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ON THE PROCEDURE OF MULTIDIMENSIONAL QUANTIZATION

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INTRODUCTION

Let G be a connected and simply connected Lie group. In order to find irreducible unitary representations of G, Kirillov's orbit method furnishes a procedure of quantization, starting from linear bundles over a G-homogeneous symplectic manifold (see [5, § 15]). In [1] and [2] Do Ngoc Diep has proposed a new procedure of quantization for the general case, starting from arbitrary irreducible G-bundles associated with the given hamiltonian mechanical system. This new procedure of quantization of Do Ngoc Diep gives us a large number of irreducible representations of G.

On the other hand, in 1980, M. Duflo ([4]) proposed three methods for constructing large subsets of the unitary dual of a Lie group. Each of them reduces Kirillov's orbit method to a specia lcontext. The first is a reduction to the case of discrete groups, with the procedure of quantization of Do Ngoc Diep as its geometrical model.

The aim of this paper is to suggest a reduction of the procedure of multidimensional quantization to the case of Lie groups whose Lie algebra is either semi-simple or reductive. Our geometrical constructions will be based on some ideas of M. Duflo (see [4]).

By using a new notion of polarization we will construct unitary representations of G by the aid of solvable or unipotent co-isotropic distributions. We will modify the usual construction of holomorphically induced representations. The representations thus obtained will be called partially invariant holomorphically induced representations and denoted by Ind $(G; \widetilde{L}, B, \sigma_0)$. They will then be illustrated as representations obtained from a natural generalization of Kirillov's procedure of quantization.

1. CO-ISOTROPIC DISTRIBUTIONS

Let us denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ its dual space, The group G acts in $\mathcal{G}^*$ by the coadjoint representation. We will simply call it a K-representation. Let $F \in \mathcal{G}^*$ be an arbitrary point in an orbit Ω, and G_F be the stabilizer of this point. Denote by $\mathcal{G}_F$ its Lie algebra, $\mathfrak{R}(F)$ the radical of $\mathcal{G}_F$ and R_F the corresponding analytic subgroup in G.

1.1. SOLVABLE CO-ISOTROPIC DISTRIBUTION.

Let S_F be the semi-simple component of G_F in its Cartan-Levi-Maltsev's decomposition $G_F = R_F . S_F$.

From the local triviality of the S_F-principal bundle $S_F \rightarrowtail R_F \setminus G \xrightarrow{k} G_F \setminus G$, where the base $\Omega \cong G_F \setminus G$ is paracompact, there exists on the bundle a connection Γ (see [6, Ch. 2, § 2]).

This means in particular that we obtain a fixed decomposition of the tangent bundle into horizontal and vertical parts

$$T(R_F \setminus G) = T_H(R_F \setminus G) \oplus T(R_F \setminus G).$$

Then the Kirillov 2-form B_Ω of K-orbit Ω induces a nondegenerate closed G-invariant 2-form $\widetilde{B}_\Omega$ on $T_H(R_F \setminus G)$ defined by the formula

$$\widetilde{B}_\Omega(f)(\widetilde{X}, \widetilde{Y}) = B_\Omega(F)\ (k_*\widetilde{X},\ k_*\widetilde{Y}),$$

where $f \in R_F \setminus G$, $k(f) = F \in \Omega$, and k_* is the linear lifting isomorphism induced from k.

DEFINITION 1.1. A smooth tangent distribution $\widetilde{L} \subset T(R_F \setminus G)$ in called a solvable co-isotropic distribution iff

i) $\widetilde{L}$ is integrable, G-invariant

ii) $\widetilde{L}$ is invariant under the action Ad of G_F.

iii) $\widetilde{L}$ is horizontal (i.e. $\widetilde{L} \subset T_H(R_F \setminus G)$).

iv) $\widetilde{L}$ is co-isotropic at $f \in R_F \setminus G$ such that $k(f) = F$ with respect to $\widetilde{B}_\Omega$, i.e.

$$(\widetilde{L}_f)^f \subset \widetilde{L}_f,$$

where $(\widetilde{L}_f)^f$ is the set of all elements $\widetilde{X} \in T_H(R_F \setminus G)$ such that $\widetilde{B}_\Omega(f)\ (\widetilde{X}, \widetilde{Y}) = 0$,

$$\forall \widetilde{Y} \in \widetilde{L}_f .$$

It follows from the definition that if $\widetilde{L}$ is co-isotropic at one point $f \in R_F \setminus G$, then all other points of $R_F \setminus G$ also have this property.

