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ON CURVES MINIMIZING POLYHEDRAL FUNCTIONALS IN $\mathbb{R}^n$

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INTRODUCTION

The problem of finding minimal surfaces in Riemannian manifolds was studied by A.T. Fomenko [5], H. Federe and W. H. Fleming [4], Dao Trong Thi [1, 2, 3], J. Simon [7] and others.

Using the language of current theory Dao Trong Thi [2] established necessary and sufficient conditions for the global minimality of currents with respect to a functional J given by a Lagrangian.

The aim of this paper is to describe the curve minimizing a functional J, where J is given by a polyhedral norm.

§1. MINIMAL CURRENTS

In this section we collect some facts on current theory that will be needed later (for details see [2, 4]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Denote by $\Lambda_k \mathbb{R}^n$ and $\Lambda^k \mathbb{R}^n$ the vector spaces of k-vectors and k-covectors, respectively. The comass of the k-covector ω is defined by

$$\| \omega \|^* = \sup \{ \omega (\xi) \,;\, \xi \in \Lambda_k \mathbb{R}^n \,,\, |\xi| = 1 \}$$

and the mass of the k-vector ξ is defind by

$$\| \xi \| = \sup \{ \omega (\xi) \,;\, \omega \in \Lambda^k \mathbb{R}^n \text{ and } \| \omega \|^* \leqslant 1 \}$$

Let M be a Riemannian manifold. We denote by $E^k M$ the vector space of all real differential k-forms on M. A current (with the compact support) on M is a real continuous linear functional on $E^k M$. For each k-current S the mass of S is defined by

$$MS = \sup \{S(\varphi);\ \varphi \in E^k M,\ \|\varphi_x\|^* \leqslant 1\ \forall\ x \in M\}$$

We denote by $E_k M$ the space of all k-currents with finite mass (and with compact support) and equip it with the weak topology. The boundary ∂S of a k-current S is $a\ (k-1)$ — current defined by $(\partial S)(\varphi) = S(\mathrm{d}\varphi)$ for every $(k-1)$ — — form φ. A current S is called closed if $\partial S = 0$ and exact if $S = \partial T$ for some current T. For each $S \in E_k M$ the complete variational measure $\|S\|$ is defined by

$$\|S\| f = \sup \{S(\varphi);\ \varphi \in E^k M,\ \|\varphi_x\|^* \leqslant f(x),\ \forall\ x \in M\}$$

for an arbitrary real nonegative continuous functional f on M.

If $S \in E_k M$ then there exists a $\|S\|$-measurable section $\vec{S}$ of the Grassman bundle $\bigwedge_k M$ with $\|\vec{S}_x\| = 1$ almost everywhere in the sense of the measure $\|S\|$ and such that

$$S(\varphi) = \int \varphi(\vec{S}_x)\ \mathrm{d}\ \|S\|\ (x) \tag{1.1}$$

for an arbitrary k-form $\varphi \in E^k M$.

Let J be a functional on $E_k M$. A current $S \in E_k M$ is called absolutely minimal with respect to J if $J(S) \leqslant J(S')$ for any $S' \in E_k M$ such that the current $S - S'$ is closed. A Lagrangian of degree k on M is any mapping $L: \bigwedge_k M \to R$ such that its rectriction on each fibre $\bigwedge_k M_x$ of the Grassman bundle $\bigwedge_k M$ is positively homogeneous. Each Lagrangian L of degree k on M defines a positively homogeneous functional J on $E_k M$ by the formula:

$$J(S) = \int L(\vec{S}_x)\ \mathrm{d}\ \|S\|\ (x). \tag{1.2}$$

Let ω be a differential k-form satisfying the following conditions;

(i) ω is exact,

(ii) $\omega(\xi) \leqslant L(\xi)$ for every $\xi \in \bigwedge_k M$.

Then the set

$$F_x(\omega) = \{\xi \in \bigwedge_k M_x;\ 1(\xi) = \omega(\xi)\}$$

is called the bunch of the minimal directions at x.

