ACTA MATHEMATICA VIETNAMICA
Vol 14 No 2 (1989)

SEMIGROUPS IN URBANIK CONVOLUTION ALGEBRAS

NGUYEN VAN THU

1. INTRODUCTION

In recent years, the potential theory of continuous convolution semigroups on locally compact groups has been extensively studied and becomes a field of considerable interest to analysis as well as probabilists. A detailed presentation of the basic topics of the theory is contained in the book [1] by C. Berg and G. Forst.

The main aim of this paper is to present a new aspect of extending the classical potential theory on $R_+ = [0, \infty)$. Namely, for a generalized convolution operation o, cf. Urbanik [4], we introduce an o-semigroup (μ_t), $t \geqslant 0$, of subprobability measures on R_+ which satisfy the natural requirement that $\mu_t \circ \mu_s = \mu_{t+s}$, $t, s \in R_+$. The Levy-Khinczyn formula for such a semigroup plays a key role in studying the related concepts like resolvent, transience, potential kernels and generalized convolution semigroups of contractions. It turns out that many results of the classical theory can be generalized to the Urbanik convolution case.

In the sequel we shall give rise to an enlargement of the fundamental concepts of the classical potential theory on R_+. In a subsequent paper we shall study the algebraic structures of a Urbanik convolution algebra which guarantee extended expositions on the subject.

2. URBANIK CONVOLUTION ALGEBRAS

Let C_b be the class of all bounded continuous real valued functions on R_+ with the uniform convergence.

Further, let M be the set of all Radon nonnegative measures on R_+ equipped with the vague topology. The subsets of M consisting of bounded, probability and subprobability measures will be denoted by N, P, Q, respectively. For a $\in$

enote the unit mass at the point a and T_a the mapping in M defined $\mu\ (a^{-1}E)$ for $a > 0$ and $T_a\ \mu = \delta_o$ for $a = 0$, where $\mu \in M$ and

continuous commutative and associative P-valued binary operation lled a *generalized convolution*, if it is distributive with respect to ations and maps $T_a (a \geqslant 0)$ with δ_0 as the unit element. Moreover, postulates the existence of positive constants C_n and a measure n δ_0 such that

$$T_{c_n}\ \delta_1^{on} \xrightarrow{W} \sigma \tag{2.1}$$

is taken in the sense of operation o and $\xrightarrow{W}$ denotes the weak measure σ is called the characteristic measure of o. The pair the *Urbanik convolution algebra*. For the basic properties of lutions we refer to standard papers by Urbanik [4, 5].

ed convolution o is called regular, if it admits a characteristic hich plays the same role as the Laplace transform for the on case. Let Ω denote the kernel of the characteristic function uestion. Then we have the formula

$$\widehat{\mu}(t) = \int \Omega(tx)\ \mu(dx) \tag{2.2}$$

ere and in the sequel we denote by $\int$ the integral over R_+. he characteristic function of the characteristic measure σ of

$$\exp\ (-t^{H}) \qquad (t \geqslant 0), \tag{2.3}$$

O is the *characteristic exponent* of the algebra in question.

operation o is weakly continuous it can be continuously of all bounded measures. Namely, we put

$$\nu = \iint \delta_x \cdot \delta_y\ \mu(dx)\ \nu(dy) \tag{2.4}$$

ing (2.2) one can define the characteristic function for we get the formula

$$\widehat{\mu \circ \nu} = \widehat{\mu}\ \widehat{\nu} \tag{2.5}$$

f $(\mu_n) \subset Q$ then the pointwise convergence of the sequence th the weak convergence of (μ_n).

s said to be *infinitely divisible*, if for every $n = 1, 2, \ldots$ $\mu_n \in N$ such that

$$\mu = \mu_n^{on} \tag{2.6}$$

presentation of infinitely divisible measures in (P, o), s that μ is infinitely divisible if and only if its charac- e form

$$\widehat{\mu}(t) = \exp(-f(t)) \qquad (t \geqslant 0) \tag{2.7}$$

where

$$f(t) = a + bt^H + \int (1 - \Omega(tx))\, m(dx) \tag{2.8}$$

where a and b are real constants, $b \geqslant 0$; m is a positive measure on R_+ with $m(\{o\}) = 0$
and

$$\int \omega(x)\, m(dx) < \infty, \tag{2.9}$$

$\omega(x)$ being a function defined by

$$\omega(x) = \begin{cases} 1 - \Omega(x) & 0 \leqslant x \leqslant x_0 \\ 1 - \Omega(x_0) & x > x_0 \end{cases} \tag{2.10}$$

where x_0 is a positive number such that $0 \leqslant \Omega(x) < 1$ for all $0 \leqslant x \leqslant x_0$.

The triple (a, b, m) in (2.8) uniquely determines μ. Further, $a = 0$ (resp. $a > 0$) if and only if $\mu \in P$ (resp. $\mu \in Q$).

Let $F(o)$ denote the set of all functions of the form (2.8) with $a \geqslant 0$. For the ordinary convolution, i.e. for $o = *$, $F(o)$ coincides with the set of all Bernstein functions, cf. Berg and Forst [1]. Hence, in general case, a function $f \in F(o)$ appearing in (2.8) will be referred to as a generalized Bernstein function or an o-Bernstein function associated with o.

