ACTA MATHEMATICA VIETNAMICA
Vol. 14 No 2 (1989)

MARCINKIEWICZ-ZYGMUND STRONG LAWS OF LARGE NUMBERS FOR TWO-DIMENSIONAL ARRAYS OF BANACH SPACE VALUED RANDOM VARIABLES

NGUYEN VAN GIANG

I. INTRODUCTION

Throughout this paper let $(E, \|\cdot\|)$ be a real separable Banach space, $\mathfrak{B}(E)$ the σ-field of Borel sets of E.

By $\mathbf{N}$ we mean the set of all positive integers. Thus $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$ is a directed set with the usual order, that is $\alpha = (i, j) \leqslant \beta = (m, n)$ if $i \leqslant m$; $j \leqslant n$; and $\alpha < \beta$ if $i < m$, $j > n$.

In the present paper we study strongly measurable random variables X_{ij}, $(i, j) \in \mathbf{N}^2$, defined on a basic probability space $(\Omega, \mathfrak{F}, P)$ with values in E. We always assume that $\{X_{ij}, \ i, j = 1, 2, \ldots\}$ are Bochner integrable and then EX stands for the Bochner integral.

It is convenientfor us to recall a couple of concepts concerning the geometric structure of E.

Let $1 \leqslant p \leqslant 2$.

A Banach space E is said to be of Rademacher type p (or briefly, R-type p) if there exists a constant C such that for every $n \in \mathbf{N}$ and for all $x_1, \ldots, x_n \in E$

$$E \| \sum_{i=1}^{n} r_i x_i \| \leqslant C(\sum_{i=1}^{n} \| x_i \|^p)^{1/p},$$

where $\{r_i, \ i = 1, 2, \ldots\}$ is a Rademacher sequence, that is, a sequence of $\{-1, 1\}$-valued independent and identically distributed random variables with $P(r_i = \pm 1) = 1/2$.

Let E be a real separable Banach space. We shall say that l_p $(1 \leqslant p \leqslant 2)$ is finitely representable in E if for every $\varepsilon > 0$ and every $n \in \mathbf{N}$ there exist $x_1, \ldots, x_n \in E$ such that for all $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$

$$(\sum_{i=1}^{n} |\alpha_i|^p)^{1/p} \leqslant \| \sum_{i=1}^{n} \alpha_i x_i \| \leqslant (1 + \varepsilon) (\sum_{i=1}^{n} |\alpha_i|^p)^{1/p}.$$

For more information about the geometry of E, see [2], [6], [7].

Now, let $\{X_\alpha, \alpha \in \Lambda\}$ be a set of E-valued random variables indexed by a set Λ. $\{X_\alpha, \alpha \in \Lambda\}$ is said to have uniformly bounded tail probabilities by tail probabilities of a real random variable X (cf. [7]) if there exists $C > 0$ such that for every $t > 0$ and every $\alpha \in \Lambda$

$$P(\| X_\alpha \| > t) \leqslant C\, P(| X | > t).$$

In the case $\Lambda = \mathbf{N}$ we have the following result due to W.A. Woyczynski [7]:

THEOREM 1.1. *Let $1 \leqslant p < 2$. Then the following properties of a Banach space E are equivalent:*

(i) *1_p is not finitely representable in E;*

(ii) *For any sequence (X_i) of zero-mean independent random variables in E with tail probabilities uniformly bounded by tail probabilities of an $X \in L^p$ if $1 < p < 2$, $\in L \log^+ L$ if $p = 1$, the series*

$$\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}}$$

converges almost surely in norm,

(iii) *For any sequence (X_i) as in* (ii)

$$\frac{S_n}{n^{1/p}} \to 0 \text{ a.s., as } n \to \infty$$

where $S_1 = X_1 + \dots + X_n$.

The aim of this paper is to extend these results to two-dimensional arrays of E-valued random variables.

2. MARCINKIEWICZ — ZYGMUND'S TYPE STRONG LAWS OF LARGE NUMBERS

The following results have been well-known from the theory of numbers (see, for example, [3]).

