ACTA MATHEMATICA VIETNAMICA
Vol. 14 No 2 (1989)

ON A CLASS OF NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH SHIFT ON COMPLEX CURVES

NGUYEN DANG TUAN

0. INTRODUCTION

Nonlinear singular integral equations (NSIE) have been studied by many authors (see, for instance, [2], [4], [9], [10], [11] and the references therein). In [4] A. I. Guseinov and H. Sh. Muchtarov introduced the generalized Hölder space $H(\omega)$, then studied the NSIE

$$u(x)=\wedge\int_a^b \frac{f[x,s,u(s)]}{s-x}\,ds \tag{1}$$

in that space and established conditions for the existence of its solutions.

The present work deals with a class of NSIE of the form

$$(M\varphi)(t)\equiv a(t)\,\varphi(t)+b(t)\,\varphi[\alpha(t)]+\frac{c(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\,d\tau+\frac{d(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-\alpha(t)}\,d\tau+$$

$$+\int_L K_M(t,\tau)\,\varphi(\tau)\,d\tau=\wedge\int_L\frac{f[t,\tau,\varphi(\tau)]}{\tau-t}\,d\tau+\mu\int_L\frac{F[\alpha(t),\tau,\varphi(\tau)]}{\tau-\alpha(t)}\,d\tau \tag{2}$$

where L is some contour in the complex plane, $\alpha(t)$ is the Carleman's shift of L. In (2), if the right hand side does not depend on $\varphi(t)$ and the coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$ satisfy the Höldr's condition on L, then we obtain the well-known linear singular integral equation (LSIE) already considered in [5] and other works. The traditional method is to reduce this equation to a LSIE not containing shift. Then we obtain the formulas for its index and Noethers property e. t. c. But if the right-hand side of (2) depends on the unknown function $\varphi(t)$ then this method is no longer applicable.

In this paper we shall prove the existence of the solution of NSIE in the generalized Hölder space $H^L(\omega)$.

The paper consists of 2 sections. In Section 1, we shall introduce the generalized Hölder space on a contour. The main result of this section is a theorem on the invariance of the generalized Hölder space for the operator defined by the left-hand side of the equation (2). In Section 2, we shall prove the existence of the solution of the equation (2) in the space $H^L(\omega)$.

First, we introduce some notions which we will need in the sequel

DEFINITION 1. 1. (see [4]). A function (σ) defined on $(0, l)$ is said to belong to the class Φ if

1) $\omega(\sigma)$ is a modulus of continuity,

2) $\sigma \int_0^l \frac{\omega(t)}{t(t+\sigma)} \, dt \leqslant D\omega(\sigma)$,

3) there exist constants m_ω, M_ω such that

$$m_\omega \frac{\omega(\sigma)}{\sigma} \leqslant \omega'(\sigma) \leqslant M_\omega \frac{\omega(\sigma)}{\sigma}.$$

From Definition 1. 1 it is easy to derive the following properties of the function $w(\sigma)$:

$$\text{(i) } \sigma_1 \leqslant \sigma_2 \text{ implies } \omega(\sigma_1)/\sigma_1 \geqslant \frac{1}{2}(\omega(\sigma_2/\sigma_2)$$

i. e. $w(\sigma)/\sigma$ is almost decreasing

(ii) $\omega(\lambda\sigma) \leqslant (\lambda + 1)\, \omega(\sigma)$ for every $\lambda > 0$

(iii) There exist numbers α, β in $(0,1)$ such that $\omega(\sigma)/\sigma^\alpha$ is almost increasing and $\omega(\sigma)/\sigma^\beta$ is almost decreasing function.

We note that condition 2) in Definition 1. 1 is equivalent to either of the following conditions :

$$2'')\ \sup_{\sigma>0} \frac{1}{\omega(\sigma)} \int_0^l \frac{\omega(t)}{t} \, dt = A_\omega < \infty;$$

$$2''')\ \sup_{\sigma>0} \frac{\sigma}{\omega(\sigma)} \int_\sigma^l \frac{\omega(t)}{t^2} = B_\omega < \infty.$$

Throughout the sequel we shall consider a curve consisting in the general case of $m+1$ simple, smooth, closed, Liapunov curves $L = L_0 + L_1 + \dots + L_m$. Besides, the plane is divided into two parts D^+ and D^-. D^+ always lies on the left-hand side by moving in the positive direction on L. We shall also fix on L a Carleman's shift $\alpha(t)$ satisfying the following conditions :

1') $\alpha(t)$ is a diffeomorphism of L on itself,

2') $\alpha(\alpha(t)) = t$, $t \in L$,

3') $\alpha(t) \neq t$,

4') $\alpha'(t) \neq 0$, $\alpha'(t) \in H_{1-\gamma}(L)$.

where $H_{1-\gamma}(L)$ is the Hölder's space with index $1-\gamma$ $(0<\gamma<1)$. Using the notations

$$(S\varphi)(t)=\frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau-t}\,d\tau \text{ - singular integral operator}$$

$(W\varphi)(t)=\varphi[\alpha(t)]$-shift operator

$$(\mathcal{S}_f\varphi)(t)=\lambda\int_L \frac{f[t,\tau,\varphi(\tau)]}{\tau-t}\,d\tau,$$

$$(\mathcal{S}_F\varphi)(t)=\mu\int_L \frac{F[t,\tau,\varphi(\tau)]}{\tau-t}\,d\tau,$$

$h^{\alpha}(t)=h[\alpha(t)]$, where $h(t)$ is a given function, the NSIE (2) can be written in the form of an operator equation

$$M\varphi\equiv a\varphi+bW\varphi+cS\varphi+dWS\varphi+D_M\varphi=\mathcal{S}_f\varphi+W\mathcal{S}_F\varphi. \qquad (2')$$

DÉFINITION 1. 2. Let $l=|L|$ be the length of the curve L and be a function defined on $[0,l]$ and belonging to the class Φ. By generalized Hölder space $H^L(\omega)$ we mean the set of all functions $u(t)$ satisfying

$$H^L_{\omega}(u)=\sup_{t_1,t_2\in L}\frac{|u(t_1)-u(t_2)|}{\omega(|t_1-t_2|)}<\infty.$$

With the norm defined by

$$\|u\|_{H^L(\omega)}=\|u\|_{C(L)}+H^L_{\omega(u)}$$

$H^L(\omega)$ becomes a Banach space.

In the sequel we need a more refined classification of the space $H^L(\omega)$.

DEFINITION 1. 3. Suppose R, K are positive numbers and $\omega(t)$ is a function of Φ. We say that the function $u(t)$, $t\in L$ belongs to $H^L(R;K;\omega)$ if

1) $|u(t)|\leqslant R,\ t\in L$

2) $|u(t_1)-u(t_2)|\leqslant K\omega(|t_1-t_2|)$.

LEMMA 1. 4. *The shift operator maps $H^L(R,K;\omega)$ into $H^L(R,K(\|\alpha'\|_c+1),\omega)$*
The proof is immediate.

