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CONVERGENCE OF CONDITIONAL EXPECTATIONS FOR UNBOUNDED RANDOM SETS, INTEGRANDS AND INTEGRAL FUNCTIONALS

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§1 — INTRODUCTION

In recent years, convergence results for the multivalued integral have been developed and used in several areas of applied mathematics: mathematical economics, optimal control, mechanics, etc.

Convergence theorems for the integrals of a sequence of random sets (measurable multifunctions or correspondences, multivalued random variables,...) were studied by Aumann [4], Schmeidler [15], Hildenbrand and Mertens [10], and Artstein [1]. These authors obtained Fatou's lemma and Lebesgue dominated convergence theorem, in the sense of Kuratowski convergence, for closed valued random sets in $\mathbb{R}^d$.

More recently, in [12], Pucci and Vitillaro proved similar results for integrably bounded random sets in a separable reflexive Banach space, in the sense of support functions convergence. In [8], Hiaï proved two kinds of multivalued Fatou's lemmas for conditional expectations: one for the strong lower limit of a sequence of random sets with, possibly unbounded, closed values in a separable Banach space; another for the weak (sequential) upper limit of a sequence of weakly compact valued random sets in a separable reflexive Banach space; from these two multivalued versions of Fatou's lemma, Hiaï deduced a version of Lebesgue's dominated convergence theorem for conditional expectations of random sets with weakly compact values, in the sense of Mosco convergence.

The main purpose of our paper is to provide multivalued versions of Fatou's lemma and Lebesgue's dominated convergence theorem for the conditional expectations and, as a special case, for the integrals, of possibly unbounded random sets; this unboundedness allows us to treat the case of epigraph convergence of normal integrands and, at last to provide an application to the convergence of certain integral functionals.

In a closely related field, a recent preprint of Artstain and Wets [2], deals with the convergence of the integrals of a fixed multifunction whose values are possibly unbounded, with respect to a weakly convergent sequence of probability measures.

In §2, we give definitions and preliminaries on random sets, Mosco's convergence and on a class of sets which plays an important role in this work: the class $\mathcal{L}_C$ of weakly locally compact closed convex sets which contains no (whole) line. §3 is concerned with Fatou's lemma for the weak upper limit of a sequence of unbounded random sets. Before stating it, some new tools concerning the class $\mathcal{L}_C$ and multifunctions whose values are in $\mathcal{L}_C$ are provided. In § 4, using the previous results and a theorem of Hiaï, we deduce a version of Lebesgue's dominated convergence theorem for conditional expectations and integrals of random sets; in this section, we also prove a monotone convergence theorem for a non increasing sequence of random sets. In §5, we treat the special case where the random sets are epigraphic multifunctions associated to normal integrands and we give an application to the convergence of certain sequences of integral functionals.

§2 — DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper, $(\Omega, \mathcal{A}, P)$ denotes an abstracip robability space, X a separable Banach space with the dual space X^* For each $C \subset X$, cl C, w-cl C and $\overline{\text{co}}\, C$ denote the norm-closure, the weak-closure and the closed convex hull of C, the *distance function* $d(.,C)$ of C and the *support function* $s(., C)$ of C are defined by

$$d(x,C) = \inf [\, \| X-y \| \,/ y \in C] \qquad x \in X$$
$$s(x^*,C) = \sup [\langle x^*,x\rangle / x \in C] \qquad x^* \in X^* .$$

Moreover, we define

$$\mathrm{m}(C) = \inf [\, \| y \| \,/ y \in C] = d(0,C),$$
$$\mathrm{M}(C) = \sup [\, \| y \| \,/ y \in C] .$$

Let $\mathbf{B}(X)$ be the Borel σ-field of X, $\mathbf{C}$ (resp. $\mathbf{C}_C$) the family of all non empty, closed (non empty, closed convex) sets, K_{cw} the family of non empty weakly compact convex sets and $\mathcal{L}_C$ the family of non empty, weakly locally compact convex sets which contain no (whole) line.

The following lemma recall how the elements of $\mathcal{L}_C$ are characterized in $\mathbf{C}_C$ (see corollary 1.15 of [5]).

LEMMA 2. 1. *If C is a nonempty element of $\mathbf{C}_C$ then the three following statements are equivalent* :

a) C belongs to $\mathcal{L}_C$.

b) there exists x_0^* in X^* such that $s(. , C)$ is finite and continuous at x_0^* for the Mackey topology.

c) there exists x_0^* in X^* such that, for any $\beta \in \mathbb{R}$, the set

$$\{x \in C / \langle x_0^*, x\rangle \geq \beta\} \text{ is weakly compact in } X.$$

The next lemma will be useful, it is an easy extension of lemma 111. 33 of [5].