LEMMA 2.1. *Let $\tau(n)$ denote the number of divisors of a positive integer n. Then we have*

(i)

$$\sum_{i=1}^{n} \tau(i) = \sum_{i=1}^{n} \left[\frac{n}{i}\right] \sim n \log n$$

for larger n. ($[x]$ denotes the integer part of x)

(ii)

$$\sum_{i=1}^{n} \frac{\tau(i)}{i} \sim (\log n)^2 .$$

for large n.

Using this lemma we can now prove:

LEMMA 2. 2. *Let X be a real random variable with $E|X| \log^+|X| < \infty$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \sum_{n=1}^{\infty} \tau(n) P(|X| \geqslant n) \leqslant E|X| \log^+|X| \leqslant C_2 \left(\sum_{n=1}^{\infty} \tau(n) P(|X|) \geqslant n) + 1\right). \quad (1)$$

Proof. Set $M(x) = \sum_{j \leqslant x} \tau(j)$, $x \geqslant 1$, $M(x) = 1$, $0 \leqslant x < 1$.

It is clear that

$$M(x) = \sum_{n=1}^{\infty} \tau(n) I_{[n,\infty)}(x) + I_{[0,1)}(x),$$

whence

$$E(M(|X|)) = \sum_{n=1}^{\infty} \tau(n) P(|X| \geqslant n) + P(0 \leqslant |X| < 1).$$

This inequality and assertion (i) of Lemma 2,1 prove (1).

Consider now an array $\{X_{ij}, i, j = 1,2,\dots\}$ of zero-mean independent random variables with values in E. The proofs of our main results will be based on the following

LEMMA 2.3. *Let $1 \leqslant p,q < 2$, $p \vee q = \max(p, q)$. Let $1_{p \vee q}$ be not finitely representable in E and let $\{X_{mn}, m,n = 1, 2\dots\}$ be E-valued zero-mean independent random variables with tail probabilities uniformly bounded by tail probabilities of a real random variable X. Setting*

$Y_{mn} = X_{mn} I(\|X_{mn}\| \leqslant m^{1/q} n^{1/p})$

we have

(i) *If $1 \leqslant p < q < 2$, then the condition*

$$E(|X|^q) < \infty$$

implies that the double series

$$\sum_{m,n} \frac{X_{mn} - EY_{mn}}{m^{1/q} n^{1/p}} \quad (2)$$

converges almost surely in norm.

(ii) *If $1 \leqslant p = q < 2$, then the condition*

$$E(|X|^p \log^+|X|) < \infty$$

implies the almost sure convergence of the following double series:

$$\sum_{m,n} \frac{X_{mn} - EY_{mn}}{(mn)^{1/p}}. \quad (3)$$

Before proving this lemma we recall some useful facts.

(1) Let $\{a_{mn}, m,n = 1,2,\dots\}$ be a sequence of elements in a Banach space E. Consider four following series

The double series

$$\sum_{m,n} a_{mn} \quad \text{(i)}$$

Two iterated series

$$\sum_m \sum_n a_{mn} = \sum_m (\sum_n a_{mn}), \tag{ii}$$

and

$$\sum_n \sum_m a_{mn} = \sum_n (\sum_m a_{mn}). \tag{iii}$$

The one-dimensional usual series

$$\sum_{i=1}^{\infty} a_i, \tag{iV}$$

where $\{a_i, i \in N\}$ is a rearrangement of the family $\{a_{mn}, m,n = 1,2,...\}$. It is well-known that the absolute convergence of one of the four above series implies the absolute convergence of the rest, and in the case where all of them have the same sum.

(2) The convergence concepts such as convergence in probability, in L^p, in distribution and almost sure convergence for two-dimensional arrays of E-valued random variables are defined similarly to those in the one-dimensional case. In the case when X_{mn}, $m,n = 1,2,...$ are independent random variables in E, one can prove that (c f.[1]) the convergence in L^p $(1 \geqslant p \leqslant 2)$ of the double series

$$\sum_{m,n} X_{mn}$$

yields its atmost sure convergence.

(3) A Banach space E is of stable-type p $(1 \leqslant p < 2)$ if and only if l_p is not finitely representable in E. And in this case, by Maurey-Pisier's Theorem, there exists $r > p$ such that E is of R-type r (see, for example, [2], [6].

