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ON THE SMOOTHNESS OF SOLUTION OF THE MIXED BOUNDARY VALUE PROBLEM FOR THE SECOND ORDER HYPERBOLIC EQUATION IN A NEIGHBOURHOOD OF AN EDGE

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1. EXISTENCE AND UNIQUENESS OF A WEAK SOLUTION

We consider the second order hyperbolic equation

$$\mathcal{L}u \equiv u_{tt} - Lu = f(x, t) \tag{1.1}$$

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\, u_{x_j}\right) + \sum_{i=1}^{n+1} a_i(x, t)\, u_{x_i} + a(x, t)\, u,$$

$$x_{n+1} = t,\ a_{ij} = a_{ji},\ \nu \xi^2 \leqslant a_{ij}\xi_i \xi_j \leqslant \mu \xi^2,\ \nu > 0,$$

where $a_{ij}(x, t)$, $a_i(x, t)$, $a(x, t)$ are real functions having infinite smoothness in the cylinder $\overline{Q_T} = \overline{G} \times [0, T]$ where $\overline{G}$ is the closure of a given bounded domain G whose boundary is a piecewise smooth face, including in $(n$-$1)$ dimensional smooth faces Γ_i $(i = 1, 2, .., m)$. Suppose that the face Γ_i can only intersect Γ_{i-1}, Γ_{i+1} along $(n$-$2)$ dimensional smooth manifolds l_{i-1}, l_{i+1} respectively. In this paper, we restrict ourselves to the case $m = 2$. Analogous results can be easily setended to the general case.

We can assume, without loss of generality that ∂G consists of two smooth faces Γ_1, Γ_2 whose intersection is l_0 denote by $\gamma(p_0)$, $(0 < \gamma(p_0) < 2\pi$, $\gamma(p_0) \neq \pi)$ the angle between Γ_1 and Γ_2 at a point $p_0 \in l_0$.

The mixed boundary value problem for equation (1.1) must satisfy the following initial conditions and boundary conditions

$$u\big|_{t=0} = \varphi(x) \quad , \tag{1.2}$$

$$u_t\big|_{t=0} = \psi(x) \quad , \tag{1.3}$$

$$u\big|_{S_1} = 0 \quad , \tag{1.4}$$

$$\frac{\partial u}{\partial N}\Big|_{S_2} = 0 \quad , \tag{1.5}$$

where $S_1 = \Gamma_1 \times [0, T]$, $S_2 = \Gamma_2 \times [0, T]$.

Before stating the existence of a weak solution of the above problem, let us introduce some notations which will be used throughout the paper.

$W^{k,l}(Q_T)$: the space of functions having generalized derivatives of variables x, t, up to order k, l, respectively, such that

$$\| u \|^2_{W^{k,l}(Q_T)} = \sum_{i+j\leqslant k+l} \iint_{Q_T} \left| \frac{\partial^{i+j} u}{\partial x^i \, \partial t^j} \right|^2 dx\, dt < +\infty \tag{1.6}$$

$\overset{\circ}{W}{}^{k,l}(Q_T)$: the closure of $C_0^\infty(Q_T)$ in $W^{k,l}(Q_T)$

$W_0^{k,l}(Q_T, S_1)$: the closure of the set of all infinite smooth functions vanishing nearly S_1.

$\widehat{W}_0^{k,l}(Q_T)$: the subspace of all functions belonging to $W_0^{k,l}(Q_T, S_1)$ and vanishing when $t = T$.

We shall denote, in particular, $W^k(Q_T)$ for the case $k = l$.

$W^k(G)$: the space of functions having generalized derivatives up to order k such that

$$\| u \|^2_{W^k(G)} = \iint_G \sum_{m=0}^{k} \left| \frac{\partial^m u}{\partial x^m} \right|^2 dx < +\infty \tag{1.7}$$

$\overset{\circ}{W}{}^k_\alpha(G)$: the space of functions such that

$$\| u \|_{\overset{\circ}{W}{}^k_\alpha(G)} = \sum_{s=0}^{k} \iint_G \rho^{\alpha+2s-2k} \left| \frac{\partial^s u}{\partial x^s} \right|^2 dx < +\infty \tag{1.8}$$

where $\rho(x)$ is a infinitely differentiable function such that $\rho(x)$ is positive outside of l_0 and equal to $r(x, l_0)$ in some neighbourhood of l_0 ($r(x, l_0)$ being the distance from a point x to l_0).

The function $u(x, t) \in W^1(Q_T)$ is called a weak solution of problem (1.1) — (1.5) if $u(x, t) \in W_0^1(Q_T, S_1)$, $u(x,0) = \varphi(x)$, and if the following integral identity is satisfied:

$$\iint_{Q_T} (-u_t \eta_t + a_{ij} u_{x_j} \eta_{x_i} + a_i u_{x_i} \eta + a\eta)\, dx dt - \iint_G \psi\eta(x, 0) dx = \iint_{Q_T} f\eta dx dt \tag{1.8'}$$

for all $\eta \in \widehat{W}_0^1(Q_T, S_1)$

THEOREM 1.1. *Suppose that the coefficients of the operator L satisfy the conditions*

$$\max_{Q_T} \left| \frac{\partial a_{ij}}{\partial t}, \frac{\partial a_i}{\partial x_i}, a_i, a \right| \leqslant \mu_1 \tag{1.9}$$

and suppose that $f \in L_{2,1}(Q_T)$, $\varphi \in \overset{\circ}{W}{}^1(G, \Gamma_1)$, $\psi \in L_2(G)$. *Then the problem (1.1) – (1.5) has a weak solution belonging to* $W^1(Q_T)$ *such that*

$$\| u \|_{W^1(Q_T)} \leqslant c(T)\left(\| \varphi \|_{W^1(G)} + \| \psi \|_{L_2(G)} + \| f \|_{L_{2,1}(Q_T)} \right) \quad (1.10)$$

where $C(T) = const > 0$ *and* $C(T)$ *does not depend on the functions* u, φ, ψ *and* f.