LEMMA 1. 5. *Let $\varphi(t)$ be a function in $H^L(R,K;\omega)$. Then $(S\varphi)(t)$ belongs to the class $H^L(R',K';\omega)$, where R', K' are some positive constants.*

Proof. First, we estimate $|(S\varphi)(t)|$. It is well-known (see [1]) that there exists a number m^* such that $|d\tau|\leqslant m^*\,dr$. We have

$$|(S\varphi)(t)| \leqslant \frac{1}{\pi}\left[\left|\int_L \frac{\varphi(\tau)-\varphi(t)}{\tau-t}\,d\tau\right| + \pi|\varphi(t)|\right] \leqslant \frac{1}{\pi}\left[2m^*K\int_0^l \frac{\omega(r)}{\tau}\,dr + \right.$$

$$\left. + \pi R\right] \leqslant 2A_\omega m^*\omega(l)\, K/\pi + R = R'.$$

o estimate $|(S\varphi)(t_1) - (S\varphi)(t_2)|$, we consider two distinct cases

a) Case 1: $t_1 \in L_i,\ t_2 \in L_j,\ 0 \leqslant i \neq j \leqslant m$

n this case, we have

$$|(S\varphi)(t_1) - S\varphi(t_2)| \leqslant \frac{1}{\pi}\left[\left|\int_L \frac{\varphi(\tau)-\varphi(t_1)}{\tau-t_1}\,d\tau\right| + \left|\int_L \frac{\varphi(\tau)-\varphi(t_2)}{\tau-t_2}\,d\tau\right| + \right.$$

$$\left. + \pi|\varphi(t_1) - \varphi(t_2)|\right]$$

rom $L_i \cap L_j = \phi$ it follows that

$$\min_{t_1 \in L_i,\, t_2 \in L_j} |t_1 - t_2| > 0$$

nd then $$\widetilde{l} = \min_{0 \leqslant i \neq j \leqslant m} \{\min_{t_1 \in L_i,\, t_2 \in L_j} |t_1 - t_2|\} > 0$$

By putting $K_0 = l/\widetilde{l}$, we obtain

$$|(S\varphi)(t_1) - (S\varphi)(t_2)| \leqslant K[4A_\omega m^*(K_0+1)/\pi + 1]\,\omega(|t_1 - t_2|).$$

b) Case 2: $t_1, t_2 \in L_j \quad 0 \leqslant j \leqslant m.$

At first, for the curve L there exists a number δ_0 such that for every t on L any circle of radius $\delta \leqslant \delta_0$ centered at t intersects L at only two points (see [8]). If $|t_1 - t_2| \geqslant \delta_0$, then it is easy to see that

$$\sup_{t_1, t_2 \in L_j} \frac{|(S\varphi)(t_1) - (S\varphi)(t_2)|}{\omega(|t_1 - t_2|)} \leqslant \frac{2R'}{C_\beta\,\delta_0^\beta},$$

where

$$C_\beta = \inf_{\sigma \in (0,l]} \frac{\omega(\sigma)}{\sigma^\beta} > 0.$$

Suppose $|t_1 - t_2| < \delta_o$. Fix an arbitrary number k, $1 < k < \delta_o/|t_1 - t_2|$. Draw a circle of radius $\delta = k|t_1 - t_2|$ centered at the point t_1. This circle obviously intersects L at two points A and B. The part L_j lying within this circle is denoted by AB.

Putting

$$v(t) = \int_L \frac{\varphi(\tau)-\varphi(t)}{\tau - t}\,d\tau$$

we have

$$v(t_1)-v(t_2)=\int\limits_L \frac{\varphi(\tau)-\varphi(t_1)}{\tau-t_1}\,d\tau-\int\limits_L \frac{\varphi(\tau)-\varphi(t_2)}{\tau-t_2}\,d\tau=\int\limits_{AB} \frac{\varphi(\tau)-\varphi(t_1)}{\tau-t_1}\,d\tau-$$

$$\int\limits_{AB} \frac{\varphi(\tau)-\varphi(t_2)}{\tau-t_2}\,d\tau+\int\limits_{L\setminus AB} \frac{[\varphi(t_2)-\varphi(t_1)]}{\tau-t_1}\,d\tau+\int\limits_{L\setminus AB} \frac{[\varphi(\tau)-\varphi(t_2)](t_2-t_1)}{(\tau-t_1)(\tau-t_2)}\,d\tau=$$

$$=A_1+A_2+A_3+A_4,$$

$$|A_1|\leqslant\int\limits_{AB}\frac{|\varphi(\tau)-\varphi(t_1)|}{|\tau-t_1|}\,|d\tau|\leqslant 2Km^*\int\limits_0^{\delta}\frac{\omega(r)}{r}\,dr\leqslant 2KA\omega m^*(k+1)\,\omega(|t_1-t_2|)\ .$$

Similarly

$$|A_2|\leqslant 2KA\omega m^*(k+1)\,\omega(|t_1-t_2|),$$

It is easy to see that

$$|A_3|\leqslant|\varphi(t_1)-\varphi(t_2)|\left|\int\limits_{L\setminus AB}\frac{d\tau}{\tau-t_1}\right|KM_1\,\omega(|t_1-t_2|),$$

$$\frac{k-1}{k}|\tau-t_1|\leqslant|\tau-t_2|\leqslant\frac{k+1}{k}|\tau-t_1| \tag{3}$$

Therefore

$$|A_4|\leqslant|t_1-t_2|\int\limits_{L\setminus AB}\frac{|\varphi(\tau)-\varphi(t_2)|}{|\tau-t_1||\tau-t_2|}\,d\tau\leqslant\frac{2(2k+1)(k+1)}{k(k-1)}Km^*B_\omega\,\omega(|t_1-t_2|)=$$

$$=B_\omega KM_2m^*\omega(|t_1-t_2|)$$

By taking

$$M^*=max\left\{2A_\omega m^*(k+1)+M_1+B_\omega M_2m^*\ ;\ \frac{2R'}{c\beta\delta_0^{\beta}}\right\}$$

we obtain

$$|(S\varphi)(t_1)-(S\varphi)(t_2)|\leqslant\frac{1}{\pi}\left[|v(t_1)-v(t_2)|+\pi|\varphi(t_1)-\varphi(t_2)|\right]\leqslant(M^*/\pi+1)\ .$$

$$.\ K_\omega(|t_1-t_2|)\ .$$

Thus we have proved that

$$|(S\varphi)(t_1)-(S\varphi)(t_2)|\leqslant K\ .\ K_s^*\ \omega(|t_1-t_2|)$$

for every $t_1, t_2\in L$ and

$$K_s^*=max\ \{4A_\omega m^*(K_0+1)/\pi+1;\ M^*/\pi+1\}.$$

Lemma 1.4 is proved.

From these lemmas it follows directly that the operator $\widetilde{M}=a^{\alpha}I-bW+$ $+c^{\alpha}S-dWS$ maps $H^L(R,\ K,\omega)$ into $H^L)R_{\widetilde{M}}\ ;\ K_{\widetilde{M}})^{\omega}$ where $R_{\widetilde{M}}$, $K_{\widetilde{M}}$ are some positive constants.

DEFINITION 1.6 The operator

$$(D\varphi)(t) = \int_L K(t,\tau)\,\varphi(\tau)\,d\tau,$$

is called operator with regular kernel of degree γ $(0 \leqslant \gamma \leqslant 1)$ if

$$K(t,\tau) = \frac{A(t,\tau)}{(\tau - t)^\gamma},$$

where $A(t,\tau)$ is some function bounded on $L \times L$.