THEOREM 1.1 *There exists a one-to-one correspondence between solvable co-isotropic distributions and Ad G_F-invariant co-isotropic subalgebras of the Lie algebra $\mathcal{G}$.*

Proof. Let $\tilde{L} \subset T(R_F \setminus G)$ be a solvable co-isotropic distribution. According to the Frobenius theorem, $\tilde{L}_f$ is a subalgebra of $T_{(f)H}(R_F \setminus G)$, and then $L_F = k_* \tilde{L}_f$ is a subalgebra of $T_F\Omega \cong \mathcal{G}_F$. Thus, the inverse image $\mathcal{B}$ of subalgebra $\mathcal{L} = L_F$ under the natural projection

$$p: \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_F \cong T_F\Omega$$

is an Ad G_F-invariant subalgebra of $\mathcal{G}$.

Denote by $\mathcal{B}^F$ the orthogonal component of $\mathcal{B}$ in $\mathcal{G}$ with respect to Kirillov's form B_F. Now we verify the coisotropic property of $\mathcal{B}$. Indeed, let

$X \in \mathcal{B}^F$. We have $B_F(X, Y) = 0$ for all $Y \in \mathcal{B} = p^{-1}(\mathcal{L})$, then

$B_\Omega(F)(\overline{X}, \overline{Y}) = 0$ for all $\overline{Y} \in \mathcal{L} = L$ and $\overline{X} \in T_F\Omega$.

This means $B_\Omega(F)(k_*\tilde{X}, k_*\tilde{Y}) = 0$ for every $\tilde{Y} \in \tilde{L}_f$ and $\tilde{X} \in T_{(f)H}(R_F \setminus G)$.

Therefore, $\tilde{X} \in (\tilde{L}_f)^f \subset \tilde{L}_f$ since $\tilde{L}$ is co-isotropic. It follows that $\overline{X} = k_*\tilde{X} \in L_F = \mathcal{L}$. Thus $X \in \mathcal{B} = p^{-1}(\mathcal{L})$ and $\mathfrak{B}^F \subset \mathfrak{B}$. This shows that $\mathfrak{B}$ is a co-isotropic subalgebra.

Suppose now that $\mathfrak{B} \subset$ of is an Ad G_F-invariant co-isotropic subalgebra. We define a smooth distribution $L \subset T\Omega$ by the formula

$$L_F = p(\mathfrak{B}),$$

$$L_{F'} = K(g).L_F; \ \forall F' = K(g)F \in \Omega.$$

By definition L is integrable, G-invariant and Ad G_F-invariant.

Denote by $\tilde{L}$ the horizontal lifting of L into the tangent bundle $T(R_F \setminus G)$; i. e. $\tilde{L} \subset T_H(R_F \setminus G)$ and $k_*(\tilde{L}) = L$. Then $\tilde{L}$ is G-invariant, integrable, horizontal and Ad G_F-invariant.

Now we verify that $\tilde{L}$ is co-isotropic. Indeed, for every $\tilde{X} \in (\tilde{L}f)^f$ we have $\tilde{B}_\Omega(f)(\tilde{X}, \tilde{Y}) = 0; \forall \tilde{Y} \in \tilde{L}_f$. By definition, this means

$$B_\Omega(F)(k_*\tilde{X}, k_*\tilde{Y}) = 0; \ \forall k_*\tilde{Y} \in L_F$$

or $\langle F, [X, Y] \rangle = 0$; $\forall Y \in \mathcal{B}$ such that $\overline{Y} = k_*\tilde{Y}$. Hence, $X \in \mathcal{B}^F \subset \mathcal{B}$ since $\mathcal{B}$ is co-isotropic. Thus $\overline{X} \in L_F = k_*\tilde{L}_f$. The latter means

$$\tilde{X} \in k_*^{-1}(L_F) \cap (T_{(f)H}(R_F \setminus G)) = \tilde{L}_f.$$

Then, we have $(\tilde{L}_f)^f \subset \tilde{L}_f$. The theorem is proved.

