ACTA MATHEMATICA VIETNAMICA
Vol. 14 No 2 (1989)

OPERATIONAL FORMULA ASSOCIATED WITH A FUNCTION DEFINED BY A GENERALIZED RODRIGUE'S FORMULA

?.N. SHRIVASTAVA* and S.S. DHILLON**

I. INTRODUCTION

In an attempt to unify classical polynomials of Mathematical Physics Singh [7] defined a function $P_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,m,A,B)$ by the Rodrigue's type formula

$$P_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,m,A,B)$$
$$= (Ax+B)^{-\alpha}(1-\lambda x^{\gamma})^{-\beta/\lambda} D^n[(Ax+B)^{\alpha+mn}(1-\lambda x^{\gamma})^{\beta/\lambda+sn}]; \quad (1.1)$$

where α, β, λ, γ, s, m, A and B are all parameters.

In continuation of this chain of unification and generalization, we consider below a generalized function $Z_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,q,A,B,m,k)$ defined by

$$Z_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,q,A,B,m,k)$$
$$= (Ax+B)^{-\alpha}(1-\lambda x^{\gamma})^{-\beta/\lambda}\, \mathcal{O}^{m+n}[(Ax+B^{\alpha+qn}(1-\lambda x^{\gamma})^{\beta/\lambda+sn}]; \quad (1.2)$$

where α, β, λ, γ, s, q, A, B, m and k are all parameters, and $\mathcal{O}=x^kD$; $D\equiv\dfrac{d}{dx}$.

The study of the type of functions defined by (1.2) is of interest as we observe that it also includes as special case associated Legendre function defined in [3]

$$P_n^m(x)=\frac{(x^2-1)^{m/2}}{2^n\quad n!}\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n \quad (1.3)$$

Also, (1.2) includes many well known classical polynomials and functions of Mathematical Physics as special cases. In particular,

$$P^{(\alpha,\beta,\lambda)}(x,\gamma,s,q,A,B)=Z_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,q,A,B,O,O) \quad (1.4)$$

— Singh [7].

$$P_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,q)=Z_n^{(\alpha,\beta,\lambda)}(x,\gamma,s,q,1,0,0,0) \quad (1.5)$$

— Shrivastava [8].

$$(-2)^n n!\,(x^2-1)^{-m/2}\,P_n^m(x) = Z_n^{(0,0,1)}(x, 1, 1, 1, 1, 1, m, 0) \tag{1.6}$$

— Associated Legendra function [3].

Other than above, we may mention the names of Generalized Hermite functions of Gould — Hopper [4], Generalized Laguerre function of Singh — Shrivastava [6], Chatterjea ([1], [2]) and Shrivastava [9].

In the present note we shall derive some operational formulas, recurrence relations and generating functions for

$Z_n^{(\alpha, \beta, \lambda)}(x, \lambda, s, q, A, B, m, k)$.

2. We mention below some well known operational relations for the operator $x^k D = O$, which will be useful in our study:

$$O^n(x^\alpha) = (\alpha)^{(k-1,\, n)}\, x^{\alpha+(k-1)n} \tag{2.1}$$

where $(\alpha)^{k,\,n} = \alpha(\alpha+k)(\alpha+2k)\dots(\alpha+nk-k)$.

$$C^{tO} f(x) = f\left[\frac{x}{\{1-(k-1)tx^{k-1}\}^{1/(k-1)}}\right]. \tag{2.2}$$

$$O^n(u.v) = \sum_{i=0}^{n} \binom{n}{i} (O^{n-i} u)\,(O^i v). \tag{2.3}$$

$$C^{tO}(u.v) = (C^{tO} u)\,(C^{tO} v). \tag{2.4}$$

$$F(O)\{x^\alpha g(x)\} = x^\alpha F(\alpha x^{k-1} + O) g(x). \tag{2.5}$$

$$F(O)\{C^{h(x)} g(x)\} = C^{h(x)} F(x^k h^1(x) + O)\, g(x); \tag{2.6}$$

where, $h^1 = \dfrac{dh}{dx}$,

The generalized rule of differentiation

$$O^n\{f(z)\} = \sum_{p=0}^{n} \frac{(-1)^p}{p!} \left(\frac{d}{dz}\right)^p f(z) \sum_{i=0}^{p} (-1)^i (p_i) z^{p-i}\, O^n z^i. \tag{2.7}$$

$$(\alpha+\beta)^{\{(k-1),\, n\}} = \sum_{i=0}^{n} \binom{n}{i} (\alpha)^{(k-1,\, n-i)}\, (\beta)^{(k-1,\, i)}. \tag{2.8}$$