LEMMA 1.7. *Assume that*

$$K_M(t,\tau) = \frac{A_M(t,\tau)}{(\tau - t)^\gamma}$$

where $A_M(t,\tau)$ is a function continuous together with its partial derivatives with respect to t, τ; γ is the positive number determined in 4') and $0 < \gamma + \beta < 1$ for any number β such that $\omega(s)/s^\beta$ is an almost decreasing function. Then the cperator D_M with the regular kernel $K_M(t,\tau)$ is a completely continuous operator mapping $H^L(w)$ into itself. Moreover, the operator with the kernel is also a completely continuous operator in $H^L(\omega)$.

Proof. First, we prove the invariance of $H^L(\omega)$ for the operator D_M. Without loss of generality, we can assume that L consists of only one closed contour.

Putting

$$K^*_M(t,\tau) = K_M(t,\tau)\,(\tau - t) \tag{4}$$

it is easy to see that

$$\left|\frac{\partial K^*_M(t,\tau)}{\partial t}(t,\tau)\right| \leqslant \frac{C_M}{|\tau - t|^\gamma},\; C_M > 0 \tag{5}$$

As in the proof of Lemma 1.5 we have

$$(D_M\varphi)(t_1) - (D_M\varphi)(t_2) = \int_{AB} \frac{K^*_M(t,\tau)}{\tau - t_1}\varphi(\tau)d\tau - \int_{AB} \frac{K^*_M(t_2,\tau)}{\tau - t_2}\varphi(\tau)\,d\tau +$$

$$+ \int_{L/AB} \frac{K^*_M(t_2,\tau) - K^*_M(t_2,\tau)}{\tau - t_1}\varphi(\tau)\,d\tau + \int_{L/AB} (t_1 - t_2)\frac{K^*_M(t_2,\tau)}{(\tau - t_1)(\tau - t_2)}\varphi(\tau)d\tau =$$

$$= I_1 + I_2 + I_3 + I_4.$$

Taking (4) into account, we obtain

$$|I_1| \leqslant \int_{L/A_B} \frac{|K^*_M(t_1,\tau)\,\|\,\varphi(\tau)}{|\tau - t_1|}\,|d\tau| \leqslant A_M\,\|\varphi\|_C\, m^* \int_0^\delta \frac{dr}{r\gamma} \leqslant$$

$$\leqslant A_M C_1 \|\varphi\|_c |t_1 - t_2)^{1-\gamma} \leqslant C_1 A_M C_B \|\varphi\|_C [\omega(|t_1 - t_2|)]^{(1-\gamma)/\beta} \leqslant$$
$$\leqslant C_2 \|\varphi\|_C \omega(|t_1 - t_2|)$$

Similarly, using (3) we have $|I_2| \leqslant C_3 \|\varphi\|_C \omega(|t_1 - t_2|)$ and from (3) and (4) it follows that

$$|I_4| \leqslant |t_1 - t_2| \int_{L/AB} \frac{|K_M^*(t_2, \tau)\|\varphi(\tau)|}{|\tau - t_1\|\tau - t_2|} |d\tau| \leqslant A_M \|\varphi\|_{C}{}^{k}_{k-1} |t_1 - t_2| \int_{L/AB} \frac{|d\tau|}{|\tau - t_1|^{1+\gamma}} \leqslant$$
$$\leqslant A_M \|\varphi\|_{C_\gamma(k-1}^{km^*} |t_1 - t_2| (\delta^{-\gamma} - l^{-\gamma}) \leqslant C_4 \|\varphi\|_C \omega(|t_1 - t_2|).$$

It remains only to estimate I_3. Let us note that the assumptions of the lemma together with (5) show that $\partial K_M^*(t, \tau)/\delta t$ is a continuous function, bounded on $L \times (L/AB)$. Expanding $K_M^*(t, \tau)$ into a Taylor's series at the point (t_1, τ), we have

$$K_M^*(t_2, \tau) = K_M^*(t_1, \tau) + \frac{\partial K_M^*(t_1, \tau)}{\partial t}(t_2 - t_1) + o(t_1, t_2),$$

hence

$$|K_M^*(t_2, \tau) - K_M^*(t_1, \tau)| \leqslant |t_1 - t_2| \left| \frac{\partial K_M^*(t_1, \tau)}{\partial t} + \frac{o(t_1 t_2)}{t_1 - t_2} \right| \leqslant \frac{c|t_1 - t_2)}{|\tau - t_1|^\gamma}$$

where c is a certain constant. Therefore,

$$|I_3| \leqslant c|t_1 - t_2| \|\varphi\|_c \int_{L/AB} \frac{|d\tau|}{|\tau - t_1|^{1+\gamma}} \leqslant c_5 \|\varphi\|_c \omega(|t_1 - t_2|).$$

Combining the two aboves estimates, we have

$$|(D_M\varphi)(t_1) - (D_M\varphi)(t_2)| \leqslant c_6 \|\mathcal{S}\|_c \omega(|t_1 - t_2|). \qquad (6)$$

Thus $H^L(\omega)$ is invariant for the operator D_M.

Now we check that D_M is a completely continuous operator in $H^L(\omega)$. Let $H \subset H^L(\omega)$ be an arbitrary set bounded in norm in $H^L(\omega)$. Obviously H is bounded in the space $C(L)$. Due to Arzela's theorem, H is a precompact set in $C(L)$. Therefore, from any sequence $\{\varphi_n\}$, $\varphi_n \in H$, we can extract a subsequence $\{\varphi_{n_k}\}$ which converges in norm in $C(L)$ to φ. It is easy to see that $\varphi \in H^L(\omega)$. Putting $\Psi_{n_k} = \varphi_{n_k} - \varphi$, we now prove that

$$\|D_M \Psi_{n_k}\|_{H^L(\omega)} \to 0, \ k \to \infty;$$

Indeed, from the assumptions of the lemma and from (4) it follows that

$$\|D_M \psi_{nK}\|_{H^L(\omega)} \leqslant (C_6 + \max_{t \in L} \int_L |K_M(t, \tau)\| d\tau|) \|\psi_{nK}\|_C \to 0, \ k \to \infty,$$

which shows that D_M si a completely continuous operator. Next let us consider the operator with the regular kernel $K_\alpha(t, \tau)$. We put

$$K^*_\alpha(t, \tau) = (\tau - t)\, K_\alpha(t, \tau).$$

Then the following estimate holds

$$|\alpha(t) - \alpha(\tau) + (\tau - t)\alpha'(t)| = |\int_\tau^t [\alpha'(\theta) - \alpha'(t)]d\theta| \leqslant A_\alpha \int_\tau^t |\theta - t|^{1-\gamma}\, |d\theta|$$

$$\leqslant A_\alpha m^* |\tau - t|^{2-\gamma}.$$

Moreover,

$$\frac{1}{|\alpha(\tau) - \alpha(t)|^2} \leqslant \frac{C'_\alpha}{|\tau - t|^2}.$$

Since $\lim\limits_{\tau\to t}\left|\frac{\tau - t}{\alpha(\tau)-\alpha(t)}\right| = \frac{1}{|\alpha'(t)|} \neq 0$, we obtain the estimate

$$\frac{\partial K^*_\alpha(t, \tau)}{\partial t} = |\tau - t|\left|\frac{\partial K_\alpha(t, \tau)}{\partial t} - K_\alpha(t, \tau)\right| = |\alpha'(\tau)|\left|\frac{\alpha(t) - \alpha(\tau) + (\tau - t)\dot\alpha(t)}{[\alpha(\tau) - \alpha(t)]^2}\right|$$

$$\leqslant \frac{\widetilde{C}_\alpha}{|\tau - t|}\,\gamma.$$

Using these estimates, we can p ove as in the first part of the proof, that the operator with the kernel $K_\alpha(t, \tau)$ is completely continuous in $H^L(\omega)$. The proof of Lemma 1. 7 is complete.