3. GENERALIZED CONVOLUTION SEMIGROUPS

Suppose that 0 is a regular generalized convolution and (μ_t), $t \geqslant 0$, is a family of subprobability measures in Q. We say that (μ_t) is a *generalized convolution semigroup* or more precisely, o-semigroup, if the following conditions are satisfied:

(i) $\mu_t \circ \mu_s = \mu_{t+s}$, $(t, s \geqslant 0)$,

(ii) $\lim_{t \to 0} \mu_t = \mu_0$ vaguely.

It should be noted that the semigroup (μ_t) is also continuous in the weak topology. Moreover, every measure μ_t is infinitely divisible. Hence and by (2.7) we infer that there exists a uniquely determined o-Bernstein function $f \in F(o)$ such that

$$\widehat{\mu}_t(u) = \exp(-tf(u)) \quad (t, u \geqslant 0). \tag{3.1}$$

In the sequel the function f appearing in (3.1) will be referred to as an o-Bernstein function associated with the semigroup (μ_t).

It is easy to prove that the set $F(o)$ is a cone which is closed under the pointwise convergence. Let m be a finite measure and $a > m(R_+)$. Consider the exponential semigroup (μ_t) defined by.

$$\mu_t = e^{-at} \sum_{K=0}^{\infty} t^k m^{ok} \Big/ k! \tag{3.2}$$

$(t \geqslant 0)$.

Then we have $\widehat{\mu}_t(u) = \exp(-at + t\,\widehat{m}(u))$ $(t, u \geqslant 0)$ which imples that for every finite measure m and $a > m(R_+)$ $a - \widehat{m}(.)$ is an o-Bernstein function.

3.1. PROPOSITION. *Let* (μ_t) *be an o-semigroup and* (ν_t) *be a* $*\alpha$ *— semigroup* $(\alpha > 0)$ (cf. Urbanik [5] for the *definition of* $*\alpha$ *— convolution*).

Then the vague integral

$$\tau_t = \int \mu_s \cdot \nu_t(ds) \quad (t \geqslant 0) \tag{3.3}$$

defines an o-semigroup.

Proof. The kernel for the characteristic function of $*_\alpha$-convolution is $\exp(-t^\alpha)$. Therefore we have

$$\widehat{\tau}_t(u) = \int \exp(-s^\alpha f(u))\,\nu_t(ds)$$
$$= \exp(-tg(f^{1/\alpha}(u)). \tag{3.4}$$

f and g being generalized Bernstein functions associated with semigroups (μ_t) and (ν_t), respectively. Consequently, (τ_t) is an o-semigroup.

An immediate consequence of the above Proposition is the following:

3.2. COROLLARY. *If* $f \in F(o)$ *and* $g\ F(*_\alpha)$ *then* $g(f^{\frac{1}{\alpha}}) \in F(o)$. *In particular, if* h *is a Bernstein function then* $h(f)$ *is an o-Bernstein function.*

The converse statement is also true. Namely, we have

3.3. PROPOSITION. *Let* g *be a function such that for every generalized convolution* o *and for every* $f \in F(o)$ *the composite function* $g(f^{1/\alpha})$ *belongs to* $F(o)$. *Then* g *is a* $*_\alpha$ *— Bernstein function.*

Proof. It follows from the fact that the function $f(x) = x^\alpha$ belongs to $*_\alpha$.

Given $\alpha > 0$ we define the resolvent R_a for the o-semigroup (μ_t) as the weak integral

$$R_a = \int e^{-at} \mu_t\, dt. \tag{3.5}$$

Then the characteristic function of R_a is given by

$$\widehat{R}_a(u) = 1/a + f(u) \qquad (u \geqslant 0), \tag{3.6}$$

f being an 0-Bernstein function associated with (μ_t).

In particular, for $u = 0$ we get

$$\widehat{R}_a(o) = 1/a + f(o) \leqslant 1/a. \tag{3.7}$$

Hence $a\,R_a$ is a subprobability measure. It is a probability measure if and only if $f(o) = 0$ i.e. (μ_t) consists of probability measures.

The family (R_a) satisfies the following resolvent equation:

$$R_a - R_b = (b - a)\,R_a \circ R_b \qquad (u, v \geqslant 0). \tag{3.8}$$

3.4. PROPOSITION. *Suppose that* (R_a) *is a family of measures such that* $(a R_a) \subset Q$ *and the equation (3.8) is satisfied. Then there exists an o-semigroup* (μ_t) *such that* (R_a) *is the resolvent for* (μ_t).

Proof. We borrow some ideas of Berg and Forst from the proof of Theorem 8. 21 [1].

Without loss of generality one may assume that $R_a \neq 0$. From (3.8) it follows that the set $H = \{ t > 0 : \widehat{R_a}(t) \neq 0 \}$ is open and independent of a. Further, since

$$\widehat{R}_a(t) = \frac{\widehat{R}_1(t)}{1 + (a-1)\widehat{R}_1(t)}$$

and $a\widehat{R}_a(t) \to 1_H(t)$ as $a \to \infty$ uniformly over compact subsets of R_+ we infer that the function 1_H m st be continuous. Therefore, $H = R_+ \setminus \{o\}$ and consequently, $\widehat{R}_a(t) \neq 0$ for al $t > 0$.