3. OPERATIONAL FORMULA

From (2.3), we obtain the operational formula

$$\left[O + \frac{Ax^k(\alpha+qn)}{(Ax+B)} - \frac{\gamma x^{k+\gamma-1}(\beta+\lambda sn)}{(1-\lambda x^\gamma)}\right]^{m+n} f =$$

$$= \sum_{i=0}^{m+n} \binom{m+n}{i} (Ax+B)^{-q(n-i)} (1-\lambda x^{\gamma})^{-s(n-i)}$$

$$\times Z_n^{(\alpha+iq,\ \beta+si\lambda,\ \lambda)} (x, \gamma, s, q, A, B, m, k) \mathcal{O}^i f. \tag{3.1}$$

When $f = 1$, we get

$$\left[\mathcal{O} + \frac{A x^k (\alpha + qn)}{(Ax+B)} - \frac{\gamma x^{k+\gamma-1} (\beta + \lambda sn)}{1 - \lambda x^{\gamma}} \right]^{m+n} . 1 =$$

$$= (Ax+B)^{-qn} (1-\lambda x^{\gamma})^{-sn} Z_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B, m, k). \tag{3.2}$$

For $m = k = 0$, (3 2) reduces to the operational formula of (1.4)

$$\left[D + \frac{A(\alpha+qn)}{(Ax+B)} \frac{\gamma x^{\gamma-1}(\beta+\lambda sn)}{(1-\lambda x^{\gamma})} \right]^{n} . 1 =$$

$$= (Ax+B)^{-qn} (1-\lambda x^{\gamma})^{-sn} P_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B) \tag{3.3}$$

4. RECURRENCE RELATIONS

From (1.2) we get

$$\Omega Z_n^{(\alpha,\beta,\lambda)}(x, \gamma, s, q, A, B, m, k)$$

$$= (Ax+B)^{-q} (1-\lambda x^{\gamma})^{-s} Z_{n+1}^{(\alpha-q,\ \beta-s\lambda,\ \lambda)}(x, \gamma, s, q, A, B, m, k,) \tag{4.1}$$

where, $\Omega = \left(\mathcal{O} + \dfrac{\alpha A x^k}{Ax+B} - \dfrac{\beta\gamma x^{k+\gamma-1}}{1-\lambda x^{\gamma}} \right)$, and repeated operation of gives

$$\Omega^t Z_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= (Ax+B)^{-qt} (1-\lambda x^{\gamma})^{-st} Z_{n+t}^{(\alpha-qt,\ \beta-st\lambda,\ \lambda)}(x, \gamma, s, q, A, B, m, k). \tag{4.2}$$

Again from (1.2), we get

$$\Omega Z_n^{(\alpha,\beta,\lambda)}(x, \gamma, s, q, A, B, m, k)$$

$$= (Ax+B)^{q} (1-\lambda x^{\gamma})^{s} Z_{n-1}^{(\alpha+q,\ \beta+s\lambda,\ \lambda)}(x, \gamma, s, q, A, B, m+2, k), \tag{4.3}$$

and repeated operation of Ω gives

$$\Omega^t Z_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= (Ax+B)^{qt} (1-\lambda x^{\gamma})^{st} Z_{n-t}^{(\alpha+qt,\ \beta+st\lambda,\ \lambda)}(x, \gamma, s, q, A, B, m+2t, k). \tag{4.4}$$

It is easily seen that

$$\Omega^n(u.v) = \sum_{i=o}^{n} \binom{n}{i} (\Omega^{n-i} u) (\mathcal{O}^i v). \tag{4.5}$$

From this we get

$$\Omega^n = \sum_{i=o}^{n} \binom{n}{i} (Ax+B)^{-q(n-i)} (1-\lambda x^{\gamma})^{-s(n-i)}$$

$$\times Z_{n-i}^{(\alpha-q(n-i),\, \beta-s\lambda(n-i),\, \lambda)}(x, \gamma, s, q, A, B, m, k)\, \mathcal{O}^i . \tag{4.6}$$

This suggests an inverse relation to (4.6) as

$$\mathcal{O}^j = \sum_{i=0}^{j} \binom{j}{i} Z_{j-i}^{(-\alpha, -\beta, \lambda)} (x, \gamma, 0, 0, A, B, m, k)\, \Omega^i . \tag{4.7}$$