Proof. Choose an increasing sequence of the domains

$$G_m = \left\{ x \in G : \text{dist}\,(x, \Gamma_1) > \frac{1}{m} \right\}$$

such that their boundaries are infinitely smooth and this sequence approaches to G. We shall set $Q_T^m = G_m[0, T]$, $S_1^m = \Gamma_1^m \times [0, T]$, $\partial G_m = \Gamma_1^m \cup \Gamma_2$. It is clear that $\Gamma_1^m \to \Gamma_1$ as $m \to \infty$.

Put

$$f_m(x, t) = \begin{cases} f(x, t) \text{ if } (x, t) \in Q_T^m, \\ 0 \quad \text{if } (x, t) \in Q_T \setminus Q_T^m. \end{cases}$$

Then $f_m(x, t) \to f(x, t)$ in $L_{2,1}(Q_T)$ as $m \to \infty$ and

$$\| f_m \|_{L_{2,1}(Q_T)} \leqslant \| f \|_{L_{2,1}(Q_T)}.$$

Put

$$\varphi_m(x) = \begin{cases} \varphi(x) \text{ if } x \in G_m, \\ 0 \quad \text{if } x \in G \setminus G_m, \end{cases}$$

$$\psi_m(x) = \begin{cases} \psi(x) \text{ if } x \in G_m \\ 0 \quad \text{if } x \in G \setminus G_m. \end{cases}$$

Then $\varphi_m \in W_0^1(G_m, \Gamma_1^m)$, $\psi_m \in L_2(G_m)$.

We consider the following problem in the domain Q_T^m

$$(u_m)_{tt} - Lu_m = f_m, \quad (1.1\text{m})$$

$$(u_m)\big|_{t=0} = \varphi_m, \quad (1.2\text{m})$$

$$(u_m)_t\big|_{t=0} = \psi_m, \quad (1.3\text{m})$$

$$(u_m)\big|_{S_1^m} = 0, \quad (1.4\text{m})$$

$$\left.\frac{\partial u_m}{\partial N}\right|_{S_2} = 0. \quad (1.5\text{m})$$

The Galerkin method will be used here to prove the theorem.

Suppose that $\{\varphi_k(x)\}$ is a fundamental system in $W_0^1(G_m, \Gamma_1^m)$ an $(\varphi_k, \mathscr{C}_l) = \delta_k^l$.

Let $\langle,\rangle$ be the scalar product in the space $L_2(Q_T)$ and set

$$u_m^N = \sum_{k=1}^{N} C_k^N(t)\, \mathscr{C}_k(x)$$

where the $C_k^N(t)_s$ satisfy the following system of equations

$$\langle (u_m^N)_{tt}, \mathscr{C}_l\rangle + \langle (a_{ij}(u_m^N)_{x_j}, a_i(u_m^N), a(u_m^N)), (\mathscr{C}_{l_{x_l}}, (\mathscr{C}_l, \mathscr{C}_l)\rangle =$$
$$= \langle f_m, \mathscr{C}_l\rangle,\ l = 1, 2, \dots, N \tag{1.11}$$

and $$\frac{d}{dt} C_k^N(t)\Big|_{t=0} = (\psi_m, \mathscr{C}_k), \tag{1.12}$$

$$C_k^N(t)\Big|_{t=0} = \alpha_k^N. \tag{1.13}$$

Observe that $\varphi^N(x) := \sum_{k=1}^{N} \alpha_k^N \varphi_k(x)$, converges in norm of $W^1(G_m)$ to a function $\varphi(x)$ as $N \to \infty$.

We shall prove that u_m^N satisfies the inequality (1.10)

Multiplying both sides of equation (1.11) by $\frac{d}{dt} C_l^N(t)$ and summing up with respect to l from 0 to N and integrating with respect to t from 0 to t both sides of the obtained equality, we get

$$\iint_{Q_t^m}\Big[\Big(u_m^N\Big)_{tt}\Big(u_m^N\Big)_t + a_{ij}\Big(u_m^N\Big)_{x_j}\Big(u_m^N\Big)_{x_{i_t}} + a_i\Big(u_m^N\Big)_{x_i}\Big(u_m^N\Big)_t + a\Big(u_m^N\Big)\Big(u_m^N\Big)_t\Big]\,dxdt =$$
$$= \iint_{Q_t^m} f_m\Big(u_m^N\Big)_t\,dxdt, \tag{1.14}$$

