the same argument as in the case $l = 1$.

Turning to the study of the asymptotic behaviour of the solution, we shall use the function $\gamma(P)$, $P \in \gamma$, introduced in Section 1.

We transform the main part of the operator L at the point $P \in \Upsilon$ into the canonical form. Consequently, $\gamma(P)$ is transformed into another angle denoted by $\omega(P)$. It is always required, that $\omega \neq \pi$.

THEOREM 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Then the weak solution of Problem (1.1) — (1.4) has the following form*

$$u(x, t) = c(t)\, r^{\pi/\omega}\, \Phi(\varphi, t) + u_1(x, t),$$

where $c(t) \in W^{l-1}([0,T])$, $(u_1)_t^{(i)} \in \overset{\circ}{W}{}_o^2(Q(t))$, $(\forall i \leqslant l-1)$,

$$\Phi(\varphi, t) \in C^\infty,\ r = \sqrt{x_1^2 + x_2^2} > 0,\ \varphi = \operatorname{arctg}\frac{x_2}{x_1},$$

$$Q(t) = Q_T \cap \{t = t\},\ t \in]0, T[.$$

Proof. 1) Assume that $l=1$. Theorem 2.1 shows that there exists a weak solution $u(x,t) \in W(Q_T)$ of Problem (1.1) — (1.4) :

By Theorem 3.1, $u_t \in W(Q_T)$. It is not difficult to show that $u(x,t)$ is a weak solution (in the sense of [5]) of the following problem:

$$\sum_{i,j=1}^{2} (aij\, u_{xi})_{xj} + F = 0 \tag{3.6}$$

$$u(x,t)\mid_{\partial Q(t)} = 0 \tag{3.7}$$

in the domain $Q(t)$ for almost all $t \in]0, T[$

where $F - u_{tt} + |u|^{\rho} u - \sum_{i=1}^{2} ai\, u_{xi} - au - f \in L^{2'}(Q(t))$.

By [2, Lemma 1], $u \in \overset{\circ}{W}{}_2^o(Q(t))$ and by [2, Lemma 3], $u \in \overset{\circ}{W}{}_2^2(Q(t))$. (3.8)

We now rewrite equation (3.6) as

$$\sum_{i,j=1}^{2} a_{ij}(0,\ t)\ u_{x_i x_j} = F - \sum_{i,j=1}^{2} [a_{ij}(x,t) - a_{ij}(0,t)]\, u_{x_i x_j} - \\ - \sum_{i,j=1}^{2} \frac{\partial a_{ij}(x,t)}{\partial x_j}\ .\ u_{x_i} = F_1 . \tag{3.9}$$

We can assume that

$$a_{ij} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Consequently, from equation (3.9) we have

$$\Delta u = \overline{F}_1$$

From the inequality $|a_{ij}(x,\ t) - a_{ij}(0,\ t)| \leqslant \text{const}\ |x|$ and (3.8) it follows that $\overline{F}_1 \in L^2(Q(t)) = \overset{\circ}{W}{}_0^0(Q(t))$.

Using [1, Theorem 1.2], we get

$$u(x,t) = C(t) r^{\pi/\omega}\, \Phi(\varphi,t) + u_1(x,t), \tag{3.10}$$

where $\Phi(\varphi,t) = \sin\dfrac{\pi\varphi}{w}\ t,\ u_1 \in \overset{\circ}{W}{}_0^2(Q(t))$,

and

$$\|u_1\|_{\overset{\circ}{W}{}_0^2(Q(t))} \leqslant C[\|\overline{F}_1\|_{\overset{\circ}{W}{}_0^0(Q(t))} + \|u\|_{\overset{\circ}{W}{}_2^2(Q(t))}] \leqslant$$

$$\leqslant C' [\,\|\overline{F}_1\|_{L^2(Q_T)} + \|u\|_{W^1_{2,0}(Q_T)}\,] \leqslant$$
$$\leqslant C'' [\,\|\overline{F}_1\|_{L^2(Q_T)} + \|u\|_{W(Q_T)}\,]. \tag{3.11}$$

It follows from (3.11) and $C(t) = (u - u_1) r^{-\pi/\omega} \left(\sin \frac{\pi\varphi}{w} t \right)^{-1}$ that $C(t) \in L^2([0, T])$.

2) Assume that the conclution of the theorem is valid for $l = l_o \geqslant 1$, we have to prove it for $l = l_o + 1$. For this purpose, we rewrite equation (1.1) as

$$Lu = u_{tt} + |u|^{\rho} u - f = F_2 .$$

By Theorem 3.1, Lemma 3.2, Remark 1 and the induction hypothesis, we have $(F_2)^{(i)}_t \in L^2 Q(t))$, $(\forall i \leqslant l_o)$, for almost all $t \in]\,0,T\,[$.

It follows from [3, Lemma 3.1] for $K = 0$ that the weak solution of Problem (1.1)—(1.4) has the form

$$u(x,t) = C(t)\, r^{\pi/\omega}\, \Phi(\varphi, t) + u_1(x,t)$$

where $C(t) \in W^{l_0}([0, T])$,

$$(u_1)^{(i)}_t \in \overset{\circ}{W}{}^2_0 (Q(t)), \ \ \forall i \leqslant l_o).$$

Remark 2. From the proof of Theorem 3.2 we see that, if $\omega < \pi$ and $l=1$, then by (3.10) and (3.11) $u \in W^2(Q_T)$. Moreover, we can show that $u \in L^{\infty}(0,T; W^2(\Omega)$.

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