We can now prove Lemma 2.3.

(i) Since

$$\begin{aligned}\sum_{m,n} P(X_{mn} \neq Y_{mn}) &= \sum_n \sum_m P(X_{mn} \neq Y_{mn}) \\ &= \sum_n \sum_m P(\|X_{mn}\| > m^{1/q} n^{1/p}) \\ &\leqslant C \sum_n \sum_m P(|X| > m^{1/q} n^{1/p}) \\ &= C \sum_n \sum_m P(|X/n^{1/p}| > m^{1/q})) \\ &\leqslant C E |X|^q \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} < \infty.\end{aligned}$$

Thus $\{X_{mn}, m, n, = 1, 2,...\}$ and $\{Y_{mn}, m, n = 1, 2,...\}$ are equivalent and to prove (2) it is sufficient to show that

$$\sum_{m,n} \frac{Y_{mn} - EY_{mn}}{m^{1/q} n^{1/p}}$$

converges almost surely.

Setting, for simplicity, $M = m^{1/q}\, n^{1/p}$ we have

$$\sum_n \sum_m \frac{E\| Y_{mn} - E\,Y_{mn}\|^r}{m^{r/q}\, n^{r/p}} \leqslant 2^{r-1} \sum_n \sum_m \frac{E\,\| Y_{mn}\|^r}{M^r}$$

$$= 2^{r-1} \sum_n \sum_m \frac{1}{M^r} \int_{\{\|X_{mn}\| \leqslant M\}} \|X_{mn}\|^r \, dP$$

$$= 2^{r-1} \sum_n \sum_m \frac{1}{M^r} \int_0^M t^r \, dP\,(\|X_{mn}\| \leqslant t)$$

$$= 2^{r-1} \sum_n \sum_m \frac{1}{M^r} \left[M^r\, P(\|X_{mn}\| \leqslant M) - r \int_0^M t^{r-1}\, P(\|X_{mn}\| \leqslant t)\, dt \right]$$

$$\leqslant 2^{r-1} \sum_n \sum_m \left[1 - \frac{r}{-M^r} \int_0^M t^{r-1}\, dt + \frac{1}{M^r} \int_0^M t^{r-1}\, P(\|X_{mn}\| > t)\, dt \right]$$

$$\leqslant 2^{r-1}\, C \sum_n \sum_m \frac{r}{M^r} \int_0^M t^{r-1}\, P\,(|X| > t)\, dt.$$

Replacing t by $s = (t/M)^r$ we get

$$\sum_n \sum_m \frac{r}{M^r} \int_0^M t^{r-1} P\,(|X| > t)\, dt$$

$$= \sum_n \sum_m \int_0^1 P(|X| > s^{1/r} M)\, ds$$

$$= \sum_n \sum_m \int_0^1 P\left(\frac{|X|}{s^{1/r}\, n^{1/p}} > m^{1/q} \right) ds$$

$$= \int_0^1 \sum_n \sum_m P\left(\frac{|X|}{s^{1/r}\, n^{1/p}} > m^{1/q} \right) ds$$

$$\leqslant E|X|^q \int_0^1 \frac{ds}{s^{q/r}} \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} < \infty.$$

Hence, this and the remark (2) above give the desired almost sure convergence of

$$\sum_{m,n} (Y_{mn} - E\,Y_{mn}) / (m^{1/q} n^{1/p}).$$

The first statement of the lemma is proved.

(ii) We have

$$\sum_{m,n} P(X_{mn} \neq Y_{mn}) = \sum_{m,n} P(\|X_{mn}\| > (mn)^{1/p})$$

$$\leqslant C \sum_{m,n} P(|X| > (mn)^{1/p}) = C \sum_{m,n} P(|X|^p > mn)$$

$$\leqslant C \sum_{j=1}^{\infty} \tau(j)\, P(|X|^p > j) \leqslant C_1\, E|X|^p \log^+ |X| < \infty,$$

that is, $\{X_{mn}\}$ is equivalent to $\{Y_{mn}\}$.