Using Lemmas 1. 4, 1. 5, 1. 7 we can derive the following

THEOREM 1. 8. *Let $a(t)$, $b(t)$, $c(t)$, $d(t)$ be continuously differentiable functions on the contour L. Suppose that $K_M(t, \tau)$ is a regular kernel satisfying the conditions of Lemma 1. 7. Then*

$$M = aI + bM + cS + dWS + D_M$$

where D_M is the operator with the kernel $K_M(t, \tau)$ acting invariantly in the space $H^L(\omega)$.

Now we study the nonlinear singular operators S_f, S_F, where the functions f, F satisfy the conditions:

$$|f(t_1, \tau_1, u_1) - f(t_2, \tau_2, u_2)| \leqslant A_f\, \omega^*(|t_1 - t_2|) + B_f\, \omega(|\tau_1 - \tau_2|) + C_f |u_1 - u_2|, \quad (7)$$

$$|F(t_1, \tau_1, u_1) - F(t_2, \tau_2, u_2)| \leqslant A_F\, \omega^*(|t_1 - t_2|) + B_F\, \omega(|\tau_1 - \tau_2|) + C_F |u_1 - u_2|, \quad (8)$$

where A_f, B_f, C_f, A_F, B_F, C_F are positive constants, $\omega, \omega^* \in \Phi$ such that,

$$\omega^*(\delta)\, ln\,(l/\delta) \leqslant \widetilde{C}\, \omega(\delta),\ \widetilde{C} > 0. \quad (9)$$

THEOREM 1.9. *Suppose that the functions f, F satisfy the conditions (7), (8), (9). Then for every $\varphi(t) = H^L(R, K; w)$, we have*

a) $(\mathcal{S}_f \varphi)(t) \in H^L(R_f, K_f, w)$

b) $(\mathcal{S}_F \varphi)(t) \in H^L(R_F, K_F, w)$

where the constants R_f, K_f, R_F, K_F *depend on* R, K, μ.

Proof. a) Put

$$g(t, \tau) = f(t, \tau, \varphi(\tau)), \quad G(t, \tau) = F(t, \tau, \varphi(\tau)). \tag{10}$$

It is easy to see that

$$|g(t_1, \tau_1) - g(t_2, \tau_2)| \leqslant A_g\, \omega^*(|t_1 - t_2|) + B_g\, \omega(|\tau_1 - \tau_2|). \tag{11}$$

$$|G(t_1, \tau_1) - G(t_2, \tau_2)| \leqslant A_G\, \omega^*(|t_1 - t_2|) + B_G\, \omega(|\tau_1 - \tau_2|) \tag{12}$$

where $A_g = A_f$, $A_G = A_F$, $B_g = B_f + KC_f$, $B_G = B_F + KC_F$

Now put

$$\widetilde{f}(t) = \int_L \frac{g(t, \tau)}{\tau - t}\, d\tau, \quad Mg = \max_{(t,\tau)\in L\times L} |g(t, \tau)|, \quad M_G = \max_{(t,\tau)\in L\times L} |G(t, \tau)|.$$

From the inequalities (11), (12), and the compactness of $L \times L$ it follows that M_g, $M_G < \infty$. First, we estimate $|\widetilde{f}(t)|$

$$|\widetilde{f}(t)| \leqslant \left| \int_L \frac{g(t, \tau) - g(t, t)}{\tau - t}\, d\tau \right| + \pi |g(t, t)| \leqslant B_g\, A_\omega\, \omega(l) + \pi M_g.$$

To estimate $|\widetilde{f}(t_1) - \widetilde{f}(t_2)|$, we consider two distinct cases

Case 1. $t_1 \in L_i$, $t_2 \in L_j$; $0 \leqslant i \neq j \leqslant m$. We have

$$|\widetilde{f}(t_1) - \widetilde{f}(t_2)| \leqslant \left| \int_L \frac{g(t_1, \tau) - g(t_1, t_1)}{\tau - t_1}\, d\tau \right| + \left| \int_L \frac{g(t_2, \tau) - g(t_2, t_2)}{\tau - t_2}\, d\tau \right| +$$
$$+ \pi |g(t_1, t_1) - g(t_2, t_2)|.$$

As in part a) of Lemma 1.5, we have

$$\left| \int_L \frac{g(t_1, \tau) - g(t_1, t_1)}{\tau - t_1}\, d\tau \right| + \left| \int_L \frac{g(t_2, \tau) - g(t_2, t_2)}{\tau - t_2}\, d\tau \right| \leqslant 4A_\omega B_g\, m^*(K_0 + 1)\omega$$
$$(|t_1 - t_2|),$$

where K_0 is determined from Lemma 1.5. Moreover,

$$\pi |g(t_1, t_1) - g(t_2, t_2)| \leqslant \pi [Ag\omega^*(|t_1 - t_2|) + Bg\omega(|\tau_1 - \tau_2|)] \leqslant$$
$$\leqslant \pi (3A_g \widetilde{C}/ln2 + B_g)\, \omega(|t_1 - t_2).$$

Therefore in this case we obtain

$$|\widetilde{f}(t_1) - \widetilde{f}(t_2)| \leqslant [4A_\omega B_g\, m^*(K_0 + 1) + \pi(3A_g \widetilde{C}/ln2 + B_g)]\, \omega(|t_1 - t_2|).$$

Case 2. Suppose t_1, $t_2 \in L_j$ $(0 \leqslant j \leqslant m)$ and AB, δ... are determined as in the proof of part b) of Lemma 1.5. Then:

$$|\widetilde{f}(t_1)-\widetilde{f}(t_2)|\leqslant\left|\int\limits_{AB}\frac{g(t_1,\tau)-g(t_1,t_2)}{\tau-t_1}d\tau\right|+\left|\int\limits_{AB}\frac{g(t_2,\tau)-g(t_2,t_2)}{\tau-t_2}d\tau\right|+$$

$$+\left|\int\limits_{L\setminus AB}\frac{g(t_1,\tau)-g(t_2,\tau)}{\tau-t_1}d\tau\right|+\left|\int\limits_{L\setminus AB}\frac{(t_1-t_2)[g(t_2,\tau)-g(t_2,t_2)]}{(\tau-t_1)(\tau-t_2)}d\tau\right|+$$

$$+\left|[g(t_2,t_2)-g(t_1,t_1)]\times\left[i\pi+\int\limits_{L\setminus AB}\frac{d\tau}{\tau-t_1}\right]\right|=|I_1|+|I_2|+|I_3|+I_4|+|I_5|.$$