LEMMA 2.2. *Let Y be a locally convex topological vector space and f a convex function defined on Y, finite and continuous at one point. If C is a convex subset of Y which intersects the interior of the domain of f (denoted int dom f) and D is a dense subset of C, then the following equality holds.*

$$\inf \{f(y)/y \in C\} = \inf \{f(y)/y \in D \cap \text{int dom} f\}.$$

We denote by J^- the σ-field on C generated by the sets

$$C^-U = \{C \in \mathrm{C}/C \cap U \neq \phi\}$$

taken for all open subsets U of X. A multifunction $F : \Omega \to \mathrm{C}$ is said to be *measurable* if it is $(\mathcal{A}, J^-)$-measurable, i.e., $F^-U = F^{-1}(\mathcal{A}^- U) \in \mathrm{C}$ for every open set U of X. A measurable multifunction is also called a *random set* (r.s.).

A function $f : \Omega \to X$ is said to be a *selection* of F if $f(\omega) \in F(\omega)$, for any $\omega \in \Omega$. A *Castaing representation* of F is a sequence $(f_n)_{n \geq 1}$ of measurable selections of F such that

$$F(\omega) = \text{cl} \{f_n(\omega)/n \geq 1\}.$$

Let us recall the following basic fact about measurability of multifunctions (theorem III.9 of [5]).

PROPOSITION 2.3. *If F is a multifunction, defined on Ω with closed values in X then the three following statements are equivalent.*

a) F is measurable

b) there exists a Castaing representation of F

c) for every $x \in X$, the function $d(x, F(.))$ is measurable.

Another type of measurability is also useful and is weaker than the previous one. A multifunction $F : \Omega \to \mathrm{C}$ is said to be *scalarly measurable* if the function $s(x^*, F(.))$ is measurable, for every $x^* \in X^*$.

For $1 \leq p < \infty$, let $L^p(\Omega, \mathcal{A}, P, X) = L^p(\Omega, X)$ denote the Banach space of (equivalence classes of) measurable functions $f : \Omega \to X$ such that the norm

$$\| f \|_p = E(\| f \|^p)^{1/p} = (\int_\Omega \| f(\omega) \|^p dp)^{1/p}$$

is finite; $L^p(\Omega, \mathbb{R})$ is denoted by L^p. For any $\mathcal{A}$-measurable r.s. F we put

$$S^1(F, \mathcal{A}) = \{f \in L^1(\Omega, X)/f(\omega) \in F(\omega) \text{ a.s.}\}$$

which is a closed set of $L^1(\Omega, X)$ and is nonempty if and only if the function

$$m(F(.)) = \inf \{ \| x \| / x \in F(.)\}$$

is in L^1.

In this case, we shall say that the r.s. F is *integrable*. On the other hand, the r.s. F is said *strongly integrable* or *integrably bounded* if the function

$$M(F(.)) = \sup \{ \| x \| / x \in F(.)\} \text{ is in } L^1.$$

The *integral* I(F) of an integrable r.s. F is defined by

$$I(F) = \{E(f)/f \in S^1(F, \mathcal{A})\}$$

where $E(f) = \int_\Omega f \, dp$ is the usual Bochner integral, since $I(F)$ is not always closed, we also put

$$E(F) = \text{cl}\, I(F).$$

Given a sub-σ-field $\mathbf{B}$ of $\mathbf{C}$, and a $\mathcal{A}$-measurable integrable r.s. F. Hiai and Umegaki [8] showed the existence of a $\mathbf{B}$-measurable integrable r.s. G such that

$$S^1(G, \mathbf{B}) = \text{cl}\,\{E(f \mid \mathbf{B}) / f \in S^1(F, \mathbf{C})\}$$

the closure being taken in $L^1(\Omega, X)$. G is the (multivalued) *conditional expectation* of F relative to $\mathbf{B}$ and is denoted by $E(F \mid \mathbf{B})$.

We recall some basic properties of multivalued conditional expectations, they can be found in [7]. If $C_1, C_2 \in \mathbf{C}$ we put $C_1 + C_2 = \overline{C_1 + C_2}$.