Now, we prove the convergence in L^p of the double series

$$\sum_{m,n} (Y_{mn} - EY_{mn}) / (mn)^{1/p}.$$

Proceeding as in the proof above we obtain

$$\sum_{m,n} E\,\|Y_{mn} - EY_{mn}\|^r / (mn)^{r/p} \leqslant 2^{r-1} C \int_0^1 \sum_{m,n} P\left(\frac{|X|^p}{s^{p/r}} > mn\right) ds$$

$$\leqslant 2^{r-1} C \int_0^1 ds \sum_{j=1}^{\infty} \tau(j)\, P\left(\frac{|X|^p}{s^{p/s}} > j\right)$$

$$\leqslant C_1 \int_0^1 E\left(\frac{|X|^p}{s^{p/r}} \log^+ \frac{|X|^p}{s^{p/r}}\right) ds$$

$$\leqslant C_2\, (E|X|^p \log^+ |X| + E|X|^p) < \infty.$$

This and the remark at the begining of the proof of this lemma yield.

The lemma is proved.

In what follows we shall need two morebasic lemmas. The first extends Toeplitz's theorem, while the second is a two-dimensional version of Kronecker's lemma.

LEMMA 2.4. *Let $\{P(m,n;\ i,j),\ i = 1,2,\ldots m;\ j = 1,2,\ldots, n;\ m,n = 1,2,\ldots\}$ be a set of real numbers satisfying the following conditions:*

(i) $P(m,n;\ i,j) = p(m,i)\, q(n,j)$ for all $i = 1,2,\ldots, m$; $j = 1,2,\ldots, n$; $m,n = 1,2,\ldots$;

(ii) $p(m,i) > 0$, $q(n,j) > 0$; and

$$\sum_{i=1}^{m} p(m,i) = 1, \qquad \sum_{j=1}^{n} q(n,j) = 1 \text{ for all } m, n = 1,2,\ldots;$$

(iii) for each fixed i $p(m,i) \to 0$ as $m \to \infty$; for each fixed j $q(n,j) \to 0$ as $n \to \infty$;

(iv) Let $\{b_{ij},\ i, j = 1, 2, \ldots\}$ be a norm — bounded set in a Banach space E with $\lim_{i,j\to\infty} b_{i,j} = b$.

Then we have

$$\lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{m} P(m, n; i, j)\, b_{ij} = b.$$

Proof. By (iv) for any $\varepsilon > 0$ there exist M, N such that

$$\| b_{ij} - b \| < \varepsilon \text{ for all } i > M,\ j > N \tag{2.1}$$

and there exists a constant B such that

$$\| b_{ij} \| \leqslant B \text{ for all } i, j = 1, 2, \ldots \tag{2.2}$$

In view of (ii)

$$\| \sum_{i=1}^{m} \sum_{j=1}^{m} P(m, n; i, j)\, b_{ij} - b \| = \| \sum_{i=1}^{m} \sum_{j=1}^{m} P(m, n; i, j)\, (b_{ij} - b) \|$$

$$\leqslant \sum_{i=1}^{M} \sum_{j=1}^{N} (P(m, n; i, j)\ \| b_{ij} - b \| + \sum_{i=1}^{M} \sum_{j=N+1}^{n} P(m, n; i, j)\ \| b_{ij} - b \|$$

$$+ \sum_{i=M+1}^{m} \sum_{j=1}^{N} P(m, n; i, j)\ \| b_{ij} - b \| + \sum_{i=M+1}^{m} \sum_{j=N+1}^{n} P(m, n; i, j)\, \| b_{ij} - b \|.$$

By (iii) and (2. 2) for large enough m, n the first sum in the right—hand side is less than ε.

Further, by (ii), (iii) and (2. 2) the second and the third ones will be less than ε when m and n are large enough.

Finally, by (2. 1) and (ii) the last sum is also less than ε, provided $m > M$, $n > N$.

Consequently, there exist M_o, N_o such that

$$\| \sum_{i=1}^{m} \sum_{j=1}^{n} P(m, n; i, j)\, b_{ij} - b \| < 4\varepsilon$$

for all $m_o > M$, $n > N_o$.

The lemma is proved.