where

$$\iint_{Q_t^m}\Big(u_m^N\Big)_{tt}\Big(u_m^N\Big)_t\,dx\,dt = \frac{1}{2}\iint_{G_t^m}\Big[\Big(u_m^N\Big)_t\Big]^2\Big|_{t=o}^{t=t} dx,$$

$$\iint_{Q_t^m} a_{ij}\Big(u_m^N\Big)_{x_j}\Big(u_m^N\Big)_{x_{i_t}}\,dxdt = \frac{1}{2}\iint_{G_t^m}\Big[a_{ij}\Big(u_m^N\Big)_{x_j}\Big(u_m^N\Big)_{x_i}\Big]\Big|_{t=o}^{t=t} dx -$$
$$- \frac{1}{2}\iint_{Q_t^m} a_{ij_t}\Big(u_m^N\Big)_{x_i}\Big(u_m^N\Big)_{x_j}\,dx\,dt.$$

Consequently,

$$y(t) = y(0) + \iint\limits_{Q_t^m} \Big[a_{ij_t}\left(u_m^N\right)_{x_i}\left(u_m^N\right)_{x_j} - 2a_i\left(u_m^N\right)_{x_i}\left(u_m^N\right)_t -$$

$$- 2a\left(u_m^N\right)\left(u_m^N\right)_t + 2f_m\left(u_m^N\right)_t\Big]\, dxdt, \tag{1.15}$$

where

$$y(t) = \iint\limits_{G_t^m}\Big[\left((u_m^N)_t\right)^2 + a_{ij}(u_m^N)_{x_j}(u_m^N)_{x_i}\Big] dx.$$

Applying the Cauchy inequality, we obtain

$$y(t) \leqslant y(0) + C_1\int_0^t y(t)dt + \mu_1 \iint\limits_{Q_t^m}(u_m^N)^2\, dxdt + 2\int_0^t \| f \|_{L_2(G_t^m)} y^{\frac{1}{2}}(t)dt. \tag{1.16}$$

We have

$$(u_m^N)(x,t) = (u_m^N)(x,o) + \int_o^t (u_m^N)_\xi(x,\xi)d\xi.$$

Then

$$\iint\limits_{G_t^m} (u_m^N)^2 dx \leqslant 2\iint\limits_{G_t^m}\left[u_m^N(x,0)\right]^2 dx + 2t\int_0^t y(t)dt. \tag{1.17}$$

From (1.16) and (1.17) it follows that

$$Z(t) \leqslant 2\, z(0) + (c_1 + 2t + \mu_1)\int_0^t z(t)dt + 2\int_0^t \| f_m \|_{L_2(G_t^m)} y^{\frac{1}{2}}(t)dt, \tag{1.18}$$

where

$$Z(t) = \iint\limits_{G_t^m}\Big[(u_m^N)^2 + (u_m^N)_t^2 + a_{ij}(u_m^N)_{x_j}(u_m^N)_{x_j}\Big] dx, \tag{1.19}$$

Put

$$\widehat{Z}(t) = \max_{0\leqslant\xi\leqslant t} Z(\xi),\ c_2 = c_1 + \mu_1.$$

It follows that

$$\widehat{Z}(t) \leqslant 2\, z(0) + (c_2 + 2t)t\,\widehat{Z}(t) + 2\,\| f_m \|_{L_{2,1}(Q_t^m)} \widehat{Z}^{\frac{1}{2}}(t) \tag{1.20}$$

As $t \leqslant \min(t_1, T)$ and $t_1 > 0$ satisfies the identity $4t_1^{\frac{5}{2}} + 2C_2 t_1 - 1 = 0$, we have from (1.20)

$$\frac{1}{2}\widehat{Z}(t) \leqslant 2Z(0) + 2\|f_m\|_{L_{2,1}(Q_t^m)} \widehat{Z}^{\frac{1}{2}}(t)$$

which implies

$$\widehat{Z}^{\frac{1}{2}}(t) \leqslant 4Z^{\frac{1}{2}}(0) + 4\|f_m\|_{L_{2,1}(Q_t^m)} \tag{1.21}$$

If $t_1 > T$, then the inequality (1.20) holds for all $t \in [0, T]$.

If $t_1 < T$, then by choosing $t = t_1$ as the initial moment, it follows from the previous arguments that

$$\widehat{Z}^{\frac{1}{2}}(t) \leqslant C_3(t) Z^{\frac{1}{2}}(0) + C_4(t)\|f_m\|_{L_{2,1}(Q_T^m)} \tag{1.21'}$$

where $C_3(t)$ and $C_4(t)$ are defined by the constants ν, μ_1 and t.