PROPOSITION 2.4. *If F and G are two integrable r.s. with closed values in X, and $\mathbf{B}$ a sub-σ-field of $\mathbf{C}$, then we have the following properties:*

a) $E(F + G \mid \mathbf{B}) = E(F \mid \mathbf{B}) + E(G \mid \mathbf{B})$ *a.s.*

b) if r is a real $\mathbf{B}$-measurable function such that rF is integrable, then $E(rF \mid \mathbf{B}) = rE(F \mid \mathbf{B})$ *a.s.*

c) *if f^* is a bounded scalarly $\mathbf{B}$-measurable function from Ω to X^* then* $s(f^*, E(F \mid \mathbf{B})) = E(s(f^*, F) \mid \mathbf{B})$ *a.s.*

d) $E(\overline{co}\,F \mid \mathbf{B}) = \overline{co}\,E(F \mid \mathbf{B})$ *a.s.*

e) Let F be $\mathbf{B}$-measurable, with values in C_c, and r a $\mathcal{A}$-measurable positive function such that rF is integrable, then

$$E(rF \mid \mathbf{B}) = E(r \mid \mathbf{B})F \quad a.s.$$

In particular, $E(F \mid \mathbf{B}) = F$.

In this paper, we use a notion of convergence, for sequences of subsets, which has been introduced by Mosco [11] and which is related to the one of Kuratowski. Let t be a topology on X and $(C_n)_{n \geq 1}$ a sequence in $\mathbf{C}$. We put

$$t\text{-li}\, C_n = \{x \in X / x = t\text{-lim}\, x_n,\ x_n \in C_n,\ \forall\, n \geq 1\} \quad \text{and}$$

$$t\text{-ls}\, C_n = \{x \in X / x = t\text{-lim}\, x_k,\ x_k \in C_{n(k)},\ \forall\, k \geq 1\}$$

where $(C_{n(k)})_{k \geq 1}$ is a subsequence of (C_n). The subsets t-li C_n and t-ls C_n are the *lower limit* and the *upper limit* of (F_n), relative to the topology t. We obviously have $t\text{-li}\, C_n \subset t\text{-ls}\, C_n$.

A sequence (C_n) *is said to converge to C, in the sense of Kuratowski* relatively to the topology t, if the two following equalities are satisfied

$$C = t\text{-li}\, C_n = t\text{-ls}\, C_n.$$

In this case we denote $C = t\text{-lim}\, C_n$.

This is true if and only if we have the next two inclusions

$$t\text{-ls}\, C_n \subset C \subset t\text{-li}\, C_n.$$

Let us denote by w (by s) the weak (the strong) topology of X. A subset C is said to be the *Mosco limit* M-lim C_n of the sequence $(C_n)_{n \geq 1}$ if

$C = w\text{-lim}\, C_n = s\text{-lim}\, C_n$.

which is true if and only if

$$w\text{-ls } C_n \subset C \subset s\text{-li } C$$

Concerning Mosco's convergence, we refer to Wijsman [18, 19], Mosco [11], Wets [17], and Attouch [2].

§3 — FATOU'S LEMMA FOR THE WEAK UPPER LIMIT OF A SEQUENCE OF RAMDOM SETS

Before giving the Fatou's lemma, we need some additional properties of the elements of $\mathcal{L}_C$, and of multifunctions with values in $\mathcal{L}_C$. Throughout this section, X^* is assumed to be endowed with the *Mackey topology* and D^* ($D_1{}^*$) denotes a countable dense subset of X^* (of the closed unit ball B^* of X^*).

LEMMA 3.1. *Let* $C \in \mathbf{C}$, $L \in \mathcal{L}_C$ and $M^* = \text{dom } s(.\,, L)$ *The two following statements are equivalent:*

a) C is contained in L

b) $s(x^*, C) \leq s(x^*, L)$ for every $x^* \in D^* \cap \text{int } M^*$

(where int denote the interior in the Mackey's topology)

Proof. Since a) $\Rightarrow$ b) is obvious, we prove b $\Rightarrow$ a). For each x in C and each x^* in $D^* \cap \text{int } M^*$ we have

$$\langle x^*, x\rangle \leq s(x^*, C) \leq s(x^*, L).$$

in order to get a), it suffices to show that for each x^* in X^*

$$\langle x^*, x\rangle \leq s(X^*, L). \tag{3.1}$$

The support function being lower semicontinuous we have for any x^*

$$s(x^*, L) = \liminf_{y^* \to x^*} s(y^*, L) = \sup_{V \in \mathcal{V}_{x^*}} \inf \,[s(y^*, L) \;/\; y^* \in V]$$

where $\mathcal{V}_{x^*}$ is a neighborhood basis of x^* of which elements are closed and convex. If $x^* \notin \text{cl } M^*$ then $s(x^*, L) = +\infty$ and (3.1) is trivially satisfied, else each V of $\mathcal{V}_{x^*}$ meets M^* and also int M^* because M^* is convex. Thanks to lemma 2.2 applied to the function $s(., L)$, we have

$$s(x^*, L) = \sup_{V \in \mathcal{V}_{x^*}} \inf \,[s(y^*, L) \;/\; y^* \in V \cap D^* \cap \text{int } M^*] \tag{3.2}$$