THEOREM 1. (see [2]) *A current $S \in E_k M$ is absolutely minimal with respect to J if and only if there exists a differential k-form satisfying the conditions (i) (ii) and such that $\vec{S}_x \in F_x(\omega)$ for almost all $x \in M$ in the sense of the measure $\|S\|$.*

Let R^n be the n-dimensional Eulidean space. Then each piecewise differentiable oriented curve S can be naturally identified with the 1-current $[S]$ which is the integration along S. Moreover for each regular point $x \in S$ we have $\vec{[S]}_x = \vec{S}_x$, where $\vec{S}_x$ is the oriented tangent vector to S at x with $\| \vec{S}_x \| = 1$.

§2. MINIMAL CURVES IN R^n.

In this section we consider a functional J given by an arbitrary polyhedral norm L.

A norm L on R^n is called polyhedral if the set

$$C_L = \{\xi \in R^n ;\ L(\xi) \leqslant 1\} \tag{2. 1}$$

is a convex polyhedron. We put

$$S_L = \{\xi \in R^n ;\ L(\xi) = 1\} \tag{2. 2}$$

Clearly, S_L is boundary of the set C_L. We denote by $H_1, H_2, \ldots, H_m$ the $(n-1)$-dimensional faces of C_L.

DEFINITION 1. A linear form ω is called a supporting form of the norm L at a point $\xi_0 \in S_L$ if

$$\omega(\xi) \leqslant L(\xi) \text{ for any } \xi \in S_L ; \tag{2. 3}$$

$$\omega(\xi_0) = L(\xi_0). \tag{2. 4}$$

Obviously, for each i $(1 \leqslant i \leqslant m)$ there exists a unique 1-form ω_i such that H_i is defined by the equation $\omega_i(\xi) = 1$. It is easy to check that ω_i is a supporting form of L at every point of H_i. Denote by $\bar{\omega}_i$ the differential 1-form defined by $(\bar{\omega}_i)_x = \omega_i$ for every $x \in R^n$. Then we have

$$F_x(\bar{\omega}_j) = \{t\xi ;\ t \geqslant 0,\ \xi \in H_j\},\ J = 1, 2, \ldots, m.$$

Clearly, the set $F_x(\bar{\omega}_i)$ is fixed when x changes. Hence we may write $F_x(\bar{\omega}_i) = F(\bar{\omega}_i)$.

LEMMA 1. *Let ω be a differential 1-form on R^n such that the conditions (i), (ii) are satisfied. Then for each $x \in R^n$ there exists i $(1 \leqslant i \leqslant m)$ such that $F_x(\omega) F(\bar{\omega}_i)$.*

Proof. Let ω be the differential 1-form mentioned in the lemma. Then there is a point $x \in S_L$ such that ω is a supporting form of L at x. It follows that there exists a face P containing x such that ω is supporting form of L at every $t \in P$. Then P is contained in some face H_i. We have $F_x(\omega) = \{t\xi\,;\ t \geqslant 0,\ \xi \in P\}$.

Since $F(\overline{\omega}_i) = \{t\xi\,;\ t \geqslant 0,\ \xi \in H_i\}$ we obtain $F_x(\overline{\omega}) \subset F(\omega_i)$.

The proof of the lemma is complete.

LEMMA 2. *Let J be an integrand given by the polyhedral norm L. Suppose that S is a piecewise differentiable oriented curve in R^n such that $\vec{S}_x \in F(\overline{\omega}_i)$ for almost $x \in S$ in the sence of the measure $\|S\|$ and for fixed i $(1 \leqslant i \leqslant m)$. Then $[S]$ is absolutely minimal 1-current with respect to J.*

Proof. Obviously, $\overline{\omega}_i$ satisfies the conditions (i) and (ii). As is well known $\vec{S}_x = \vec{[S]}_x$. Hence $[S]$ is absolutely minimal by Theorem 1. The lemma is proved.