Consequently

$$\widehat{Z}^{\frac{1}{2}}(t) \leqslant C(T)\left[Z^{\frac{1}{2}}(0) + \|f_m\|_{L_{2,1}(Q_T^m)} \right]. \tag{1.22}$$

From (1.19) and (1.22) it follows that

$$\iiint\limits_{Q_T^m} \left[(u_m^N)^2 + (u_m^N)_t^2 + (u_m^N)_x^2 \right] dxdt \leqslant C(T)\left[Z^{\frac{1}{2}}(0) + \|f\|_{L_{2,1}(Q_T^m)} \right]. \tag{1.23}$$

We now estimate Z (0):

$$Z(0) = \iiint\limits_{G_0^m} \left[(u_m^N)_t(x,0) \right]^2 dx + \iiint\limits_{G_0^m} \Big\{ \left[(u_m^N)(x,0) \right]^2 +$$

$$+ a_{ij} (u_m^N)_{x_i}(x,0)(u_m^N)(x,0) \Big\} dx, \tag{1.24}$$

where

$$\iiint\limits_{G_0^m} \left[(u_m^N)(x,0) \right]^2 dx = \iiint\limits_{G_0^m} \left[\sum_{k=1}^{N} (\psi_m, \varphi_k)\, \varphi_k(x) \right]^2 dx =$$

$$\sum_{k=1}^{N} (\psi_m, \varphi_k) \leqslant \|\psi_m\|_{L_2(Q_T^m)}. \tag{1.25}$$

$$\iint_{G_0^m} \left\{ \left[(u_m^N)(x,0) \right]^2 + a_{ij} (u_m^N)_{x_i} (x,0)(u_m^N)(x,0) \right\} dx \leqslant$$

$$\leqslant C \iint_{G_0^m} \left[\varphi_m^2(x) + \varphi_{m_x}^2(x) \right] dx = C \| \varphi_m \|_{W_2^1(G_m)} \tag{1.26}$$

From (1.24), (1.25), (1.26) we get

$$Z(0) \leqslant C \left[\| \varphi_m \|_{W_2^1(G_m)} + \| \psi_m \|_{L_2(G_m)} \right], \tag{1.27}$$

where C is a constant not depending on N and m.

From (1.23) and (1.27) it follows that

$$\left\| u_m^N \right\|_{W^1(Q_t^m)} \leqslant C \left[\| \varphi_m \|_{W_2^1(G_m)} + \| \psi_m \|_{L_2(G_m)} + \| f_m \|_{L_{2,1}(Q_T^m)} \right]. \tag{1.28}$$

Moreover

$$\left\| u_m^N \right\|_{W^1(Q_T^m)} \leqslant C \tag{1.29}$$

Hence, by passing to a subsequence if necessary, we may assume that the sequence $\{u_m^N\}$, weakly converges to some element $u_m \in W_0^1(Q_T^m, S_1)$ in $W^1(Q_T^m)$. This convergence is uniform with respect to t in the sense of norm in $L_2(G_m)$.

From (1.28) it follows that

$$\| u_m \|_{W^1(Q_T^m)} \leqslant C [\| \varphi_m \|_{W_2^1(G_m)} + \| \Psi_m \|_{L_2(G_m)} + \| f_m \|_{L_{2,1}(Q_T^m)} \tag{1.30}$$

Next, we shall prove that the function $u_m(x,t)$ is a weak solution of the problem (1.1_m) — (1.5_m) in the domain Q_T^m.

Indeed, we put

$$\eta = \sum_{k=1}^{N} d_k(t) \varphi_k(x), \text{ where } d_k(t) \in W^1([0,T]),\ d_k(T) = 0$$

From (1.14) we obtain

$$\iiint_{Q_T^m} \left[-\left(u_m^N \right)_t \eta_t + a_{ij} \left(u_m^N \right)_{x_j} \eta_{x_i} + a_i \left(U_m^N \right)_{x_i} \eta + a \left(u_m^N \right) \eta \right] dxdt -$$

$$- \iint_{G_m} \left(u_m^N \right)_t \eta \Big|_{t=0} dx = \iint_{Q_T^m} f_m \, \eta \, dxdt, \tag{1.31}$$

for all $\eta = \sum_{k=1}^{N} d_k(t) \varphi_k(x)$

Put

$$\mathcal{M}_N = \left\{ \eta : \eta = \sum_{k=1}^{N} d_k(t)\, \varphi_k(x),\ d_k(t) \in W^1([0,\ T]),\ d_k(T) = 0 \right\}$$

But $\overline{\mathcal{M}} = \bigcup_{N=1}^{\infty} \mathcal{M}_N = \widehat{W}^1_{2,0}\left(Q_T^m,\ S_1^m\right)$, so the function $u_m^N(x,t)$ satisfies the integral identity (1.31) for all $\eta \in \widehat{W}^1_{2,0}(Q_T^m)$.

Passing to limit under the sign of integral as $N \to \infty$ we have

$$\iint_{Q_T^m} [-(u_m)_t \eta_t + a_{ij}(u_m)_{xj}\eta_{xi} + a_i(u_m)_{xi}\eta + a(u_m)\eta]\, dxdt -$$
$$- \iint_{G_m} (u_m)_t \ \eta \Big|_{t=0} dx = \iint_{Q_T^m} f_m \eta\, dxdt \tag{1.32}$$

The initial condition $u_m \Big|_{t=0} = \varphi_m(x)$ is satisfied because $u_m^N \to u_m$ in $L_2(G_m)$ and $u_m^N(x,0) \to \varphi_m(x)$ in $L_2(G_m)$. Moreover, since $u_m^N(x,0) = \sum_{k=1}^{N} \alpha_k^N \varphi_k(x) \to \varphi_m(x)$ in $W^1(G_m)$, we get $u_m^N(x,0) \to \varphi_m(x)$ in $L_2(G_m)$. This shows that, $u_m(x,t)$ is a weak solution of the problem (1.1m) — (1.5m).