Thus, if the following inequality

$$\langle y^*, x\rangle \leq s(y^*, L)$$

is satisfied for every $y^* \in D^* \cap \text{int } M^*$, we can deduce from (3.2) that

$$\langle x^*, x\rangle = \lim \langle y^*, x\rangle \leq s(x^*, L)$$

where the limit is taken for $y^* \to x^*$ and $y^* \in D^* \cap \text{int } M^*$, which is the desired conclusion.

Q.E.D.

The next lemma will be useful in the first part of the proof of Fatou's lemma for unbounded r.s ; the part b) extends lemma 1.1(1) of [8] to the case of unbounded sets.

LEMMA 3.2. a) *If* $(C_n)_{n\geq 1}$ *is a sequence in* $\mathbf{C}$ *then*

$$s(x^*, \text{ w-ls } C_n) \leq \limsup_{n\to+\infty} s(x^*, C_n) \quad \text{for each } x^* \in x^*$$

b) *Moreover, if* L *is an element of* $\mathcal{L}_C$ *which contains all the* C_n, *we have* $\limsup s(x^*, C_n) \leq s(x^*, \overline{co}\,(\text{w-ls } C_n))$ *for each* $x^* \in$ *int dom* $s(., L)$.

Proof. a) If $x \in w\text{-}ls\ C_n$ then $x = w - \lim x_k$ where $x_k \in C_{n(k)}$ and $(C_{n(k)})$ is a subsequence of (C_n). Thus

$$\langle x^*, x\rangle = \lim_{k\to+\infty} \langle x^*, x_k\rangle \leqslant \limsup_{k\to+\infty} s(x^*, C_{n(k)}) \leq \limsup_{n\to+\infty} s(x^*, C_n)$$

b) Take $x^* \in$ int dom $s(.\,, L)$. There exists a subsequence $(C_{n(k)})$ of C_n such that $\lim s(x^*, C_{n(k)}) = \limsup s(x^*, C_n)$. For any $k \geqq 1$ it is possible to find $x_k \in C_{n(k)}$ satisfying

$$s(x^*, C_{n(k)}) - 1/k \leq \langle x^*, x_k\rangle \leq s(x^*, C_{n(k)}).$$

It follows that

$$\lim \langle x^*, x_k\rangle = \limsup s(x^*, C_n)\,. \tag{3.3}$$

Since all the C_n are included in $\mathcal{L}$ and $x^* \in$ int dom $s(.\,, L)$ we see that $\limsup s(x^*, C_n)$ is finite. Further, by lemma 2.1, there exists $\beta \in \mathbf{R}$ verifying

$$\{x_k \,/ k \geqslant 1\} \subset \{x \in L / \langle x^*, x\rangle \geqslant \beta\} \qquad 3.4)$$

and such that the right hand side of (3.4) is weakly compact. Now, let $(x_{k(i)})_{i\geq 1}$ a subsequence of (x_k) such that $x = w\text{-}\lim x_{k(i)}$ for some $x \in X$. because x belongs to w–ls C_n, relation (3.4) shows

$$\limsup s(x^*, C_n) = \langle x^*, x\rangle \leq s(x^*, \overline{co}\,(\text{w-ls } C_n))$$

Q.E.D

The following definition will be useful. Let $C \in C_C$ and $x_o \in C$. Recall that the *asymptotic* (or *recession) cone* of C is the greatest convex cone Γ such that $x_o + \Gamma \subset C$. This cone which does not depend on x_o is denoted by $As(C)$. We also have

$$As(C) = \bigcap_{t>0} t(C - x_o)$$

and $As(C)$ is the polar cone of dom $s(.\,, C)$

The following simple lemma will be useful in the second part of the proof of Fatou's lemma for unbounded r.s.

LEMMA 3.3. *Let* $D \in \mathrm{C}_c$ *and* (C_n) *a sequence in* C_c. *Denote w-ls* C_n *by* C *and assume the two following conditions hold*

i) $C_n \subset D$ *for every* $n \geqslant 1$.

ii) $As(D) = w\text{-}li\ As(C_n)$.

Then $As(\overline{co}\ C) = As(D)$.