Suppose that G is an algebraic Lie group. Let $F \in \mathcal{G}^*$. Denote by U_F the unipotent radical of G_F and by, $\mathcal{U}(F)$ its Lie algebra. Let Q_F be the reductive component of G_F in its Cartan—Levi's decomposition $G_F = U_F . Q_F$.

Using the Q_F — priucipal bundle $Q_F \longrightarrow U_F \setminus G \longrightarrow G_F \setminus G$ we can construct a nondegenerate closed G-invariant 2-form $\tilde{B}_\Omega$ on the horizontal component $T_H(U_F \setminus G)$ of $T(U_F \setminus G)$.

DEFINITION 1.2 A smooth distribution $L \subset T(U_F \setminus G)$. is called a *unipotent co-isotropic distribution* if it is integrable, G-invariant, invariant under the action Ad of G_F, horizontal and co-isotropic at f with respect to $\tilde{B}_\Omega$.

The following theorem can be proved in a similar way as Theorem 1.1.

THEOREM 1.2 *There is a one-to-one correspondence between unipotent co-isotropic distribution and Ad* G_F — *invariant co-isotropic subalgebras of the Lie algebra* $\mathcal{G}$.

2. (σ, η_F) -POLARIZATIONS

DEFINITION 2.1. A point $F \in \mathcal{G}^*$ is called *r-admissible* (r for radical) if the e exists a character η_F (i.e. one-dimensional representation) of R_F such that its derivative is the restriction of $\sqrt{-1} . F$ to $\mathcal{R}(F)$.

Denote by $Y^{irr}(F)$ the set of all equivalent classes of irreducible unitary representations of G_F such that the restriction of each of them to R_F is a multiple of the representaion η_F. Then there is a one-to-one correspondence between $Y^{irr}(F)$ and the set of all equivalent classes of irreducible projective representations of the group $R_F \setminus G_F$ (its Lie algebra is semi-simple).

Remark and definition 2.1' Let G be an algebraic Lie group. Note that the Lie subalgebra $\mathcal{U}(F)$ of the unipotent radical U_F is unipotent. So, from the diffeomorphic property of exponential map it follows that there exists a character θ_F of U_F such that its derivative is the restriction $\sqrt{-1} . F \mid \mathcal{U}(F)$. Hence we can say that every point $F \in \mathcal{G}^*$ is u *-admissible*.

Denote by $Z^{irr}(F)$ the set of all equivalent classes of irreducible unitary representations of G_F such that the restriction of each of them to U_F is a multiple of the character θ_F. There is a one-to-one correspondence between $Z^{irr}(F)$

and the set of all equivalent classes of irreducible projective representations of $U_F \backslash G_F$.

Since the reductive component Q_F of G_F has only a trivial covering, we can identify $Z^{irr}(F)$ with a subset of the set of all equivalent classes of irreducible unitary representations of Q_F (see [4]).

In order to find irreducible representations, we consider the important generalization of the co-isotropic distribution by « going over to the complex domain ». This means that we define the co-isotropic distribution $\tilde{L}$ in such a way that $\tilde{L}_f$ is a complex subspace of $(T_{(f)H}(R_F \backslash G))_{\mathbf{C}}$.

Then Theorems 1.1 and 1.2 are also valid for the complex case.

Let $\tilde{L} \subset (T_H(R_F \backslash G))_{\mathbf{C}}$ be a co-isotropic distribution such that $\tilde{L} \cap \bar{\tilde{L}}$ and $\tilde{L} + \bar{\tilde{L}}$ are the complexifications of some real distributions. In this case, the corresponding complex subalgebra $\mathcal{P} \subset \mathcal{G}_{\mathbf{C}}$ (see Theorem 1.1) satisfies the condition: $\mathcal{P}^F \cap \overline{\mathcal{P}}^F$ and $\mathcal{P}^F + \overline{\mathcal{P}}^F$ are the complexifications of the real Lie subalgebras $\mathcal{P}^F \cap \mathcal{G}$ and $\mathcal{M}^F = (\mathcal{P}^E + \overline{\mathcal{P}}^E) \cap \mathcal{G}$. Denote by B^F and M^F the corresponding analytic subgroups in G.