LEMMA 3. *Let a and b be two points in R^n. Denote by $[a.\ b]$ the oriented straight segment with endpoints a and b. Then $[a,b]$ is absolutely minimal with respect to J. Moreover $J([a,b]) = L(b - a)$.*

Proof. Clearly, there exists i such that $(b - a) \in F(\overline{\omega}_i)$ From Lemma 2 it follows that $[a,b]$ is an absolutely minimal curve in R^n. We have

$$\begin{aligned} J([a,b]) &= \int L(\overrightarrow{[a.\ b]}_t)d\,\|[a,\ b]\|\,(t) = \\ &= \int_{[a,b]} L(\overrightarrow{[a,\ b]}_t)\ d\,\|[a,b]\|\,(t) = \\ &= L(\overrightarrow{[a,b]}_t)\,\|[a,b]\|\,([a,b]) = \\ &= |\,b - a\,|\ L(\overrightarrow{[a,\ b]}_t) = L(b - a) \end{aligned} \tag{2.5}$$

Thus the proof of the lemma is complete.

THEOREM 2. *Let J be an integrand given by a polyhedral norm L. A piecewise differentiable oriented curve S in R^n is absolutely minimal with respect to J if and only if there exists an index i $(1 \leqslant i \leqslant m)$ such that $\vec{S}_x \in F(\overline{\omega}_i)$ for every regular point $x \in S$.*

Proof. By Lemma 2, it is sufficient to prove the necessary condition. Suppose that S is absolutely minimal. If there exist two points $x \in S$, $y \in S$ such that $\vec{S}_x$, $\vec{S}_y$ do not belong to the same set $F(\overline{\omega}_i)$ for each i $(1 \leqslant i \leqslant m)$, then a contradiction will be obtained.

Since S is absolutely minimal there exists a 1-form ω for which the conditions (i), (ii) are satisfied. Moreover $\vec{S}_t \in F_t(\omega)$ for every regular point $t \in S$. In particular, $\vec{S}_x \in F_x(\omega)$ and $\vec{S}_y \in F_y(\omega)$. By Lemma 1, it follows that $F_x(\omega) \subset F(\bar{\omega}_j)$, $F_y(\omega) \subset F(\bar{\omega}_k)$, where according to our assumption $j \neq k$ and $\vec{S}_x \notin F(\bar{\omega}_k)$, $\vec{S}_y \notin F(\bar{\omega}_j)$.

Obviously, the set $F(\bar{\omega}_j)$ is closed. Since $\vec{S}_y \notin F(\bar{\omega}_j)$, there exists a point $z \in S$ such that $(z-y) \notin F(\bar{\omega}_j)$. Denote by $S_{(x,y)}$ the part of S joining x and y. Since S is minimal, $S_{(x,y)}$ is minimal, too. On the other hand $[x, y]$ is also minimal.

According to [2], we have $\overrightarrow{[x,y]}_t \in F_t(\omega)$ for almost all $x \in R^n$ in the sense of the measure $\|[x,y]\|$. Hence for any sphere $O_{(x,\varepsilon)}$ of center x and radius ε there exists a point $t \in [x,y] \cap O_{(x,\varepsilon)}$ such that $\overrightarrow{[x,y]}_t \subset F_t(\omega)$. Whence we may assume that $\overrightarrow{[x,y]}_x \in F_x(\omega)$. Analogously, $\overrightarrow{[y,z]}_y \in F_y(\omega)$, $\overrightarrow{[x,z]}_x \in F_x(\omega)$. Since $F_x(\omega) \subset F(\bar{\omega}_j)$ and $F_y(\omega) \subset F(\bar{\omega}_k)$ we obtain $(y-x) \in F(\bar{\omega}_j)$, $(z-x) \in F(\bar{\omega}_j)$ and $(z-y) \in F(\bar{\omega}_k)$.

From the minimality of S, $[x, y]$, $[x, z]$ and $[y, z]$ we have

$$J(S_{(x,y)}) = J([x,y]) = L(y-x), \tag{2.6}$$

$$J(S_{(y,z)}) = J([y,z]) = L(z-y), \tag{2.7}$$

$$J(S_{(x,z)}) = J([x,z]) = L(z-x). \tag{2.8}$$

Further

$$J(S_{(x,z)}) = J(S_{(x,y)}) + J(S_{(y,z)}) = L(y-x) + L(z-y). \tag{2.9}$$

Let P be a 2-dimensional plane in R^n defined by the origin 0 and the vectors $(y-x)$, $(z-x)$. The set $Q = P \cap C_L$ is a convex polygon. The edge d_i of the polygon Q is contained in the set $P \cap H_i$. Put $V_i = F(\bar{\omega}_i) \cap P$ for every i $(1 \leqslant i \leqslant m)$. Then $(y-x) \in V_j$, $(z-y) \in V_k$, $(z-x) \in V_j$.