Proof. From i) it trivially follows $\mathrm{As}(\overline{\mathrm{co}}\ C) \subset \mathrm{As}(D)$. Conversely, let h be a non nul element of $As(D)$ and $x \in C$. Thanks to ii), there exists a sequence $(h_n)_{n \geqq 1}$ in X verifying $h = w\text{-}\lim h_n$ and $h_n \in As(C_n)$, for every $n \geqq 1$. By the def nition of C, there also exists a sequence (x_k) in X such that $x = w\text{-}\lim x_k$ and $x_k \in C_{n(k)}$ where $(C_{n(k)})$ is a subsequence of (C_n). Thus for any $t \geqslant 0$, we have

$$x + th = w\text{-}\lim (x_k + th_{n(k)}).$$

Since $x_k + th_{n(k)} \in C_{n(k)}$ for every $k \geqq 1$, we see that $x + th \in C$, which shows $h \in As(\overline{\mathrm{co}}\ C)$.

Q. E. D.

The next proposition will enable us to state Fatou's lemma for conditional expectation of the weak upper limit, in a wider general t . It can be v ewed as a measurable parametrization of lemma 3.1 and prov.des a tool which may be useful in other situations.

PROPOSITION 3.4. *Recall that* X^* *is endowed with the Mackey topology and that* D_1^* *is a dense subset of the closed unit ball* B^* *of* X^*. *Let* F *be an* $\mathcal{A}$*-measurable and integrable r.s. with values in* $\mathcal{L}_c$ *and denote* $M^*(\omega) = dom\ s(., F(\omega))$.

Then, there exists a sequence $(g_k^*)_{k \geqq 1}$ *of measurable functions from* Ω *to* D_1^* *verifying :*

i) $\{g_k^*(\omega)/k \geqq 1\}$ *is a dense subset of* $B^* \cap \mathrm{int}\ M^*(\omega)$.

ii) for every $k \geqq 1$, *the real measurable function* $s(g_k^*(.), F(.)) \in L^1$.

Proof. It will be achieved in three steps.

Step one. We begin by constructing a measurable function $f^*\colon \Omega \to D_1^*$ such that $f^*(\omega) \in \mathrm{int}\ M^*(\omega)$, for all $\omega \in \Omega$, and $s(f^*(.), F(.))$ is integrable. We first note that

$$-m(F(\omega)) = \inf\ [s(x^*, F(\omega))/x^* \in B^*]$$

and by lemma 2.2,

$$-m(F(\omega)) = \inf\ [s(x^*, F(\omega))/x^* \in D_1^* \cap \mathrm{int}\ M^*(\omega)]. \tag{3.5}$$

If the function $r(.) \in L^1$ and is > 0, we define the multifunction Y^* on Ω by

$$Y^*(\omega) = \{x^* \in D_1^* \cap \mathrm{int}\ M^*(\omega)/s(x^*, F(\omega)) \leq -m(F(\omega)) + r(\omega)\} \tag{3.6}$$

By (3.5) and the greatest lower bound property, $Y^*(\omega)$ is non empty. If we put

$D_1^* = \{x_n^* / n \geq 1\}$ and $A_n = \{\omega \in \Omega\ /\ x_n^* \in Y^*(\omega)\}$ for all $n \geq 1$, we can write

$$A_n = A_n^1 \cap A_n^2$$

where

$$A_n^1 = \{\omega \in \Omega \,/\, x_n^* \in \text{int}\, M^*(\omega)\} \text{ and}$$

$$A_n^2 = \{\omega \in \Omega \,/\, m(F(\omega)) - r(\omega) + s(x_n^*, F(\omega)) \leq 0\}.$$

We have $A_n \in \mathcal{A}$, for any n, because, on one hand, proposition 5.1.7 of [6] shows that $A_n^1 \in \mathcal{A}$, for each n, and on the other hand, the measurability of F implies that the real function

$$m(F(.)) - r(.) + s(x_n^*, F(.))$$

is measurable and, therefore, that A_n^2 belongs to $\mathcal{A}$. Moreover, we have

$$\Omega = \cup_n A_n.$$

Indeed, $F(\omega) \in \mathcal{L}_c$ for all ω, thus it follows from lemma 2.1 that the subset of $M^*(\omega)$

$$\{x^* \in X^* \,/\, s(x^*, F(\omega)) \leq -\, m(F(\omega)) + r(\omega)\}$$

is convex and that its non empty interior meets B^*, and also D_1^*. Next, we define inductively a measurable partition of Ω by putting

$$B_1 = A_1,\ B_2 = A_2 \setminus A_1,\ \ldots,\ B_n = A_n \setminus (A_1 \cup \ldots \cup A_{n-1}), \ldots$$

At last, the desired function is obtained by putting

$$f^* = \Sigma_n\, x_n^* \chi B_n$$

where χB_n denote the indicator function of B_n.