Similarly, we construct the complex unipotent co-isotropic distribution $\tilde{L} \subset (T_H(U_F \backslash G))_{\mathbf{C}}$. Now suppose that the subalgebra $\mathfrak{B} = \mathcal{P} \cap \mathcal{G}$ is algebraic co-isotropic.

DEFINITION 2.2. A solvable (resp. unipotent) co-isotropic distribution $\tilde{L}$ is called *closed* iff all the subgroups B^F, M^F, and the semi-direct products $B = G_F.B^F$, $M = G_F.M^F$ are closed in G.

DEFINITION 2.3. Let $\tilde{\delta}$ be some fixed irreducible unitary representation of G_F in a separable Hilbert space $\tilde{V}$ such that its restriction to R_F(resp. U_F) is a multiple of the character η_F (resp. θ_F).

The triplet $(\tilde{L}, \rho, \sigma_0)$ is called *a* $(\tilde{\sigma}, \eta_F)$*-solvable* (resp. $(\tilde{\sigma}, \theta_F)$*-unipotent*) polarization. $\tilde{L}$ is called *a weakly Lagrange distribution* iff

(i) σ_0 is an irreducible representation of the group B^F in a Hilbert space V' such that

(a) The restriction $\sigma_0 \big|_{G_F \cap B^F} = \tilde{\sigma} \big|_{G_F \cap B^F}$.

(b) The point σ_0 in the dual $\widehat{B}_F$ is fixed under the natural action of the subgroup G_F. Recall that the subgroup G_F normalizes B^F, thus G_F acts naturally on the dual $\widehat{B^F}$ of the subgroup B^F, (see [1])).

(ii) ρ is a representation of the complex Lie subalgebra $\mathcal{P}^F$ in V' which satisfies E. Nelson's condition and $\rho \mid_{\mathfrak{B}_F} = d\sigma_0$ where $d\sigma_0$ is the representation of the Lie subalgebra $\mathfrak{B}^F$ in V', corresponding to σ_0.

PROPOSITION 2.1. *Suppose that $F \in$ of* is r-admissible, $\widetilde{L}$ is closed and $(\widetilde{L}, \rho, \sigma_0)$ is either a $(\widetilde{\sigma}, \eta_F)$-solvableor $(\widetilde{\sigma}, \theta_F)$-unipotent) polarization. Then*

1) *There exists a structure of mixed manifold of type (k, l) on the space B/G, where $k = \dim G - \dim M$, $l = \frac{1}{2}(\dim M - \dim B)$.*

2) *There exists a unique irreducible representation σ of the subgroup $B = G_F . B^F$ such that*

$\sigma \mid G_F = \widetilde{\sigma}$, $\sigma|_{B^F} = \sigma_0$ and $\rho \mid_{\mathfrak{B}^F} = d\sigma$.

PROOF. 1) The assertion follows from Theorem 1 in [5, § 13.4]

2) Notice that $\mathfrak{B}$ is invariant under the action Ad of G_F and G_F acts naturally on the dual $\widehat{B^F}$ of the subgroup B^F. From the assumptions, σ_0 is fixed under the natural action of G_F. The formula

$$(x, b) \overset{\tau}{\mapsto} \widetilde{\sigma}(x)\, \sigma_0(b)\, ;\; x \in G_F,\; b \in B^F,$$

defines a representation of the product $G_F \times B^F$ in the space $V = \widetilde{V} \otimes V'$.

Indeed, since σ_0 is fixed under the natural action of G_F in $\widehat{B^F}$ we have

$$\begin{aligned}\tau(x, b)\, \tau(x', b') &= \widetilde{\sigma}(x)\, \sigma_0(b)\, \widetilde{\sigma}(x')\, \sigma_0(b') \\ &= \widetilde{\sigma}(xx')\, [\widetilde{\sigma}(x')^{-1} \sigma_0(b) \widetilde{\sigma}(x')]\, \sigma_0(b') \\ &= \sigma(xx')\, [\sigma_0(b)\, \sigma_0(b')] \\ &= \sigma(xx')\, \sigma_0(bb')).\end{aligned}$$

On the other hand, by definition we have $\tau((x, b), (x'\, b')) = \tau(xx', bb') = \widetilde{\sigma}(xx', \sigma_0(bb')$. It is clear that the representation τ is trivial on the kernel of the surjection

$$\begin{aligned} G_F \times B^F &\longrightarrow B = G_F .\; B^F, \\ (x, b) &\longrightarrow x . b. \end{aligned}$$

Thus, there exists a unique representation of the semi-direct product $B = G_F . B^F$. We denote this representation by σ. Obviously, σ is an irreducible representation and $\sigma \mid G_F = \widetilde{\sigma}$, $\sigma|_{B^F} = \sigma_0$. The proposition is proved.