First assume that $d_j \nparallel d_k$. Let d'_j, d'_k be the straight lines defined by the conditions:

$$d'_j \ni 0 \quad \text{and} \quad d'_j \parallel d_j ,$$
$$d'_k \ni 0 \quad \text{and} \quad d'_k \parallel d_k .$$

Putting $e_k = d'_j \cap d_k$, $e_j = d'_k \cap d_j$ we have for each vector $\xi \in P$

$$\xi = \xi_j e_j + \xi_k e_k , \text{ where } \xi_j , \xi_k \in R \tag{2.10}$$

and $L(\xi) = \max\{\xi_j , \xi_k\}$ if $\xi \in V_j \cup V_k$. (2.11)

Suppose now that

$$(y - x) = \alpha_j e_j + \alpha_k e_k ,$$

$$(z - y) = \beta_j e_j + \beta_k e_k .$$

Then $(z - x) = (\alpha_j + \beta_j) e_j + (\alpha_k + \beta_k) e_k$, where $\alpha_j \geqslant \alpha_k$,

$(\alpha_j + \beta_j) \geqslant (\alpha_k + \beta_k)$, $\beta_j < \beta_k$.

From (2. 11) we obtain

$$L(y - x) + L(z - y) = \alpha_j + \beta_k , \tag{2.12}$$

$$L(z - x) = \alpha_j + \beta_j . \tag{2.13}$$

Consequently,

$$\alpha_j + \beta_k = \alpha_j + \beta_j$$

Whence $\beta_k = \beta_j$. (2. 14)

Since $\beta_k > \beta_j$, the equality (2. 14) cannot be satisfied. We get a contradiction.

Now suppose that $d_j \parallel d_k$. We denote by e_j any vector on d_j , by e_k a conlinear vector to d_j : Then for each $\xi \in P$ we have

$$\xi = \xi_j e_j + \xi_k e_k$$

and $L(\xi) = | \xi_j |$ if $\xi \in V_j \cup V_k$.

Using an argument analogous to the previous one we shall get a contradiction.

This completes the proof.

Example. Let L be a norm given by the formula

$$L(\xi) = \max \{ |\xi_1|, \dots, |\xi_n| \}, \tag{2.15}$$

where $\xi = (\xi_1, \dots, \xi_n)$.

Then the set $C_L = \{\xi \in R^n ; L(\xi) \leqslant 1\}$ is a convex polyhedron. The notation being as above, we have

$$H_1 = \{\xi \in R^n ; \xi_1 = 1, |\xi_j| \leqslant 1 \text{ for } j \neq 1\},$$

$$\dots\dots\dots$$

$$H_n = \{\xi \in R_n ; \xi_n = 1, |\xi_j| \leqslant 1 \text{ for } j \neq n\},$$

$$H_{n+1} = \{\xi \in R^n ; \xi_1 = -1, |\xi_j| \leqslant 1 \text{ for } j \neq 1\},$$

$$\dots\dots\dots$$

$$H_{2n} = \{\xi \in R^n ; \xi_n = -1, |\xi_j| \leqslant 1 \text{ for } j \neq n\}.$$

and

$$\overline{\omega}_1 = dx_1,$$

$$\overline{\omega}_2 = dx_2,$$

$$\dots\dots$$

$$\overline{\omega}_n = dx_n,$$

$$\overline{\omega}_{n+1} = -dx_1,$$

$$\dots\dots$$

$$\overline{\omega}_{2n} = -dx_n.$$

Hence

$$F_x(\overline{\omega}_i) = \{t\xi, \ t \geqslant 0, \ 1 = \xi_i \geqslant |\xi_j| \ \forall j \neq i\}$$

for $1 \leqslant i \leqslant n$ and

$$F_x(\overline{\omega}_{n+i}) = \{t\xi, t \geqslant 0, 1 = -\xi_i \geqslant |\xi_j| \ \forall j \neq i\}$$

for $1 \leqslant i \leqslant n$.

Using Theorem 2 we can obtain all globally minimal curves for the integrand J given by the norm (2.15) in R^n.

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