Step two. Now, we define a sequence of measurable selections of Y^* by

$$f_n^* = x_n^* \chi A_n + f^* \chi A'_n$$

where $A'_n = \Omega \setminus A_n$. Firstly, we note that the real function $s(f_{n^*}(.),\ F(.))$ is integrable because, by the definition of y^*, f_n^* satisfies

$$-m(F(\omega)) \leq s(f_n^*(\omega), F(\omega)) \leq -\, m(F(\omega)) + r(\omega).$$

Secondly, we observe that $Y^*(\omega) = \{f_n^*(\omega) \,/\, n \geq 1\}$. These two facts will be used in the third step.

Step three. For every integer $j \geq 1$, replace the strictly positive integrable function $r(.)$, introduced in the first step of the proof, by the function $jr(.)$. If we denote by Y_j^* the corresponding multifunction defined in the first step, we have.

$$Y_j^*(\omega) = \{x^* \in D_1^* \cap \text{int}\, M^*(\omega) \,/\, s(x^*, F(\omega)) \leq -\, m(F(\omega)) + jr(\omega)\}$$

The results of the second step show that, for each j, there exists a sequence $(f_n^{j*})_{n \geqq 1}$ of measurable selections of Y_j^* verifying

$$Y_j^*(\omega) = \{f_n{}^{j*}(\omega) \,/\, n \geqq 1\}.$$

and such that each function $s\,(f_n^{j*}(.), F(.))$ is integrable, for $j,\ n \geqq 1$. Thus, the ralations

$$D_1{}^* \cap \text{int}\, M^*(\omega) = \bigcup_j Y_j{}^*(\omega) = \{f_n{}^{j*}(\omega) \,/\, j, n \geqq 1\}$$

show that the right hand side is dense in $B^* \cap \text{int}\, M^*(\omega)$. At last, the functions $g_k{}^*$ of the statement are obtained by reindexing the $f_n{}^{j*}$ with a suitable bijection from $\mathbf{N}^* \times \mathbf{N}^*$ onto $\mathbf{N}^*$.

Q.E.D.

We are now in position to state Fatou's lemma for the conditional expectation of the weak upper limit of a sequence of unbounded r.s.

THEOREM 3.5. *Let B be a sub-σ-field of $\mathcal{A}$, $(F_n)_{n \geqq 1}$ a sequence of integrable r.s with values in $\mathcal{L}_c$ and define $F = w\text{-ls}\, F_n$. Suppose the following hypotheses (H1) and (H2) hold*

(H1) the function $\liminf m\,(F_n) \in L^1$

(H2) there exist

a) an $\mathcal{A}$-measurable and integrably bounded r. s G, with values in K_{cw}

b) an $\mathcal{A}$-measurable bounded function $r\,(.)$

c) a $\mathbf{B}$-measurable integrable r.s. H, with values in $\mathcal{L}_c$, verifying

c1) $F_n(\omega) \subset G(\omega) + r(\omega)\, H(\omega)$ a.s. for every $n \geq 1$

c2) $As\,(H(\omega)) = w\text{-li}\, As\,(F_{\Phi(\omega,n)}(\omega))$ a.s. where Φ is function from $\Omega \times \mathbf{N}^*$ to $\mathbf{N}^*$ such that $\Phi(\omega,.)$ is strictly increasing, a,s. in ω.

Then, under the foregoing hypotheses, we have

i) the multifunction F is measurable and integrable

ii) $w\text{-ls}\, E(F_n \mid \mathbf{B})(\omega) \subset E(\overline{co}\, F \mid \mathbf{B})(\omega)$ a.s

Proof of i). The multifunctions $F = w\text{-ls}\, F_n$ and $w\text{-ls}\, E\,(F_n \mid \mathbf{B})$ are $\mathcal{A}$-measurable and $\mathbf{B}$-measurable, respectively. This can be seen by proposition 6.3.9 of [8] where the measurability of multifunctions defined on an abstract measurable space and with values in $\mathcal{L}_c$, is discussed. We also note that by lemmas 6.1.7 and 6.3.8 of [8], these two multifunctions have nonempty, weakly closed values. In order to prove that F is an integrable r. s, we use (H1) and lemma 6.4.7 of [8] which asserts the inequality

$$\liminf m\,(F_n(.)) \geqq m\,(F(.)) \tag{37}$$

Proof of ii) We proceed in two steps.