We have thus proved that the inequality (1.30) holds for the solution $u_m(x,t)$.

Extend $u_m = 0$ out of Q_T^m. Then from (1.30) it follows that

$$\| u_m \|_{W^1(Q_T)} \leqslant C\, [\, \| \varphi \|_{W^1(G)} + \| \Psi \|_{L_2(G)} + \| f \|_{L_{2,1}(Q_T)}] \tag{1.33}$$

Because the sequence $\{u_m\}$ is bounded in $W^1(Q_T)$, there exists a subsequence $\{u_{m_k}\}$ which weakly converges to some function $u(x,t)$ in $W_0^1(Q_T,\ S_1)$. Therefore, (1.33) implies that

$$\| u \|_{W^1(Q_T)} \leqslant C\, [\, \| \varphi \|_{W^1(G)} + \| \Psi \|_{L_2(G)} + \| f \|_{L_{2,1}(Q)}]. \tag{1.34}$$

Now, we shall prove that $u(x,t)$ is a weak solution of problem (1.1) – (1.5) in the domain Q_T. Indeed, since $u_{m_k}(x,0) \to u(x,0)$ in $L_2(G)$ and $u_{m_k}(x,0) = \varphi_{m_k}(x) \to \varphi(x)$ in $L_2(G)$, we obtain $u(x,t)\,|_{t=0} = \varphi(x)$.

Because u_{m_k} is a weak solution of problem $(1.1_{m_k})-(1.5_{m_k})$, the following integral identity

$$\iint_{Q_T^{m_k}} [-(u_{m_k})_t \eta_t + a_{ij}(u_{m_k})_{xj}\eta_{xi} + a_{ij}(u_{m_k})_{xi}\eta + a(u_{m_k})\eta]\,dxdt -$$

$$- \iint_{G^{m_k}} \psi_{m_k}\eta(x,0)\,dx = \iint_{Q_T^{m_k}} f_{m_k}\eta\,dxdt, \tag{1.35}$$

holds for all $\eta \in \widehat{W}_0^1(Q_T^m k, S_1^m k)$. In addition, since $u_{m_k}=0$ and $f_{m_k}=0$ out of $Q_T^m k$, we have

$$\iint_{Q_T} [-(u_{m_k}\eta_t + a_{ij}(u_{m_k})_{xj}\eta_{xi} + a_{ij}(u_{m_k})_{xi}\eta + a(u_{m_k})\eta]\,dxdt -$$

$$- \iint_G \psi_{m_k}\eta(x,0) = \iint_{Q_T} f_{m_k}\eta\,dxdt,$$

for all $\eta \in \widehat{W}_0^1(Q_T^m k, S_1^m k)$. Now take an arbitrary function $\eta \in W_0^1(Q_T, S_1)$. The set $\{u(x,t) \in C^\infty(Q_T),\ u=0 \text{ nearly } S_1\}$ being dense in $\widehat{W}_0^1(Q_T, S_1)$, there exists a subsequence $\{\eta_s\}$ belonging to $C^\infty(\bar{Q}_T)$, $\eta_s = 0$ nearly S_1 and converging to the function η in $W_0^1(Q_T, S_1)$. It follows that there exists a sufficiently large m_k such that $\eta \in \widehat{W}_0^1(Q_T^m k, S_1^m k)$. Consequently, we have (1.36) for $\eta_s \in \widehat{W}_0^1(Q_T^m k, S_1^m k)$.

In this equality, passing to limit under the sign of integral as $m_k \to \infty$, we obtain

$$\iiint_{Q_T}\Big[-u_t\eta_{st} + a_{ij}u_{x_j}\eta_{sx_i} + a_{ij}u_{x_i}\eta_s + au\eta_s\Big]\,dxdt -$$

$$-\iint_G \varphi\eta_s(x,0)\,dx = \iint_{Q_T} f\eta_s\,dxdt. \tag{1.37}$$

In equality (1.37), passing to limit under the sign of integral as $s \to \infty$, we obtain

$$\iiint_{Q_T}\Big[-u_t\eta_t + a_{ij}u_{x_j}\eta_{x_i} + a_{ij}u_{x_i}\eta + au\eta\Big]\,dxdt - \iint_G \varphi\eta(x,0)\,dx = \iint_{Q_T} f\eta\,dxdt,$$

for all $\eta \in \widehat{W_0^1 Q_T, S_1})$. (1.38)

We have thus proved that, the function $u(x,t)$ is a weak solution of the problem (1.1) — (1.5). Moreover, it satisfies the inequality (1.10).