2. INDUCED REPRESENTATION OBTAINED FROM THE SOLVABLE OR UNIPOTENT POLARIZATION

Suppose that $\sigma : B \to AutV$ is the representation obtained in Proposition 2.1. Denote by $\overline{\mathcal{E}}_{V,\sigma} = G_{B,\sigma} x V$ the smooth G-bundle over $B\backslash G$ associated with σ (see [5, § 13]). Similarly, we a construct the G-bundles $\mathcal{E}_{V,\sigma} = G \underset{G_F, \sigma_{G_F}}{\times} V$ and

$$\widetilde{\mathcal{E}}_{V,\sigma} = G \underset{R_F, \sigma_{R_F}}{\times} V.$$

To obtain an unitary representation we apply the usual construction of unitary G-bundle (see[1], [5, § 13.2]).

Suppose that $\triangle_G$, $\triangle_B$ are the modular functions of the groups G and B, respectively. Let $\delta^2(h) = \triangle_B(h)/\triangle_G(h)$ $h \in B$ be the non-unitary character of B. We consider the G-bundle $\mu = G \underset{R_F, \delta^2}{\times} \mathbf{C}$ associated with the non-unitary character δ^2 of the subgroup R_F. Denote by $\mu^{1/2} = G \underset{R_F, \delta}{\times} \mathbf{C}$ the G-bundle associated with the character $\delta = (\triangle_B/\triangle_G)^{1/2}$. The bundle $\widetilde{\mathcal{E}}_{V\delta,\sigma} = \widetilde{\mathcal{E}}_{V,\sigma} \otimes \mu^{1/2}$ is a G-bundle over $R_F\backslash G$. It is called an unitarization of $\widetilde{\mathcal{E}}_{V,\sigma}$.

According to the construction, the unitary G-bundle $\widetilde{\mathcal{E}}_{V,\delta\sigma}$ can be identified with the set of pairs $(g, v) \in G \times V$ factorized by the equivalence relation: $(g, v) \sim (g', v')$ iff there exists $h \in R_F$ such that $g' = hg$ and $v' = \delta(h)\sigma(h)v$. Then we have an isomorphism of vector spaces (see [5, § 13]):

$$\Gamma(\widetilde{\mathcal{E}}_{V,\delta\sigma}) \cong C^\infty(G : V, \delta\sigma, R_F)$$
$$\tilde{s} \mapsto f_{\tilde{s}},$$

where $C^\infty(G ; V, \delta\sigma, R_F)$ is the space of smooth functions f on G taking values in V and satisfying the following equation

$$f(hg) = \delta(h)\sigma(h)f(g) : \forall h \in R_F,\ g \in G.$$

Similarly, we obtain the unitarization $\mathcal{E}_{V,\delta\sigma}$ and $\mathcal{E}_{V,\delta\sigma}$ of $\mathcal{E}_{V,\sigma}$ and $\mathcal{E}_{V,\sigma}$, respectively.

A section $\tilde{s} \in \Gamma(\widetilde{\mathcal{E}}_{V,\delta\sigma})$ is said to be S_F-equivariant iff $f_{\tilde{s}}(hg) =$

$$= \sigma(h)\,\sigma(h) f_{\tilde{s}}(g),\ \forall h \in S_F,\ g \in G.$$

PROPOSITION 3.1. *There exists isomorphisms of vector spaces*

$$\Gamma_{S_F}(\widetilde{\mathcal{E}}_{V,\delta\sigma}) \cong \Gamma(\mathcal{E}_{V,\delta\sigma})$$

and

$$\Gamma_{S_F \cdot B^E}(\widetilde{\mathcal{C}}_{V,\delta\sigma}) \cong \Gamma(\overline{\mathcal{C}}_{V,\delta\sigma},$$

where $\Gamma_{S_F}(\widetilde{\mathcal{C}}_{V,\delta\sigma})$ *and* $\Gamma_{S_F \cdot B^F}(\widetilde{\mathcal{C}}_{V,\delta\sigma})$ *are the vector spaces of* S_F*-equivariant and* $S_F \cdot B^F$ *-equivariant sections of bundle* $\widetilde{\mathcal{C}}_{V,\delta\sigma}$, *respectively.*