From (4.6) and (4.7) we get the following relations

$$Z_{t+n}^{(\alpha-qn,\, \beta-\lambda sn,\, \lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= \sum_{i=0}^{n} \binom{n}{i} (Ax+B)^{qi} (1-\lambda x^{\gamma})^{si} Z_{n-i}^{(\alpha-q(n-i),\, \beta-s\lambda(n-i),\, \lambda)}(x, \gamma, s, q, A, B, m, k)$$

$$\times \mathcal{O}^i\, Z_t^{(\alpha, \beta, \lambda)} (x, \gamma, s, q, A, B, m, k), \tag{4.8}$$

and

$$\mathcal{O}^j Z_t^{(\alpha, \beta, \lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= \sum_{i=0}^{j} \binom{i}{j} Z_{j-i}^{(-\alpha, -\beta, \lambda)} (x, \gamma, 0, 0, A, B, m, k)$$

$$\times Z_{t+i}^{(\alpha-qi,\, \beta-si\lambda,\, \lambda)}(x, \gamma, s, q, A, B, m, k). \tag{4.9}$$

5. GENERATING FUNCTIONS

We see that (1.2) can be written as

$$Z_n^{(\alpha, \beta, \lambda)}(x, \gamma, s, q, A, B, m, k) = (Ax+B)^{-\alpha} (1-\lambda x^{\gamma})^{-\beta/\lambda} \left(\frac{d}{du}\right)^{m+n}$$

$$\times [A((1-k)u)^{1/(1-k)}+B\}^{\alpha+qn} \{1-\lambda((1-k)u)^{\gamma/(1-k)}\, \beta/\lambda + sn] \tag{5.1}$$

where $u = \dfrac{x^{1-k}}{1-k}$

Using the modified form of Lagrange's expansions Theorem [5] we have

$$\frac{F(P)}{1-t\Phi'(p)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n [(\Phi(x))^n F(x)], \tag{5.2}$$

where $P = x + t\Phi(p)$ and $\Phi(p)$ is derivable at $p = x$ and $\Phi(x) = 0$. Now

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Z_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= (Ax + B)^{-\alpha} (1 - \lambda x^{\gamma})^{-\beta/\lambda} \sum_{n=0}^{\infty} \frac{t^n}{n!}\left(\frac{d}{du}\right)^{m+n} [\{A((1-k)u)^{1/1-k} + B\}^{\alpha+qn}$$

$$\times\{1 - \lambda((1-k)u)^{\gamma/(1-k)}\}^{\beta/\lambda+sn}]. \qquad (5.3)$$

Hence taking

$$\Phi(u) = \{A((1-k)u^{1/1-k} + B\}^q \{1 - \lambda((1-k)u)^{\gamma/1-k}\}^s$$

and

$$F(u) = \{A((1-k)u)^{1/1-k} + B\}^{\alpha} \{1 - \lambda((1-k)u)^{\gamma/1-k}\}^{\beta/\lambda},$$

we get the desired generating function as

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} Z_n^{(\alpha,\beta,\lambda)} (x, \gamma, s, q, A, B, m, k)$$

$$= (Ax + B)^{-\alpha}(1 - \lambda x^{\gamma})^{-\beta/\lambda} (x^k D_x)^m [\{A((1-k)p)^{1/1-k} + B\}^{\alpha}$$

$$\times\{1 - \lambda((1-k)p)^{\gamma/1-k}\}^{\beta/\lambda} [1 - t\{((1-k)p)^{k/1-k}$$

$$\times(A((1-k)p)^{1/1-k} + B)^{q-1} (1 - \lambda((1-k)p)^{\gamma/1-k})^{s-1}$$

$$\times (Aq(1 - \lambda((1-k)p)^{\gamma/1-k}) - \lambda\gamma s((1-k)p)^{(\gamma-1)/1-k}$$

$$(A((1-k)p)^{1/1-k} + B))\}]^{-1}], \qquad (5.4)$$

where

$$p = \frac{x^{1-k}}{1-k} + t\{A((1-k)p)^{1/1-k} + B\}^q \{1 - \lambda((1-k)p)^{\gamma/1-k}\}^s.$$

This generating function is a generalization of so many well known generating functions.