The Theorem 1.1 is thus proved

We consider the following problem

$$\mathcal{L}u = f(x,t), \tag{2.1}$$

$$u\Big|_{t=0} = 0, \tag{2.2}$$

$$u_t\Big|_{t=0} = 0, \tag{2.3}$$

$$u\Big|_{S_1} = 0, \tag{2.4}$$

$$\frac{\partial u}{\partial N}\Big|_{S_2} = 0, \tag{2.5}$$

in the domain Q_T, where $a_{ij} = a_{ji}$, $\nu\xi^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \mu\xi^2$, $\nu > 0$ (2.6)

THEOREM 2.1. *Suppose that the following conditions are satisfied*

i) $\max\limits_{Q_T}\left|\frac{\partial^k a_{ij}}{\partial t^k}, \frac{\partial^{k-1} a_{ijx}}{\partial t^{k-1}}, \frac{\partial^{k-1} a_i}{\partial t^{k-1}}, \frac{\partial^{k-1} a}{\partial t^{k-1}}\right| \leqslant \mu_1$, $k \leqslant l+1$

ii) $\frac{\partial_k f}{\partial t^k} \in L_{2,1}(Q_T)$, $k \leqslant l-1$, *and* $\frac{\partial^k f}{\partial t^k}\Big|_{t=0} = 0$, $k \leqslant l$. (2.7)

Then the weak solution of the problem (2.1)–(2.5) has generalized derivatives up to order l with respect to t. In addition this solution belongs to $W^1(Q_T)$ and satisfies the following inequality

$$\left\|\frac{\partial^l u}{\partial t^l}\right\|_{W^1(Q_T)} \leqslant C\sum_{k\leqslant l}\left\|\frac{\partial^k f}{\partial t^k}\right\|_{L_{2,1}(Q_T)}, \tag{2.8}$$

where the constant C does not depend on u and $f(x,t)$.

Proof. From inequality (1. 30) it follows that

$$\| u_m^N \|_{W^1(Q_T^m)} \leqslant C \| f_m \|_{L_{2,1}(Q_T^m)}. \tag{2.9}$$

We shall prove the following inequality by induction with rerpect to l.

$$\left\|\frac{\partial^l (u_m^N)}{\partial t^l}\right\|_{W^1(Q_T^m)} \leqslant C\sum_{k\leqslant l}\left\|\frac{\partial^k f_m}{\partial t^k}\right\|_{L_{2,1}(Q_T^m)} \tag{2.10}$$

Indeed, for $l = 0$, this follows from (2.9). Observe now from (2.11) that

$$\langle (u_m^N)_{tt}, \varphi_s\rangle + \iint\limits_{G_m^t}[a_{ij}(u_m^N)_{x_j}(u_m^N)_{x_j}\varphi_{sx_i} + a_i(u_m^N)_{x_j}\varphi_s + a(u_m^N)\varphi_s]dx =$$

$$= \langle f_m, \mathscr{C}_s\rangle, \; s = 1, 2, \ldots, N \tag{2.11}$$

Differentiating both sides of (2.11) with respect to t up to order l, multiplying the obtained equality by $\frac{d^{l+1}C_s^N}{dt^{l+1}}$ and summing up with respect to the index l, we obtain

$$\left\langle \frac{\partial^{l+2}u_m^N}{\partial t^{l+2}}, \frac{\partial^{l+1}u_m^N}{\partial t^{l+1}} \right\rangle + \iint\limits_{G_m^t} \left(a_{ij}(u_m^N)_{x_j}\right)_t^{(l)} \left((u_m^N)_{x_j}\right)_t^{(l)} dx + \iint\limits_{G_m^t} \left(a_i(u_m^N)_{x_i}\right)_t^{(l+1)} dx +$$

$$+ \iint\limits_{G_m^t} \left(a(u_m^N)\right)_t^{(l)} (u_m^N)_t^{(l+1)} dx = \left\langle (f_m)_t^{(l)} (u_m^N)_t^{(l+1)} \right\rangle, \tag{2.12}$$

Observe that

$$\left\langle \frac{\partial^{l+2}u_m^N}{\partial t^{l+2}}, \frac{\partial^{l+1}u_m^N}{\partial t^{l+1}} \right\rangle = \frac{1}{2}\frac{d}{dt}\left\| \frac{\partial^{l+1}u_m^N}{\partial t^{l+1}} \right\|_{L_2(G_m^t)} . \tag{2.13}$$

$$\iint\limits_{G_m^t} \left(a_{ij}(u_m^N)_{x_i}\right)_t^{(l)} \left((u_m^N)_{x_i}\right)_t^{(l)} dx = \frac{1}{2}\frac{\partial}{\partial t}\left[a_{ij}\left((u_m^N)_{x_j}\right)_t^{(l)} \left((u_m^N)_{x_i}\right)_t^{(l)} \right] +$$

$$+ l\frac{\partial}{\partial t}\left[a_{ij}\left((u_m^N)_{x_j}\right)_t^{(l-1)} \left((u_m^N)_{x_i}\right)_t^{(l)} \right] - \frac{3}{2} a_{i_{jt}}\left((u_m^N)_{x_j}\right)_t^{(l)} \left((u_m^N)_{x_j}\right)_t^{(l)} -$$

$$- a_{ijt}\left((u_m^N)_{x_j}\right)_t^{(l-1)} \left((u_m^N)_{x_i}\right)_t^{(l)} + \sum_{k\leqslant l-2}\binom{k}{l}\frac{\partial}{\partial t}\left[(a_{i_j})_t^{(l)} \left((u_m^N)_{x_j}\right)_t^{(k)} \right] -$$

$$- \sum_{k\leqslant l-2}\binom{k}{l}(a_{ij})_t^{l-k+1}\left((u_m^N)_{x_j}\right)_t^{(k)} \left((u_m^N)_{x_i}\right)_t^{(l)} -$$

$$- \sum_{k\leqslant l-2}\binom{k}{l}(a_{ij})_t^{(l-k)}\left((u_m^N)_{x_j}\right)_t^{(k+1)} \left((u_m^N)_{x_j}\right)_t^{(l)}, \tag{1.14}$$

$$\iint\limits_{G_m^t} (a_i(u_m^N)_{x_i})_t^{(l+1)} dx = \sum_{k\leqslant l}\binom{k}{l}\left(a_{ij}\right)_t^{(l-k)} \left((u_m^N)_{x_i}\right)_t^{(k)} \left(u_m^N\right)_t^{(l+1)} . \tag{2.15}$$