Let us estimate each term of the right-hand side

$$|I_1|\leqslant B_g\int\limits_{AB}\frac{\omega(|\tau-t_1|)}{|\tau-t_1|}|d\tau|\leqslant A_\omega B_g\widetilde{M}_1\,\omega(|t_1\ t_2|),$$

$$|I_2|\leqslant A_\omega B_g\widetilde{M}_2\,\omega(|t_1-t_2|),$$

$$|I_3|\leqslant A_g\int\limits_{L\setminus AB}\frac{\omega^*(|t_1-t_2|)}{|\tau-t_1|}|d\tau|\leqslant 2A_g m^*\omega^*(|t_1-t_2|)\int\limits_\delta^l\frac{dr}{r}\leqslant$$

$$\leqslant 2A_g m^*\omega^*(|t_1-t_2|)\times ln(l/\delta)\leqslant\widetilde{C}A_g\widetilde{M}_3\,\omega(|t_1-t_2|),$$

$$|I_4|\leqslant B_g|t_1-t_2|\int\limits_{L\setminus AB}\frac{\omega(|\tau-t_2|)}{|\tau-t_1||\tau-t_2|}|d\tau|\leqslant B_g B_\omega\widetilde{M}_4\,\omega(|t_1-t_2|).$$

From the inequality

$$\left|i\pi+\int\limits_{L\setminus AB}\frac{d\tau}{\tau-t_1}\right|\leqslant\widetilde{M}_5$$

we obtain

$$|I_5|\leqslant\widetilde{M}_5(3A_g\widetilde{C}/ln2+B_g)\,\omega(|t_1-t_2|).$$

Thus

$$|\widetilde{f}(t_1)-\widetilde{f}(t_2)|\leqslant[A_g\widetilde{C}\widetilde{M}_3+A_\omega B_g(\widetilde{M}_1+\widetilde{M}_2)+B_gB_\omega\widetilde{M}_4+3A_g\widetilde{C}/ln_2+B_g)\widetilde{M}_5]\cdot$$
$$\cdot\,\omega(|t_1-t_2|),$$

where $\widetilde{M}_i$ are positive constants.

Setting

$$M\widetilde{f}=\max\{4A_\omega B_g m^*(K_0+1)+\pi(3A_g\widetilde{C}/ln_2+B_g)[A_\omega B_g(\widetilde{M}_1+\widetilde{M}_2)+A_g\widetilde{C}\widetilde{M}_3+$$
$$+B_gB_\omega\widetilde{M}_4+(3A_g\widetilde{C}/ln_2+B_g)\widetilde{M}_5]\},$$

$$R_f=|\Lambda|(A_\omega B_g\omega(l)+\pi M_g,\qquad K_f=|\Lambda|M_{\widetilde{f}}$$

we can easily see that

$$(\mathcal{S}_f\varphi)(t)\in H^L(Rf,\ Kf;\ \omega)$$

By an argument similar to that used in part a) we have

$$(\mathcal{S}_F\varphi)(t) \in H^L(R_F, K_{\tilde{F}}, \omega)$$

The theorem is proved.

2. EXISTENCE OF SOLUTIONS OF N.S.I.E. WITH SHIFT IN THE GENERALIZED HÖLDER SPACE

We first consider the case where the shift $\alpha(t)$ preserves the orientation of the curve L. It is well-known (see [1]) that for any operator

$$M = aI + bW + cS + dWS + D_M$$

there exists a so-called regular operator of the form

$$\tilde{M} = a^{\alpha}I - bW + c^{\alpha}S - dwS\,,$$

such that

$$M\tilde{M} = \tilde{m}I + \tilde{n}S + D_{M\tilde{M}}\,,$$

where

$$\tilde{m} = aa^{\alpha} + cc^{\alpha} - bb^{\alpha} - dd^{\alpha}\,,$$

$$\tilde{n} = ac^{\alpha} + ca^{\alpha} - bd^{\alpha} - db^{\alpha}\,.$$

Then we can write the equation (2) in the form

$$\tilde{m}\varphi + \tilde{n}S\varphi + D_{M\tilde{M}}\,\varphi = \mathcal{S}_f\,\tilde{M}\varphi + W\mathcal{S}_F\tilde{M}\varphi\,.$$

Therefore, our main task is to find solutions of the equation (13).

Let us denote $\Delta_1 = \tilde{m} - \tilde{n}$, $\Delta_2 = \tilde{m} + \tilde{n}$ and assume that $\Delta_1(t) \neq 0$; $\Delta_2(t) \neq 0$ on L. Put

$$\mathcal{X}_{M\tilde{M}} = \text{ind}\,\frac{\Delta_1(t)}{\Delta_2(t)}$$

and let $\alpha(M)$, $\beta(M)$ be the numbers of the eigenfunctions of the operator M and its adjoint operator respectively.

It is easy to see that the operator defined by (13) is an operator with the regular kernel satisfying the conditions of Lemma 1. 7. In fact, it is shown in 1] that the operator (SW — WS) is an operator with regular kernel

$$K_{\alpha}(t, \tau) = \frac{1}{\tau - t} - \frac{\alpha'(\tau)}{\alpha(\tau) - \alpha(t)}.$$

From Lemma 1.7 and the conditions of Theorem 1.8 we deduce that $K_{M\tilde{M}}(t, \tau)$ is a linear combination of bounded regular kernels of degree γ.

In the remainder we assume that $\mathcal{X}_{M\tilde{M}} \geqslant 0$. Then by the Vekua-Carleman's method, the equation (13) can be transformed into an integral Fredholm's equation :

$$\varphi(t) + \int_L \mathcal{L}_{M\tilde{M}}(t,\tau)\,\varphi(\tau)\,d\tau = f_1(t), \tag{14}$$

where

$$\mathcal{L}_{M\tilde{M}}(t,\tau) = \tilde{m}(t)\,K_{M\tilde{M}}(t,\tau) - \frac{\tilde{n}(t)\,Z(t)}{\pi i}\int_L \frac{K_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau_1 - t)}\,d\tau_1$$

$$f_1(t) = \tilde{m}(t)\,[(\mathcal{S}_f\,\tilde{M}\varphi)(t) + (W\mathcal{S}_F\tilde{M}\varphi)(t)] -$$

$$-\frac{\tilde{n}(t)\,Z(t)}{\pi i}\int_L \frac{\mathcal{S}_f\,\tilde{M}\varphi)(\tau) + (W\mathcal{S}_F\tilde{M}\varphi)(\tau)}{Z(\tau)(\tau - t)}\,d\tau + \tilde{n}(t)\,Z(t)\,P_{\mathcal{X}_{M\tilde{M}-1}}(t).$$

$$Z(t) = \Delta_1(t)\,X^-(t) = \Delta_2(t)\,X^+(t) = e^{\Gamma(t)}\left|\sqrt{t^{\mathcal{X}_{M\tilde{M}}}\pi(t)}\right.,$$

$$\pi(t) = \prod_{k=1}^{m}(t - Z_k)^{\mathcal{X}_k}_{(\text{see [1]})}.$$

LEMMA 2. 1. *The operator with the regular kernel* $\mathcal{L}_{M\tilde{M}}(t,\tau)$ *is completely continuous in* $H^L(w)$.

Proof. First, we prove that $\mathcal{L}_{M\tilde{M}}(t,\tau)$ is a regular kernel of degree γ. Indeed, $K^*_{MM}(t,\tau) = (\tau - t)\,K_{MM}(t,\tau)$ is an operator with a regular kernel of degree γ satisfying the conditions of Lemma 1. 7.