LEMMA 2. 5. *Let E be a Banach space, $\{x_{ij}, i, j = 1, 2, \ldots\}$ be a set of elements in E and $\{a_{ij}, i, j = 1, 2, \ldots\}$ a set of real numbers satisfying the conditions*

$$a_{ij} = \alpha(i)\, \beta(j) \text{ with } 0 < \alpha(i),\ \beta(j) \uparrow \infty.$$

Then, the convergence of the series

$$\sum_{i,j} \frac{x_{ij}}{a_{ij}}$$

and the condition

$$\sup_{m, n} \| B_{mn} \| < \infty,$$

where

$$B_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{x_{ij}}{a_{ij}}$$

imply that

$$\frac{1}{a_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \to 0 \text{ as } m, n \to \infty.$$

Proof. Set

$$B = \sum_{i,j} \frac{x_{ij}}{a_{ij}}, \quad A_{mn} = \frac{1}{a_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}.$$

Writing $B_{ij} = 0$ if i or $j = 0$ we have

$$x_{ij} = a_{ij}(B_{ij} - B_{i-1j} - B_{ij-1} + B_{i-1\,j-1}),$$

and so

$$A_{mn} = \frac{1}{a_{mn}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}(B_{ij} - B_{i-1j} - B_{ij-1} + B_{i-1j-1})$$

$$= B_{mn} - \frac{1}{a_{mn}} \sum_{i=0}^{m-1} B_{in}(a_{i+1n} - a_{in}) +$$

$$+ \frac{1}{a_{mn}} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} B_{ij}(a_{i+1j+1} - a_{i+1j} - a_{ij+1} + a_{ij}) -$$

$$- \frac{1}{a_{mn}} \sum_{j=0}^{n-1} B_{mj}(a_{mj+1} - a_{mj}) =$$

$$= B_{mn} - \sum_{i=0}^{m-1} B_{in} \frac{\alpha(i+1) - \alpha(i)}{\alpha(m)} +$$

$$+ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} B_{ij} \frac{\alpha(i+1) - \alpha(i)}{\alpha(m)} \cdot \frac{\beta(j+1) - \beta(j)}{\beta(n)} -$$

$$- \sum_{j=0}^{n-1} B_{mj} \frac{\beta(j+1) - \beta(j)}{\beta(n)}.$$

Using the well known Toeplitz's theorem, Lemma 2.4 and the hypothesis we can show that all terms in the right-hand side of the last expression above converge to the same limit B. Thus

$$A_{mn} \to 0 \text{ as } m,n \to \infty.$$

The proof of the lemma is complete.

We now turn to the maint result of the present paper.

THEOREM 2.1. *Let $1 \leqslant p,q < 2$, E be a Banach space in which $1_{p \vee q}$ is not finitely representable. Let $\{X_{mn}, m,n = 1,2,\dots\}$ be a two-dimensional array of E-valued independent zero-mean random variables with tail probabilities uniformly bounded by tail probabilities of a real random variable X such that $E|X|^q < \infty$ if $1 \leqslant p < q < 2$, $E|X|^p \log^+|X| < \infty$ if $1 < p = q < 2$, and $E|X|(\log^+|X|)^2 < \infty$ if $p = q = 1$. Then*

(i) *The double series*

$$\sum_{m,n} \frac{X_{mn}}{m^{1/q}n^{1/p}}$$

converges almost surely in norm.

(ii)

$$\frac{1}{m^{1/q}\,n^{1/p}} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} \to 0 \qquad \text{a.s.}$$

Proof. We need only prove (i) because by Lemma 2.5 (ii) follows immediately from (i). Moreover, in view of Lemma 2.3 it is sufficient to prove the absolute convergence of the double series

$$\sum_{m,n} \frac{E\,Y_{mn}}{m^{1/q}n^{1/p}}.$$

We shall distinguish three cases.