Step one: Consider the B-measurable r.s. H: by Proposition 3.4 we know that there exists a sequence $(g_k^*)_{k \geq 1}$ of B-measurable functions verifying the properties i) and ii) of this proposition. It will be convenient to put

$u_k(\omega) = S(g_k^*(\omega), w\text{-}ls\ E(F_n \mid \mathbf{B})(\omega))$ for $k \geq 1$ and $\omega \in \Omega$.

By using prop. 2. 4c) and lemma 3.2 a) we get

$$u_k(\omega) \leq \limsup s(g_k^*(\omega), E(F_n \mid \mathbf{B})(\omega)) \tag{3.8}$$

$$\leq \limsup E(s(g_k^*, F_n) \mid \mathbf{B})(\omega) \quad \text{a.s. for each } k \geq 1$$

From (H1) it follows, for all $k, n \geq 1$ and a.s.

$$s(g_k^*(\omega), E_n(\omega)) \leq s(g_k^*(\omega), G(\omega)) + r(\omega)\, s(g_k^*(\omega), H(\omega)) \tag{3.9}$$

Thanks to the properties of the functions g_k^* and to the hypotheses on G, H, r, we can see that the right hand side of (3.9) defines a real integrable function, for every $k \geq 1$. Therefore, it is possible to apply the elassical Fatou's lemma to the sequence $(s(g_k^*, F_n))_{n \geq 1}$ which allows us to rewrite (3.8) in the following form

$$u_k(\omega) \leq E(\limsup s(g_k^*, F_n) \mid \mathbf{B})(\omega) \quad \text{a.s. for each } k \geq 1.$$

Then lemma 3.2 b) and prop. 2.4c) give, for every $k \geq 1$ and a.s.,

$$u_k(\omega) \leq E(s(g_k^*, \overline{\text{co}}\, F) \mid \mathbf{B})(\omega) \leq s(g_k^*(\omega), E(\overline{\text{co}}\, F \mid \mathbf{B})(\omega))$$

and, expliciting the u_k

$$s(g_k^*(\omega), w\text{-}lsF(F_n \mid \mathbf{B})(\omega)) \leq s(g_k^*(\omega), E(\overline{\text{co}}\, F \mid \mathbf{B})(\omega)) \tag{3.10}$$

Step two: By prop 3.4 i), we know that $\{g_k^*(\omega)/k \geq 1\}$ is a dense subset of int dom $s(.,H(\omega))$ a.s. Moreover it is clear that

$$\text{int dom } s(.\,, H\omega)) = \text{int dom } s(.,\, G(\omega) + r(\omega)H\,\omega)).$$

In order to prove that relations (3.10) imply the desired conclusion, it suffices in view of lemma 3. 1 applied with $L = E(\overline{\text{co}}\, F \mid \mathbf{B})(\omega)$ and $C = w\text{-}ls\ E(E_n \mid \mathbf{B})(\omega)$, to show the following equality

$$\text{int dom } s(.\,, H(\omega)) = \text{int dom } s(.\,, E(\overline{\text{co}}F \mid \mathbf{B})(\omega)) \quad \text{a. s.}$$

which, by polarity, is equivalent to

$$As(H(\omega)) = As(E(\overline{\text{co}}F \mid \mathbf{B})(\omega)) \quad \text{a. s.} \tag{3.11}$$

If we define the multifunctions T and T_n, for $n \geq 1$, by

$$T_n(\omega) = F_{\phi(\omega,n)}(\omega) \quad \text{and} \quad T = w\text{-}\text{ls}T_n$$

it is clear, by ($H2$) c) that $T(\omega) \subset X(\omega)$ a. s.

Now, consider the multifunction $As(H)$. Like the multifunction H, it is B-measurable. This fact can be proved by using the equality

$$As(H(\omega)) = \bigcap_j 1/j[H(\omega) - h(\omega)]$$

where $j \in \mathbf{N}^*$ and h is a fixed element of $S_H^1(\mathbf{B})$, and by invoking theorem 5.2.10. of [8] which is a result about preservation of measurability of multifunctions by countable intersection. By (H2). we have for each $n \geq 1$

$$As(G(\omega) + r(\omega)\, H(\omega)) = As(H(\omega)) = w\text{-}li\ As(T_n(\omega)) \quad \text{a. s.}$$