Proof. The assertion follows from the definition of S_F -equivariant section and the construction of the unitary G-bundles $\widetilde{\mathcal{C}}_{V,\delta\sigma}$ and $\mathcal{C}_{V,\delta\sigma}$.

The space $\Gamma_{S_F}(\widetilde{\mathcal{C}}_{V}, \delta\sigma)$ is too large to yield an irreducible representation. By dint of using the $(\widetilde{\sigma},\eta_F)$-solvable polraization $(\widetilde{L}, \rho, \sigma_0)$ we will restrict this space. From Proposition 2.1 it follows that if $\widetilde{s} \in \Gamma_{S_F \cdot B^F}(\widetilde{\mathcal{C}}_{V}, \delta\sigma)$ then $\| \widetilde{S} \|_V^2$ is a B-equivariant section of μ. Then the integral

$$\| \widetilde{S} \|^2 = \int_{B\setminus G} \| \widetilde{S} \|_V^2 \, d\mu_{B\setminus G}(x)$$

is justified and we can define the scalar product of every pair of sections of this type by the formula

$$\langle \widetilde{s_1}, \widetilde{s_2} \rangle = \int_{B\setminus G} \langle \widetilde{s_1}(x), \widetilde{s_2}(x) \rangle_V \, d\mu_{B\setminus G}(x).$$

By fixing a connection $\overline{\Gamma}$ on the principal bundle $B \rightarrowtail G \twoheadrightarrow B/G$, we obtain the connection $\overline{\nabla}$ on the bundle $\overline{\mathcal{C}}_{V,\delta\sigma}$ Using the natural projection π: $G_F\setminus G \to B\setminus G$ and the projection k: $R_F\setminus G \to G_F\setminus G$ (see[6, §6]), we obtain the connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{C}}_{V,\delta\sigma}$. The following diagram holds

$$\begin{array}{ccccccc} & (\mathcal{C}_{V,\delta\sigma}; \overline{\nabla} & & (\mathcal{C}_{V,\delta\sigma}; \nabla) & & (\widetilde{\mathcal{C}}_{V,\delta\sigma}; \widetilde{\nabla}) & G \times V \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ B \overset{\overline{\Gamma}}{\rightarrowtail} G \twoheadrightarrow B\setminus G & & \overset{\pi}{\longleftarrow} & G_F\setminus G & \overset{k}{\longleftarrow} & R_F\setminus G & \longleftarrow - G \end{array}$$

DEFINITION 3.1. A section $\widetilde{s} \in \Gamma_{S_F \cdot B^F}(\widetilde{\mathcal{C}}_{V,\delta\sigma})$ is called *partially invariant partially holomorphic* if its corresponding function $f_{\widetilde{s}}$ satisfies the following equation

$$[L_X + \rho(x) + d\sigma(x)] f = 0; \forall x \in \mathcal{P} F.$$

Denote by $\mathcal{H}$ the Hilbert space which is the completion of the space of all partially invariant partially holomorphic square-integrable section $\widetilde{s}$ of G-bundle $\widetilde{\mathcal{C}}_{V,\delta\sigma}$.

The natural unitary representation of G in $\mathcal{H}$ will be called the partially invariant holomorphically induced representation and denoted by Ind $(G; \tilde{L}, B, \rho, \sigma_0)$.