In particular, for associated Legendre function, (5.4) reduces to

$$\sum_{n=0}^{\infty} (-\omega)^n P_n^m(x) = (x^2 - 1)^{-m/2} D_x^m[(1 + \omega^2 + 2x(\omega)^{-1/2}]$$

$$= (x^2 - 1)^{-m/2}(-1)^m \left(\frac{1}{2}\right)_m (2\omega)^m (1 + \omega^2 + 2x(\omega)^{-1/2-m}. \qquad (5.5)$$

Further, using (4.6) and (2.2), we get

$$C^{t\Omega} f(x) = (Ax + B)^{-\alpha}(1 - \lambda x^{\gamma})^{-\beta/\lambda} \theta^m [Ax(1 - (k-1)tx^{k-1})^{-1/k-1} + B]^{\alpha}$$

$$[1 - \lambda x^{\gamma}(1 - (k-1)tx^{k-1})^{\gamma/k-1}]^{\beta/\lambda}. f\{x(1 - (k-1)tx^{k-1})^{-1/k-1}\}, \qquad (5.6)$$

or

$$C^{t\Omega} f(x) = (Ax + B)^{-\alpha}(1 - \lambda x^{\gamma})^{-\beta/\lambda}$$

$$x[D_{\omega}^m \Psi(x, t, \omega)]_{\omega=0} f\{x(1 - (k-1)tx^{k-1})^{-1/k-1}\}. \qquad (5.7)$$

where

$$\Psi(x, t, \omega) = [Au(1-(k-1)t(xu)^{k-1})^{-1/k-1} + B]^{\alpha}$$
$$\times [1 - \lambda x^{\gamma} u^{\gamma} (1-(k-1)t(xu)^{k-1})^{-\gamma/k-1}]^{\beta/\lambda},$$

and

$$u = (1-(k-1)\omega x^{k-1})^{-1/k-1}.$$

Taking $f(x) = Z_n^{(\alpha, \beta, \lambda)}(x, \gamma, s, q, A, B, m, k)$, relation (5·6) gives another generating relation

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} (Ax+B)^{-qj} (1-\lambda x^{\gamma})^{-sj} Z_{n+j}^{(\alpha-qj, \beta-sj\lambda, \lambda)}(x, \gamma, s, q, A, B, m, k)$$
$$= Ax+B)^{-\alpha}(1-\lambda x\gamma)^{-\beta/\lambda} O^m[Ax(1-(k-1)tx^{k-1})^{-1/k-1} + B]^{\alpha}$$
$$\times[1-\lambda x^{\gamma}(1-(k-1)tx^{k-1})^{-\gamma/k-1}]^{\beta/\lambda}$$
$$\times Z_n^{(\alpha, \beta, \lambda)}[x(1-(k-1)tx^{k-1})^{-1/k-1}, \gamma, s, q, A, B, m, k]. \quad (5.8)$$

In particular, for associated Legendre function, (5.8) reduces to

$$\sum_{j=o}^{\infty} \binom{n+j}{j} w^j P_{n+j}^m(x)$$
$$= (x^2-1)^{m/2} \left\{x - \frac{\omega}{2}(1-x^2))^2 - 1\right\}^{-m/2} P_n^m\left\{x - \frac{\omega}{2}(1-x^2)\right\}; \quad (5.9)$$

(5. 5) and (5. 9) appear to be new generating functions.

REFERENCES

[1] S.K. Chatterjea, *On generalization of Laguerre polynomials*, Rendi, Sem. Mat. Univ. padova, 34(1964), 180–190.

[2] S.K. Chatterjea, *Some operational formulas connected with a function defined by generalized Rodrigue's formula*, Acta Math., Tomus 17, 3–4, (1966), 379–385.

[3] E.T. Copson, *An introduction to the theory of functions of a complex variable*. Oxford University Press, New Delhi, 1974 (India ed.).

[4] H.W. Gould and A.T. Hopper, *Operational formulas connected with two generalization of Hermite polynomials*, Duke Math. Jour. 29 (1962), 51–64.

[5] E.C. Poole, *Introduction to the theory of linear differential equation*, Dover, 1960.

[6] R.P. Singh and K.N. Shrivastava, *A note on generalization of Laguerre and Humbert polynomials*, La Ricerca, 1963, 1–11.

[7] A. Singh, *Generalized Rodrigue's formula for classical polynomials and related operational relation*, Ph. D., Thesis, 1981, Bundelkhand University, Jhansi.

[8] P.N. Shrivastava, *Classical polynomials – A unified presentation*, pub. DE L'INST. Math. (Beograd) Nouvelle series, 23(37), 1978, 169–177.

[9] P.N. Shrivastava, *Generalized Rodrigue's formula and related operational relation*, p.A.M.S., XVI (1981), 7–14.

Received June 12, 1987

* DEPARTMENT OF MATHEMATICS AND STATISTICS
BUNDELKHAND UNIVERSITY, JHANSI (U. P.) 284001 (INDIA)

** DEPARTMENT OF MATHEMATICS AND STATISTICS
62 ISLAM GANJ, NAGRA JHANSI (U. P.) 284003 (INDIA).