Without loss of generality, we assume that L consists of only one closed contour. Draw the circle $O(t,\delta)$ of radius σ centered at the point t. $O(t,\delta)$ intersects L at two points A, B and by AB we denote the part of L lying within $O(t,\delta)$.

Suppose τ is a point of AB, so obviously $\delta = k/\tau - t/$ with some $k > 1$. We have

$$\int_L \frac{K_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau_1 - t)}\,d\tau_1 = \int_L \frac{K^*_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau - \tau_1)(\tau_1 - t)}\,d\tau_1 = \frac{1}{\tau - t}$$

$$\int_{AB} \frac{K^*_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau - \tau_1)}\,d\tau_1 + \int_{AB} \frac{K^*_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau_1 - t)}\,d\tau_1 + (\tau - t)\int_{L\setminus AB} \frac{K^*_{M\tilde{M}}(\tau_1,\tau)}{Z(\tau_1)(\tau - \tau_1)(\tau - t)}\,d\tau_1 =$$

$$= \frac{1}{\tau - t}(I_1 + I_2 + I_3).$$

Putting $m_z = \min_{t \in L} |z(t)| > 0$, and taking the inequality (3) into account, we obtain:

$$|I_1| \leqslant \int_{AB} \frac{C_{M\tilde{M}} |\tau - \tau_1|^{1-\gamma}}{|Z(\tau_1)||\tau - \tau_1|} |d\tau_1| \leqslant \frac{C_{M\tilde{M}}}{m_2} \left(\frac{k}{k-1}\right)^{\gamma} \int_{AB} \frac{|d\tau_1|}{|\tau_1 - t|^{\gamma}} \leqslant N_1 |\tau - t|^{1-\gamma},$$

$$|I_2| \leqslant (C_{M\tilde{M}}/m_2) \int_{AB} \frac{|\tau - \tau_1|^{1-\gamma}}{|\tau_1 - t|} |d\tau_1| \leqslant \left(\frac{k+1}{k}\right)^{1-\gamma} \frac{C_{M\tilde{M}}}{m_2} \int_{AB} \frac{|d\tau_1|}{|\tau_1 - t|^{\gamma}} \leqslant N_2 |\tau - t|^{1-\gamma}$$

$$|I_3| \leqslant \frac{C_{M\tilde{M}} |\tau - t|}{m_2} \int_{L \setminus AB} \frac{|\tau - \tau_1|^{1-\gamma}}{|\tau - \tau_1||\tau_1 - t|} |d\tau_1| \leqslant \frac{C_{M\tilde{M}} |\tau - t|}{m_2} \left(\frac{k+1}{k}\right)^{1-\gamma} \frac{k m^*}{k-1} \left| \int_{\delta}^{l} \frac{dr}{r^{2-(1-\gamma)}} \right| \leqslant$$

$$\leqslant N_3 |\tau - t|^{1-\gamma}$$

Consequently,

$$\left| \int_L \frac{K_{M\tilde{M}}(\tau_1, \tau)}{(\tau_1 - t) Z(\tau_1)} d\tau_1 \right| \leqslant \frac{1}{|\tau - t|} N |\tau - t|^{1-\gamma} = \frac{N}{|\tau - t|^{\gamma}}.$$

That is $\int_L \frac{K_{M\tilde{M}}(\tau_1, \tau)}{Z(\tau_1)(\tau_1 - t)} d\tau_1$ is a regular kernel and then $\mathcal{L}_{M\tilde{M}}(t, \tau)$ is also a regular kernel of degree γ. Since $K_{M\tilde{M}}(t, \tau)$ satisfies the conditions of Lemma 1.7, $\mathcal{L}_{M\tilde{M}}(t, \tau)$ also satisfies the same conditions in any region not containing singular points. From Lemma 1, 7 we conclude that the operator with the kernel $\mathcal{L}_{M\tilde{M}}(t, \tau)$ is completely continuous in $H^L(\omega)$. Lemma 2.1 is thus proved.

LEMMA 2.2. *The homogeneous Fredholm integral equation*

$$\varphi(t) + \int_L \mathcal{L}_{M\tilde{M}}(t, \tau) \varphi(\tau) d\tau = 0 \tag{15}$$

has no eigenfunction if and only if

$$\alpha(M\tilde{M}) = \mathcal{X}_{M\tilde{M}} \tag{16}$$

where $\alpha(M\tilde{M})$ is the number of linearly independent solutions of the equation $M\tilde{M}\varphi = 0$.

Proof. a) Necessity. Suppose that the equation (15) has only trivial solution. Then the homogeneous equation (14) is solvable without any condition. We have to prove that in this case, the equation

$$M\tilde{M}\varphi = f \tag{17}$$

is solvable for every right-hand side $f(t)$. Indeed, if we put :

$$f_1(t) = \widetilde{m}(t)f(t) - \frac{\widetilde{n}(t)\, Z(t)}{\pi i}\int_L \frac{f(\tau)}{Z(\tau)}\frac{d\tau}{\tau - t} + \widetilde{n}(t)\, Z(t) P_{\mathcal{X}_{M\widetilde{M}}-1}(t) \;,$$

then by the assumptions of the lemma there exists $\varphi(t) \in H^L(\omega)$ such that

$$\varphi(t) + \int_L {}_{M\widetilde{M}}(t, \tau)\, \varphi(\tau) d\tau = f_1(t),$$

i. e. $\varphi(t)$ is a solution of the equation (17). So (17) is solvable for every right-hand side $f(t) \in H^L(\omega)$ and $\beta(M\widetilde{M}) = 0$. This proves the equality (16).

b) Sufficiency. Suppose that the equality (16) holds and $f(t)$ is any function of $H^L(\omega)$. Put

$$\widetilde{m}(t)\, f(t) + \widetilde{n}(t)(sf)(t) \equiv g(t).$$

Since $f(t)$ is a solution of the equation (18), $f(t)$ must be of the form

$$f(t) = \widetilde{m}(t)cglt) - \frac{\widetilde{n}(t)Z(t)}{\pi i}\int_L \frac{g(\tau)}{Z(\tau)}\frac{d\tau}{\tau - t} + \widetilde{n}(t)\, Z(t)\, P'_{\mathcal{X}_{M\widetilde{M}-1}}(t),$$

The assumption implies that there exists a solution $\varphi^*(t)$ of the equation $(M\widetilde{M}\varphi)(t) = g(t)$. It is easy to see that $\varphi^*(t)$ is also the solution of the equation

$$\varphi(t) + \int_L \mathcal{L}_{M\widetilde{M}}(t, \tau)\varphi(\tau)d\tau = f(t).$$

This fact shows that the non-homogeneous Fredholm's equation (19) is solvable for every right-hand side. Therefore, the homogeneous equation (15) has only the trivial solution. The lemma is proved.

Remark 2.3. The class of all singular integral operators without shift satisfying the equality (16) of Lemma 2.2 is very wide, because it is well-known (see [1]) that in most cases the number of solutions of full equation is not greater than that of its corresponding characteristic equation. In particular, in the case of nonnegative index, the number of solutions of the characteristic equation coincides with the index.