Remark. Suppose that $\sigma: B \to AutV$ is the representation obtained in Proposition 2.1. Let G be an algebraic Lie group. Then we can construct the smooth G—bundles

$$\overline{\mathcal{E}}_{V,\sigma} = G \underset{B,\sigma}{\times} V, \quad \mathcal{E}_{V,\sigma} = G \underset{G_F, \sigma_{G_F}}{\times} V, \quad \hat{\mathcal{E}}_{V,\sigma} = G \underset{U_F, \sigma_{U_F}}{\times} V$$

The construction of the unitary G—bundle discussed in the preceeding paragraph gives us an unitary representation of G. We also denote this unitary representation by Ind $(G; \tilde{L}, B, \rho, \sigma_0)$.

THEOREM 3. 1. *The representation Ind $(G; \tilde{L}, B, \rho, \sigma_0)$ of the Lie group G in the space $\mathcal{H}$ is equivalent to the representation of this group by right translations in the space $C^\infty(G : \tilde{L}, B, \rho, \sigma_0)$ of smooth functions f on G with values in V satisfying the following system of equations*

$$(1) \quad f(hg) = \delta(h)\sigma(h)f(g); \forall h \in B = G_F.B^F, \ g \in G$$

$$(2) \quad [L_x + \rho(x) + d\delta(x)]f = 0; \ \forall x \in \gamma^F,$$

where L_x is the Lie derivation along the vector ield on G corresponding to x.

PROOF. According the definition, the partially invariant partially holomorphic sections $\tilde{s}$ are identified with smooth functions $f_{\tilde{s}}$ on G which satisfy the conditions (1) and (2). Then the action of $g \in G$ on a section $\tilde{s}$ is identified with the action by right translations of the function $f = f_{\tilde{s}}$, (see [2]). The space of partially invariant partially holomorphic square-integrable sections is then identified with the space $C^\infty(G; V, \tilde{L}, B, \rho, \sigma)$ of smooth functions $f : G \to V$ satisfying (1) and (2). The assertion of our theorem is proved.

4. UNITARY REPRESENTATION ARISING IN THE PROCEDURE OF MULTIDIMENSIONAL QUANTIZATION

In this section applying the procedure of multidimensional quantization in [3], we will propose a mechanical interpretation of the representation Ind $(G; \tilde{L}, B, \sigma, \delta_0)$.

As a model of the quantum system we choose the Hilbert space $\mathcal{H}$ in Section 3. We define the procedure of quantization as follows:

$$\widehat{(\cdot)} \; : \; \overset{\infty}{C}(\Omega) \longrightarrow \mathcal{B}(\mathcal{H})$$
$$f \longrightarrow \widehat{f} = f + \frac{\hbar}{i} \widetilde{\nabla}_{\tilde{\xi}_f}$$

where $\widetilde{\nabla}_{\tilde{\xi}_f}$ is the covariant derivation associated with the connection $\widetilde{\nabla}$ on the G-bundle $\mathcal{E}_{V,\delta\sigma}$. We recall that $\widetilde{\nabla}_{\tilde{\xi}_f}$ is defined by the formula

$$\widetilde{\nabla}_{\tilde{\xi}_f} = L_{\tilde{\xi}_f} + \frac{i}{\hbar} \alpha (\tilde{\xi}_f),$$

where α is 1-form of connection $\widetilde{\nabla}$, $L_{\tilde{\xi}_f}$ is the Lie derivation along $\bar{\xi}_f$ which is the horizontal lifting of strictly hamiltonian vector field ξ_f corresponding to f.

By a similar argument as in [3] we obtain the following result.

THEOREM 4.1. *The three following conditions are equivalent:*

(i) *The application $f \to \widehat{f}$ is a procedure of quantization*

(ii) *Curv* $\widetilde{\nabla} (\tilde{\xi}, \tilde{\eta}) = - \frac{i}{\hbar} \widetilde{B}_\Omega (\tilde{\xi}, \tilde{\eta}). \mathrm{I}$

(iii) $d_{\widetilde{\nabla}}\, \alpha (\tilde{\xi}, \tilde{\eta}) = - \widetilde{B}_\Omega (\tilde{\xi}, \tilde{\eta}). \mathrm{I}$,

where $\tilde{\xi}, \tilde{\eta}$ are the horizontal lifting of strictly hamiltonian field ξ, η on Ω and $\widetilde{B}_\Omega$ is 2-form defined in Section 1.