1. *Case* $1 \leqslant p < q < 2$. Since $E\,X_{mn} = 0$, setting $M = m^{1/q}n^{1/p}$ we have

$$\sum_n \sum_m \frac{\|E\,Y_{mn}\|}{m^{1/q}n^{1/p}} \leqslant \sum_n \sum_m \frac{E\,\|Y_{mn}\|}{M}$$

$$= \sum_n \sum_m \frac{1}{M} \int_M^{\infty} t dP\,(\|X_{mn}\| \leqslant t)$$

$$= -\sum_n \sum_m \frac{1}{M} \int_M^{\infty} t dP\,(\|X_{mn}\| > t)$$

$$= \sum_n \sum_m \left[P(|X_{mn}| > M) + \frac{1}{M}\int_M^{\infty} P(\|X_{mn}\| > t)\,dt\right]$$

$$\leqslant C \sum_n \sum_m \left[P(|X| > M) + \int_M^{\infty} P(|X| > t)\,\frac{dt}{M}\right]$$

$$= C \sum_n \sum_m \left[P(|X| > m^{1/q}\,n^{1/p}) + \int_1^{\infty} P(|X| > sm^{1/q}\,n^{1/p})\,ds\right]$$

$$\leqslant C\,E\,|X|^q \left(1 + \int_1^{\infty} \frac{ds}{s^q}\right) \sum_n \frac{1}{n^{q/p}} < \infty.$$

2. *Caes* $1 < p = q < 2$.

Proceeding as in the proof above we get in this case

$$\sum_{m,n} \| E\, Y_{mn} \| / (mn)^{1/p}$$

$$\leqslant C \sum_{m,n} [P(|X| > (mn)^{1/p}) + \int_1^\infty P(|X| > s(mn)^{1/p})ds]$$

$$< C_1 E |X|^p \log^{\pm} |X| < \infty.$$

3. *Case* $p = p = 1$.

Since $E\, X_{mn} = 0$, by integration by parts we obtain

$$\sum_{m,n} \| EY_{mn} \| / (mn)$$

$$\leqslant \sum_{m,n} \frac{1}{mn} \int_{mn}^{\infty} t dP(\| X_{mn} \| \leqslant t)$$

$$= \sum_{m,n} \left[P(\| X_{mn} \| > mn) + \frac{1}{mn} \int_{mn}^{\infty} P(\| X_{mn} \| > t)\, dt \right]$$

$$\leqslant C_1 \left[E |X| \log^+ |X| + \sum_{m,n} \frac{1}{mn} \sum_{j=mn}^{\infty} P(|X| > j) \right]$$

$$= C_1 \left[E |X| \log^+ |X| + \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{\tau(i)}{i} P(|X| > j) \right]$$

$$\sim C_1\, E |X| \log^+ |X| + \sum_{j=1}^{\infty} (\log j)^2 P(|X| > j)$$

$$\leqslant C[E |X| \operatorname{Log}^+ |X| + E |X| (\log^+ |X|^2] < \infty.$$

The proof of the theorem is complete.

Acknowledgement. The author would like to thank Prof. Nguyen Zuy Tien for introducing him to the subject and Prof. Dinh Quang Luu for useful discussions.

REFERENCES

[1] J-P; Gabriel, *An inequality for sums of independent random variables indexed by finite dimensional filtering sets and its applications to the convergence of the series*, Ann. Probab., 1977, 5, (1977), 779—786.

[2] B.Maurey et G.Pisier, *Séries des variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58, (1976), 58, 45—90.

[3] G.Polya und G.Szego, *Aufgaben und lehrsatze aus der analysts* 1926, Berlin, Verlag von Julius Springer (in Russian, M. 1956).

[4] R.T.Smythe, *Strong Laws of Large Numbers for r-dimensional arrays of random variables, Ann. Probab.*, 1 (1973), № 1, 164—170.

[5] R.T.Smythe, *The sums of independent random variables on the partially ordered sets*, Ann. Probab. 2 (1974), 5, 906—917.

[6] W.AW.hyczynski, *Geometry and martingales in Banach spaces, Part II Independent increments*, Advances in Probability 4, 1978, 267—518.

[7] W.A.Woyczynski, *On Marcinkiewicz-Zygmund laws of large numbers in Banach spaces and related rates of convergence, Probability and Mathematical Statistics*, 1 (1980), F.2, 117—131.

Received July 15, 1988

NHÀ XUẤT BẢN ĐẠI HỌC
FACULTY OF MATHEMATICS UNIVERSITY OF HANOI