Henceforth, the conditions of Lemma 2.2 are assumed to be satisfied. Then the equation (14) is equivalent to the non-linear singular integral equation with shift

$$\varphi(t) = f_1(t) - \int_L R_{M\widetilde{M}}(t, \tau)\, \varphi(\tau)d\tau, \tag{20}$$

where the function $R_{M\widetilde{M}}(t, \tau)$ is the resolvent of the equation (14) which is definitely represented via the kernel $\mathcal{L}_{M\widetilde{M}}(t, \tau)$. As we know (see [1], [6], [7]) the function $R_{M\widetilde{M}}(t, \tau)$ is a sum of iterated kernels and the iteration betters their properties. Therefore the functional properties of the resolvent are the same as those of the kernel. In other words, the operator

$$(B\varphi)(t) = \int_L R_{M\widetilde{M}}(t, \tau)\, \varphi(\tau)\, d\tau$$

is completely continuous on $H^L(\omega)$.

Denoting by $\mathcal{L}\varphi$ the operator defined by the right-band side of (20) we have

$$\mathcal{L}\varphi = \widetilde{m}(t)\,[(\mathcal{S}_f \widetilde{M}\varphi)(t) + (W\mathcal{S}_F \widetilde{M}\varphi)(t)] - \frac{\widehat{n}(t)Z(t)}{\pi i}\int_L \frac{(\mathcal{S}_f \widehat{M}\varphi)(\tau) + (W\mathcal{S}_F \widehat{M}\varphi)(\tau)}{Z(\tau)(\tau - t)}\, d\tau$$

$$+ \widehat{n}(t)Z(t)P\mathcal{X}_{M\widetilde{M}-1}(t) - \int_L R_{M\widetilde{M}}(t, \tau)\,\{\widetilde{m}(\tau)\,[(\mathcal{S}_F \widetilde{M}\varphi)(\tau) + (W\mathcal{S}_F \widehat{M}\varphi)(\tau)] -$$

$$- \frac{\widetilde{n}(\tau)Z(\tau)}{\pi i}\int_L \frac{(\mathcal{S}_F \widetilde{M}\varphi)(\tau_1) + (W\mathcal{S}_F \widetilde{M}\varphi)(\tau_1)}{Z(\tau_1)(\tau_1 - \tau)}\, d\tau_1 + \widetilde{n}(\tau)Z(\tau)\, P\mathcal{X}_{M\widetilde{M}-1}(\tau)\}\, d\tau \quad (22)$$

Since the operator $B\varphi$ is completely continuous in $H^L(\omega)$ for every $\varphi(t) \in H^L(R, K; \omega)$, we have

$$\| B\varphi \|_{H^L(\omega)} \leqslant M_\omega \|\varphi\|_{H^L(\omega)} \leqslant (R + K)M_\omega,$$

where M_ω is the norm of the linear operator. Now for every function $\Psi(t) \in H^L(R_{\widetilde{M}}, K_{\widetilde{M}}; \omega)$ we put

$$\Gamma(t) = (\mathcal{L}\Psi)(t).$$

Then with the aid of Lemmas 1.4, 1.5 and Theorem 1.9 we obtain

$$|\Gamma(t)| \leqslant A(R, K)\,|\Lambda| + B(R, K)\,|\mu| + (1 + B_R)\,\|\widetilde{n}\|_C \|Z\|_C M_P, \quad (23)$$

$$|\Gamma(t_1) - \Gamma(t_2)| \leqslant \{C(R, K)\,|\Lambda| + D(R, K)(\mu) + (1 + M_\omega)[\|\widetilde{n}\|_C \|Z\|_C \widetilde{M}_P +$$

$$+ H^L_\omega(\widetilde{n}Z)M_P] + M_\omega(1 + B_R)\|\widetilde{n}\|_C \|Z\|_C M_P\}\,\omega(|t_1 - t_2|), \quad (24)$$

where

$$M_P = \max_{t\in L} |P\mathcal{X}_{M\widetilde{M}-1}(t)|, \quad B_R = \max_{t\in L}\int_L |R_{M\widetilde{M}}(t, \tau)\, d\tau|, \quad \widetilde{M}_P = H^L_\omega(P\mathcal{X}_{M\widetilde{M}-1}),$$

$A(R, K)$, $B(R, K)$, $C(R, K)$, $D(R, K)$ being positive constants depending only on R and K.

If we choose

$$R > (1 + B_R)\,\|\widetilde{n}\|_c \|z\|_c M_p,$$

$$K > (1 + M_\omega)[\|\widetilde{n}\|_c \|z\|_c \widetilde{M}_p + H^L_\omega(\widetilde{n}z)\, M_p] + M_\omega(1 + B_R)\|\widetilde{n}\|_c \|z\|_c M_p,$$

then there exist numbers Λ and μ such that

$$A(R, K)\,|\Lambda| + B(R, K)\,|\mu| + (1 + B_R)\,\|\widetilde{n}\|_c \|z\|_c M_p \leqslant R, \quad (25)$$

$$C(R, K)\,|\Lambda| + D(R, K)\,|\mu| + (1 + \mu_\omega)[\|\widetilde{n}\|_c \|z\|_c \widetilde{M}_p + H^L_\omega(\widetilde{n}z)\, M_p] +$$

$$+ M_\omega(1 + B_R)\,\|\widetilde{n}\|_c \|z\|_c M_p \leqslant K \quad (26)$$

Let us endow $H^L(R, K; \omega)$ with the metric of the space of continuous functions

$$\rho_{HL}(u, v) = \rho_{C(L)}(u, v) = \max_{t \in L} |u(t) - v(t)| = \|u - v\|_c. \tag{27}$$

It is easy to see that, $H^L(R, K; \omega)$ is made into a closed, convex, compact and complete metric space.

LEMMA 2.4. *Suppose all assumptions of Theorem 1.8, Lemma 1.7 and the inequalities (7), (8), (9), (25), (26) are satisfied. Then the operator $\mathcal{L}_\varphi$ continuously acts on $H^L(R, K; \omega)$*

Proof. If $\varphi(t) \in H^L(R, K; \omega)$ then $\Psi(t) \in H^L(R_{\tilde{M}}, K_{\tilde{M}}; \omega)$, where $\Psi(t) = (\tilde{M}\varphi)(t)$. For that reason, from (23), (24) and the assumptions of the lemma it follows that $(\mathcal{L}_\varphi)(t) \in H^L(R, K; \omega)$. It remains to prove the continuity of the operator φ.

Putting

$$f_1(t, \tau, \varphi(\tau)) = f(t, \tau, \varphi(\tau)) - f(\tau, \tau, \varphi(\tau)) \tag{28}$$

we have

$$(\mathcal{S}_f\varphi)(t) = \Lambda\int_L \frac{f_1(t, \tau, \varphi(\tau))}{\tau - t}\, d\tau + \Lambda\int_L \frac{f(t, \tau, \varphi(\tau)}{\tau - t}\, d\tau = (\mathcal{S}_{f1}\varphi)(t) + \pi\Lambda(Sf_\varphi)(t),$$

where $f_\varphi(t) = f(t, t, \varphi(t))$.