Putting $\psi(t) = \dfrac{1 - a\widehat{R}_a(t)}{\widehat{R}_a(t)}$

and taking into account (3. 8) we infer that $\psi(t)$ is independent of a and

$$a\widehat{R}_a(t)\,\psi(t) = a(1 - a\widehat{R}_a(t)) \, .$$

Since the right–hand side of the above equation is an o—Bernstein function and $a\widehat{R}_a(t) \to 1$ $(t > 0)$ the limit $\psi(t) = \lim_{a\to\infty} a\widehat{R}_a(t)\,\psi(t)$

$$= \lim_{a\to\infty} a(1 - a\widehat{R}_a(t))$$

is an o-Bernstein function. Finally, define (μ_t) as an o–semigroup with the associated o-Bernstein function ψ. Then (μ_t) has the desired property.

4. GENERALIZED TRANSLATION INVARIANT SEMIGROUPS

Let o be a generalized convolut'on. Given $a \geqslant 0$ we define an operator τ_a acting on Borel functions by the formula

$$\tau_a f(x) = \int f(u)\, \delta_a \circ \delta_x \,(du) \qquad (4.1)$$

$(x \in R_+)$ provided the integral on the right-hand side exists. Let $\mathcal{D}(\tau_a)$ denote the domain of τ_a. It is clear that all bounded Borel functions belong to $\mathcal{D}(\tau_a)$ and $\mathcal{D}(\tau_a)$ is a linear space.

In the ordinary case the operator τ_a in (4.1) is reduced to the following

$$\tau_a f(x) = f(x + a) \qquad (x \in R_+). \qquad (4.2)$$

Thus, for $o = *$, τ_a is the usual translation. Hence in general case the operator τ_a will be referred to as *generalized translation* or more precisely, *o-translation*. In the sequel we shall consider τ_a on C_b only.

Let A be a linear operator defined on C_b. Then it is called *o-translation invariant* if for all $a \geqslant 0$ and $f \in \mathcal{D}(A)$ $\tau_a f \in \mathcal{D}(A)$ and

$$A(\tau_a f) = \tau_a (Af). \tag{4.3}$$

Let μ be a measure in N. It is easy to see that the operator τ_μ defined by

$$\tau_\mu f(x) = \int f(u)\, \mu o \delta_x (du) \tag{4.4}$$

is o-translation invariant. Conversely, any 0-translation invariant linear bounded operator A is given in this way:

4.1. LEMMA. *Let A be an o-translation invariant linear bounded positive operator on* C_b. *There exists a uniquely determined measure* μ *in* N *such that* $A = \tau_\mu$, τ_μ *being given by* (4.4).

Proof. Since $Af(o)$ is a nonnegative functional on C_b we infer that there exists a measure $\mu \in N$ satisfying the equation

$$Af(o) = \int f(u)\, \mu(du) \qquad (f \in C_b). \tag{4.5}$$

Therefore, for every $x \geqslant 0$ we have

$$\begin{aligned} A\,(\tau_x f)\,(o) &= \iint f(u)\, \delta_a \,o\, \delta_z (du)\, \mu\,(da) \\ &= \int f\,(u)\, \mu \,o\, \delta_x\,(du). \end{aligned} \tag{4.6}$$

It is clear by (4.5) that the measure μ is unique.

Suppose that μ and ν are measures in N. Then, by (4. 7),

$$\tau_\mu \tau_\nu = \tau_\nu \tau_\mu = \tau_{\mu o \nu}. \tag{4.7}$$

Consequently, if (μ_t) is an o-semigroup then operators S_t, $t \geqslant 0$, defined by

$$S_t = \tau_{\mu_t} \qquad (t \geqslant 0) \tag{4.8}$$

is a strongly continuous contraction semigroup on C_b.

Conversely, if (S_t) is a strongly continuous contraction semigroup on C_b then, by Lemma 4. 1, there exists a unique o-semigroup (μ_t) such that S_t is given by (4. 8). Thus we have the following theorem:

4. 2. THEOREM. *There exists a one-to-one correspondence between o-semigroups* (μ_t) *and strongly continuous o-translation invariant contraction semigroups* (S_t) *on* C_b.

REFERENCES

[1] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian groups*, Berlin Springer 1975.

[2] W. Feller. *An introduction to probability theory and its applications volume II*, John Wiley and Sons Inc., New York 1956.

[3] A. M. Yosida. *Functional Analysis*. Berlin 1962.

[4] K. Urbanik. *Generalized convolutions*. Studia Math 23 (1964).

[5] K. Urbanik. *Generalized convolutions III*.
Preprint N^{0}1, Math. Institute, Wroclaw University 1984.

Received November 15, 1988

INSTITUTE OF MATHEMATICS, P. O. BOX 631 BO HO, HANOI VIET NAM