Integrating both sides of (2.12) with respect to t from 0 to t and using the Cauchy inequality together with the induction hypothesis, we obtain

$$\left\| \frac{\partial^{l+1} u_m^N}{\partial t^{l+1}} \right\|_{L_2(G_m^t)} + \left\| \frac{\partial^l (u_m^N)_{xi}}{\partial t^l} \right\|_{L_2(G_m^t)} \leqslant \frac{1}{2} \left\| \frac{\partial^l (u_m^N)_{xi}}{\partial t^l} \right\|_{L_2(G_m^t)} +$$

$$+ \frac{1}{4T} \left\| \frac{\partial^l (u_m^N)_{xi}}{\partial t^l} \right\|_{L_2(Q_t^m)} + \frac{1}{4T} \left\| \frac{\partial^{l+1} u_m^N}{\partial t^{l+1}} \right\|_{L_2(Q_t^m)} +$$

$$+ C_1 \int_0^t \left\| \frac{\partial^l (u_m^N)_{xi}}{\partial t^l} \right\|_{L_2(G_m^t)} + C_2 \int_0^t \left\| \frac{\partial^l f_m}{\partial t^l} \right\|_{L_2(G_m^t)} \left\| \frac{\partial^{l+1} u_m^N}{\partial t^{l+1}} \right\|_{L_2(G_m^t)} dt +$$

$$+ C_3 \sum_{k=0}^{l-1} \left\| \frac{\partial^k f_m}{\partial t^k} \right\|_{L_{2,1}(Q_t^m)} . \qquad (2.16)$$

Let us put

$$\frac{\partial (u_m^N)}{\partial t^l} = \vartheta_m^N, \quad \frac{\partial^l f_m}{\partial t^l} = g_m .$$

By the same argument as in the proof of Theorem 1.1 we have

$$\left\| \vartheta_m^N \right\|_{W^1(Q_T^m)} \leqslant C(T) \left[\left\| g_m \right\|_{L_{2,1}(Q_T^m)} + \sum_{k=0}^{l-1} \left\| \frac{\partial^k f_m}{\partial t^k} \right\|_{L_{2,1}(Q_T^m)} \right. . \qquad (2.17)$$

or

$$\left\| \frac{\partial^l (u_m^N)}{\partial t^l} \right\|_{W^1(Q_T^m)} \leqslant C \sum_{k \leqslant l} \left\| \frac{\partial^k f_m}{\partial t^k} \right\|_{L_{2,1}(Q_T^m)} . \qquad (2.18)$$

Extend $u_m = 0$ out of Q_T^m, $f_m = 0$ out of Q_T^m. It follows from (2.20) that

$$\left\| \frac{\partial^l (u_m^N)}{\partial t^l} \right\|_{W^1(Q_T)} \leqslant C \sum_{k \leqslant l} \left\| \frac{\partial^k f}{6 t^k} \right\|_{L_{2,1}(Q_T)} , \qquad (2.19)$$

and then,

$$\left\| \frac{\partial^l u}{\partial t^l} \right\|_{W^1(Q_T)} \leqslant C \sum_{k \leqslant l} \left\| \frac{\partial^k f}{\partial t^k} \right\|_{L_{2,1}(Q_T)} \qquad (2.20)$$

The proof of the theorem is thus complete.

For the asymiotic property of the solution, we shall use the function $\gamma(p_0)$, introduced in section 1 where $p_0 \in l_0$, $\Gamma_1 \cap \Gamma_2 = l_0$.

We transform the main part of the operator L at the point $p_0 \in l_0$ into canonical form. Consequently, $\gamma(p_0)$ is transformed into another angle which is denoted by $\omega(p_0)$. It is always required that $\omega \neq \pi$.

THEOREM 2.2. *Suppose that the following conditions are satisfied*

$$\frac{\partial^i f}{\partial t^i} \in L_{2,1}(Q_T),\ i \leqslant l,\ \left.\frac{\partial^i f}{\partial t^i}\right|_{t=0} = 0,\ i \leqslant l-1,\ \omega \neq \frac{\pi}{j+1},\ j = 0,1,\dots,l,$$

$0 < \omega < 2\pi$. *For the differential pairs* $(m_1, s_1), (m_2, s_2)$ *such that* $\frac{m_i}{\omega} + s_i < l + 1$, $(i = 1,2)$, *it is always required that* $\frac{m_1\pi}{\omega} + s_1 \neq \frac{m_2\pi}{\omega} + s_2$ *where* m_i *and* s_i *are integral numbers,* $m_i > 0$, $s_i > 0$, $(i = 1,2)$.