For $\varphi_1(t), \varphi_2(t) \in H^L(\omega)$ and every fixed positive number ν of the interval $(0,1)$, the following inequalities holds

$$|f_1(t, \tau, \varphi_1(\tau)) - f_1(t, \tau, \varphi_2(\tau))| \leqslant 2A_f^{1-\nu}C_f^\nu[\omega^*(|\tau - t|)]^{1-\nu}|\varphi_1(\tau) - \varphi_2(\tau)|^\nu.$$

The Riss-Chvedelidze's theorem shows that S is continuous in the metric (27) i. e. $\|S\varphi\|_c \leqslant \|S\|_c\|\varphi\|_c$ for every $\varphi(t) \in H^L(\omega)$. On the other hand, it is not difficult to see that the integral

$$\int_L \frac{[\omega^*(|\tau - t|)]^{1-\nu}}{(\tau - t)}\,|d\tau|$$

converges. Indeed, by the property (iii) there exists $\alpha^* \in (0, 1)$ such that $\omega^*(r) \leqslant C_* r^{\alpha^*}$ for every $r \in (0, l)$. Consequently,

$$\int_L \frac{[\omega^*(|\tau - t|)]^{1-\nu}}{|\tau - t|}\,|d\tau| \leqslant C_*^{1-\nu} m^* \int_0^l \frac{r^{\alpha^*(1-\nu)}}{r}\,dr = C_*^{1-\nu} m^* \frac{l^{\alpha^*(1-\nu)}}{\alpha^*(1-\nu)}.$$

Putting

$$r_{1f}(\Lambda) = 2|\Lambda| A_f^{1-\nu}C_f^\nu \frac{m^* C_*^{1-\nu}}{\alpha^*(1-\nu)} l^{\alpha^*(1-\nu)}\quad r_{2f}(\Lambda) = |\Lambda|\pi\|S\|_c C_f,$$

we have

$$\rho_{H^L}(\mathcal{S}_f \varphi_1, \mathcal{S}_f \varphi_2) \leqslant r_{1f}(\lambda)\,[\rho_{H^L}(\varphi_1, \varphi_2)]^{\nu} + r_{2f}(\lambda)\, \rho_{H^L}(\varphi_1, \varphi_2)$$

Similarly

$$\rho_{H^L}(\mathcal{S}_F \varphi_1, \mathcal{S}_F \varphi_2) \leqslant r_{1F}(\mu)[\rho_{H^L}(\varphi_1, \varphi_2)]^{\nu} + r_{2F}(\mu)\, \rho_{H^L}(\varphi_1, \varphi_2).$$

It is obvious that the linear operator W is continuous in the metric (27) in $H^L(\omega)$. Furthermore, for the operator $\tilde{M}\varphi = a^{\alpha}\varphi - bW\varphi + cS\varphi - dWS\varphi$ the following estimate holds

$$\| \tilde{M}\varphi \|_c \leqslant (\| a \|_c + \| b \|_c + \| c \|_c \, \| s \|_c + \| d \|_c \, \| s \|_c) \| \varphi \|_c = \tilde{M}_c \, \| \varphi \|_c.$$

Finally we obtain

$$\rho_H L(\mathcal{L}\varphi_1, \mathcal{L}\varphi_2) \leqslant (1 + B_R)(\| \tilde{m} \|_c + \| \tilde{n} \|_c \, \| Z \|_c \, \| S \|_c / m_2)\{[r_{1f}(1) + r_{1f}(\mu)]\tilde{M}_c^{\nu} \times [\rho_H L(\varphi_1, \varphi_2)]^{\nu} + [r_{2f}(1) + s_{2f}(\mu)]\, \tilde{M}_c \, \rho_H L(\varphi_1, \varphi_2)\}.$$

This inequality shows that the operator $\mathcal{L}\varphi$ maps continuously $H^L(R, K; \omega)$ into itself. The lemma is proved.

Now we are in a position to prove the following

THEOREM 2.5. *Suppose that all assumptions of Theorem 1.8, Lemma 1.7 and the inequalities* (7), (8), (9) *are fulfiled. If* $|1|$, $|\mu|$ *sotisfy the inequalities* (25) (26) *then the equation* (2) *has solutions in the generalized Hölder space* $H^L(\omega)$.

Proof. From Lemmas 1.4, 1.5, 1.7, 2.1, 2.2, 2.4 and Shauder fixed point principle, it follows that the equation (20) and hence the equation (13) has at least a solution $\varphi(t) \in H^L(R, K : \omega)$. If we put $\psi(t) = (\tilde{M}\varphi)(t)$, then $\psi(t)$ is a solution of the equation (2) and belongs to the class $H^L(R_{\tilde{M}}, K_{\tilde{M}}; \omega)$. The theorem is proved.

So far we have assumed that the shift $\alpha(t)$ preserves the orientation of the curve L. The case where the shift $\alpha(t)$ changes the orientation of L can be treated analogously. Moreover let us note that the operator $SW + WS$ is an operator with regular kernel and the regular operator is

$$\tilde{M}\varphi = a^{\alpha}\varphi - bW\varphi - c^{\alpha}S\varphi - c^{\alpha}S\varphi - dWS\varphi,$$

where

$$\tilde{m}(t) - \hat{n}(t) = \Delta_1(t) = \Delta(t)$$

$$\tilde{m}(t) + \hat{n}(t) = \Delta_2(t) = \Delta(\alpha(t)),$$

$$\Delta(t) = (a^{\alpha}(t) + c^{\alpha}(t))\,(c(t) - a(t) - C(t) + d(t))\,(d^{\alpha}(t) - b^{\alpha}(t)) \neq 0$$

The author is greatly indebted to Nguyen Van Mau for his suggestion on the problem and his attention to this work.

REFERENCES

[1] Gakhov F. D. *Boundary value problems.* Moscow « Nauka » – 1977 (Russian).

[2] Guseinov A.I. *On a class of NSIE.* Izvestia Acad. Nauk SSSR Sc. Math. 12 (1948), 193-212 (Russian).

[3] Gochberg I. TS, Gruprik N. IA *Introduction to the theory of one dimensional singular integral operators.* Kishivev « Shtintsa » — 1973 (Russian).

[4] Guseinov A.I., Muehtarov H. Sh. *Introduction to the theory of nonlinear singular integral equations.* Moscow « Nauka » – 1980 (Russian).

[5] Litvinchuk G.S. *Boundary value problems and singular integral equations with shift.* Moscow « Nauka » – 1977 (Russian).

[6] Mihlin S.G. *Lectres on linear integral eqnations.* Moscow « Physmathgys » – 1959 (Russian).

[7] Mihlin S.G. *Linear partial differential equations.* Moscow « Vyshaja Shkola » — 1977 (Rnssian).

[8] Muskhelishvili N. I. *Singular integral equations.* MoscoW « Nauka » – 1968 (Russian).

[9] Mukhtarov H. M. *Study of a NSIE with special Cauchy's kernel in the class* $H^R_{\alpha-\sigma,\sigma}$ (a, b). Doklady Acad. Nauk. SSSR 182, 3 (1968), 497 — 499 (Russian).

[10] Mukhtarov V. M. *Study of NSIE in functional spaces with weight.* Func. Anal. 1982, 77 — 79 (Russian).

[11] Oinarov R., Otelbaev M. *On the resolvability of a class of NSIE.* Izvestia Acad. Nauk. Ka/SSR Phys — Math. 5 (1983), 44 — 46 (Russian).

Received November 6, 1988

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HANOI — VIETNAM.