Having this procedure of quantization, we obtain the following representation of the Lie algebra $\mathcal{G}$ in space $\mathcal{B}(\mathcal{H})$:

$$\Lambda : \mathcal{G} \to \mathcal{B}(\mathcal{H})$$
$$x \to \Lambda(x) = \frac{i}{\hbar} \cdot \widetilde{f}_x ;$$

where $x \in \mathcal{G}$ and $f_x \in C^\infty(\Omega)$ is the generating function of the hamiltonian field ξ_x corresponding to x.

If G is connected and simply connected, we obtain a unitary representation T of G defined by

$$T(\exp x) = \exp(\Lambda(x)); \; x \in \mathcal{G}.$$

We say that it is the representation of G arising from the procedure of multidimensional quantization.

THEOREM 4.2. *The representation λ arising from the procedure of multidimensional quantization coincides with the representation*

$$\psi : \mathcal{G} \to \mathcal{B}(\mathcal{H})$$
$$x \to \psi(x) = L_{\tilde{\xi}} + \frac{i}{\hbar} \alpha_1(\tilde{\xi}_x),$$

where α_1 is the differential 1-form associated with the representation $\frac{\hbar}{i}(d\sigma + d\delta)$, $\hbar = \frac{h}{2\pi}$ is the Planck's constant. (In [3], ψ is just the covariant derivation of the representation Ind $(G;\ \tilde{L},\ B,\ \rho,\ \sigma_0)$).

Proof. The covariant derivation associated with $\tilde{V}$ is defined by the formula

$$\tilde{\nabla}_{\tilde{\xi}_x} = L_{\tilde{\xi}_x} \frac{i}{\hbar} \alpha(\tilde{\xi}_x);\ x \in \mathcal{G},$$

where α is the differential 1-form of $\tilde{\nabla}$ with values in the Lie algebra of the structural group of the G-bundle $\tilde{\mathcal{E}}\ V,\ \delta\sigma$. Then we have $\alpha = \alpha_1 - \beta$, where β is defined as follows:

Let $F \in \Omega$. The function $\varphi_x \equiv \langle\ .,\ x\ \rangle$,

$$\varphi_x : \Omega \to \mathbf{R}$$
$$F \to \langle F,\ x\rangle$$

is the generating function of ξ_x. On the other hand, we can consider $\langle F,.\rangle$ as a differential 1-form β on G by setting

$$\langle\beta,\ \xi\rangle (F) = \langle F,\ \xi\,(e)\rangle,$$

where ξ is an arbitrary vector field on G and e is the unit element of G. Then $\varphi_x(F) = \langle F,\ x\rangle = \langle \beta,\ \tilde{\xi}_x\rangle (F)$. It follows that $\varphi_x = \beta(\tilde{\xi}_x)$.

By the definition of ψ, $\psi(x) = L_{\tilde{\xi}_x} + \frac{i}{\hbar}\alpha_1(\tilde{\xi}_x)$. Since

$$\varphi_x = \beta(\tilde{\xi}_x),\ \text{we have}$$

$$\psi(x) = L_{\tilde{\xi}_x} + \frac{i}{\hbar}\varphi_x - \frac{i}{\hbar}\beta(\tilde{\xi}_x) + \frac{i}{\hbar}\alpha_1(\tilde{\xi}_x),$$
$$= L_{\tilde{\xi}_x} + \frac{i}{\hbar}\varphi_x + \frac{i}{\hbar}[\alpha_1(\tilde{\xi}_x) - \beta(\tilde{\xi}_x)].$$

Therefore, $\alpha_1 - \beta = \alpha$. Hence,

$$\psi(x) = \frac{i}{\hbar}\left[\varphi_x + \frac{\hbar}{i} L_{\tilde{\xi}_x} + \alpha(\tilde{\xi}_x)\right],$$
$$= \frac{i}{\hbar}\left\{\varphi_x + \frac{\hbar}{i}\left[L_{\tilde{\xi}_x} + \frac{i}{\hbar}\alpha(\tilde{\xi}_x)\right]\right\}$$
$$= \frac{i}{\hbar}\left[\varphi_x + \frac{\hbar}{i}\tilde{\nabla}_{\tilde{\xi}_x}\right] = \frac{i}{\hbar}\hat{\varphi}_x.$$

The theorem is proved.

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