Then the weak solution of problem (2.1) — (2.5) has the form

$$u(x, t) = C(t)\, r^{\frac{\pi}{\omega}} \phi + u_1(x, t),$$

$$\text{where}\quad C(t) \in W^{l-1}([0, T]),\quad \frac{\partial^l u_i}{\partial t^i} \in \overset{\circ}{W}{}^2(Q(\bar t)),\ \ i \leqslant l,\ \bar t \in [0, T],$$

$$Q(\bar t) = \{(x_1, x_2, \bar t) \in Q_T\}$$

The function $\phi(\varphi, t)$ *does not depend on solution. It is an infinitely differentiable function of polar angle in the 2-dimensional coordinates system having center at the point* $p \in p_0 \times [0, T]$ *and disposing in the plane orthogonal to* $[0, T]$.

Proof. We shall prove the theorem by induction with respe.t to l.

For $l = 1$, we shall prove that

$$u(x, t) = C(t)\, r^{\frac{\pi}{\omega}} \phi(\varphi, t) + u_1(x, t) \tag{2.21}$$

where $C(t) \in L_2([0, T])$, $u_1(x, t) \in \overset{\circ}{W}{}^2_0(Q(\bar t))$.

Take a denumerably dense set $\{\varphi_k(x)\}$ in $W^1_0(Q(t), \Gamma_1)$ a function $\psi(t) \in C^\infty_0([0, T])$ and put $\eta(x, t) = \varphi_k(x)\psi(t) \in \widehat{W^1_0}(Q_T, S_1)$.

We derive from (1.8) for $\psi = 0$ that

$$\iint_{Q_T}(a_{ij} u_{x_j}\varphi_{kx_i} + a_i u_{x_i}\varphi_k + au\varphi_k + f\varphi_k)\psi(t)\,dx\,dt = \iint_{Q_T} u_t \psi_t \varphi_k\, dx\,dt. \tag{2.22}$$

By Theorem 2.1, $u_{tt} \in L_2(Q_T)$. Therefore

$$\iint_{Q_T} u_t \psi_t \varphi_k\, dx\,dt = -\iint_{Q_T} u_{tt}\varphi_k \psi\, dx\,dt \tag{2.23}$$

It follows from (2. 24) and (2. 25) that

$$\int_0^T \psi(t)dt \{ \iint_{Q(t)} (u_{tt} \varphi_k + a_{ij} u_{x_j} \varphi_{kx_i} + a_i u_{x_i} \varphi_k + au\varphi_k + f\varphi_k)dx\} = 0 \quad (2.\ 24)$$

Since (2. 26) holds for all $\psi \in C_0^{\infty}([0, T])$ being dense in $L_2([0, T])$, then

$$\iint_{Q(\bar{t})} (u_{tt}\varphi_k + a_{ij}u_{x_j}\varphi_{kx_i} + a_i u_{x_i}\varphi_k + au\varphi_k + f\varphi_k)\, dx = 0 \quad (2.\ 25)$$

for all $\bar{t} \in E(\varphi_k)$ where mes $[0, T] \setminus E(\varphi_k) = 0$.

Because $\{\varphi_k\}$ is a denumerably dense set in $\overset{o}{W}{}_0^1\ (Q(t), \Gamma_1)$.

$$\iint_{Q(\bar{t})} (u_{tt}\varphi + a_{ij} u_{x_j}\varphi_{x_i} + a_i u_{x_i}\varphi + au\varphi + f\varphi)dx = 0$$

for all $\varphi \in W_0^1(Q(\bar{t}, \Gamma_1))$, mes $\bigcup_{k=1}^{\infty} [0, T] \setminus E(\varphi_k) = 0$, $\bar{t} \in \bigcup_{k=1}^{\infty} E(\varphi_k)$.

Consequently, $u(x, t)$ is a weak solution of the following problem in the $Q(\bar{t})$ for almost of all $\bar{t} \in [0, T]$

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} (a_{ij} u_{x_i}) + \sum_{i=1}^{2} a_i u_{x_i} + cu = u_{tt} - f \equiv F(x,t), \quad (2.\ 26)$$

$$u\Big|_{\Gamma_t^1} = 0 \quad (2.\ 27)$$

$$\frac{\partial u}{\partial N}\Big|_{\Gamma_t^2} = 0 \quad (2.\ 28)$$

where $\Gamma_t^1 = S_1 \cap Q(\bar{t})$, $\Gamma_t^2 = S_2 \cap Q(\bar{t})$.

Since $f, f_t \in L_{2,1}\ (Q_T)$, $u_{tt} \in L_2(Q_T)$. Consequently, $F(x, t) \in L_2(Q(\bar{t}))$, $u \in W_0^1(Q_T, S_1)$. From [4, Theorem 3. 1] it follows that $u \in \overset{o}{W}{}_2^1(Q(t))$.

Using the know results [2] for the boundary value problems for the elliptic equations, we have

$$u(x, t) = C(t)\, r^{\frac{\pi}{\omega}}\ \phi(\varphi, t) + u_1(x, t) \quad (2.29)$$

where $C(t) \in L_2([0, T])$, $u_1(x, t) \in \overset{o}{W}{}^2(Q(t))$, $\phi(\varphi, t) = \sin \frac{\pi\varphi}{\omega} t$.

Using the same argument as in the proof of [3, Theorem 2. 1] and taking account of (2. 31), we obtain the desired conclusions of Theorem 2. 2.

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