Therefore, lemma 3.3 applied to the sequence $(T_n(\omega))_{n\geq 1}$ shows

$$As(\overline{\text{co}}\, T(\omega)) = As(H(\omega)) \quad \text{a.s.} \tag{3.12}$$

Moreover, the inclusions

$$T(\omega) \subset F(\omega) \subset G(\omega) + r(\omega) H(\omega) \quad \text{a.s.}$$

imply

$$\overline{\text{co}}\, T(\omega(\subset \overline{\text{co}}\, F(\omega) \subset G(\omega) + r(\omega) H(\omega) \quad \text{a.s.}$$

Hence, using (3. 12) we get

$$As(\overline{co})\, F(\omega)) = As\,(H(\omega)) \quad \text{a.s.} \tag{3. 13}$$

Finally, noting that the operations $As\,(.)$ and $E(.\mid \mathbf{B})$ are monotone we can write

$$\begin{aligned} As(H) = E(As(H) \mid \mathbf{B}) &= E(As(\overline{co}\,F) \mid \mathbf{B}) \\ &\subset As\,E\,(\overline{co}\,F) \mid \mathbf{B}) \\ &\subset As\,E\,(G + rH \mid \mathbf{B}) \\ &= As\,E(H \mid \mathbf{B}) \\ &= As(H) \quad \text{a.s.} \end{aligned}$$

These relationships show that

$$As(H(\omega)) = As(E(\overline{co}\,F \mid \mathbf{B})(\omega)) \quad \text{a.s}$$

which is nothing else than (3. 11) and gives the desired conclusion.

Q.E.D,

The next theorem concerns the particular case $\mathbf{B} = \{\Omega, \varnothing\}$; here, conditional expectation relative to $\mathbf{B}$, reduces to integral. Using a result of Truffert [16], it is possible to weaken (H1) and the first part of (H2).

THEOREM 3.6. *Assume* $\mathbf{B} = \{\Omega, \varnothing\}$. *Let* $(F_n)_{n\geqq 1}$ *a sequence of integrable r. s. with values in* $\mathcal{L}_c$ *and define* F = *w-ls* F_n. *Suppose the following hypotheses (H1) and (H2) hold*

(H1·) the sequence $(m(F_n))_{n\geqq 1}$ *is uniformly integrable*

(H2·) there exist

a) a sequence $(G_n)_{n\geqq 1}$ *of* $\mathcal{A}$*-measurable and integrably bounded r.s. with values in* $\mathcal{K}_{cw}$ *and such that the sequence* $(M(G_n))_{n\geqq 1}$ *is uniformly integrable*

b) an uniformly integrable sequence $(r_n)_{n\geqq 1}$ *of* $\mathcal{A}$*-measurable positive functions*

c) an element L *of* $\mathcal{L}_c$, *verifying*

c1) $F_n(\omega) \subset G_n(\omega) + r_n(\omega)\, L$ *a.s. for every* $n \geqq 1$.

c2) $As\,(L) = w - li\ As(F_{\varphi(\omega,\ n)}(\omega))$ *a.s. where* φ *is function from* $\Omega\times\mathbf{N}^*$ *to* $\mathbf{N}^*$ *such that* $\varphi\,(\omega,\,.)$ *is strictly increasing, a.s. in* ω. *Under the foregoing hypotheses, we have* *i) the r.s. F is integrable*

ii) w-ls $E(F_n) \subset E(\overline{co}\, F)$ *a.s.*

Proof. i) By (3.7) and classical Fatou's lemma we get

$$\liminf E(m(F_n)) \geq E\,(\liminf m(F_n)) \geq E(m(F))$$

hence (*H1*) imply that $m(F)$ is integrable.

ii) We first note that the $\mathbf{B}$-measurable multifunction H is constant and equal to L, thus the sequence $(g_k{}^*)_{k\geq 1}$ can be replaced by the countable set $D^* \cap \operatorname{int} \operatorname{dom} s(.,\, L)$. Here, inequality becomes

$$\begin{aligned} s(x^*, F_n(\omega)) &\leq s(x^*, G_n(\omega)) + r_n(\omega)\, s(x^*, L) \\ &\leq \| x^* \| M(G_n(\omega)) + r_n(\omega)\, s(x^*, L) \end{aligned} \tag{3.14}$$

This last inequality shows that if $x^* \in \operatorname{dom} s(.,\; L)$, the sequence $(s(x^*)\, F_n\, (\omega)^+)_{n\geq 1}$ is uniformly integrable because the right hand side of (3.14) define a uniformly integrable sequence of real functions,for $n \geq 1$ (for any real function u we set $u^+ = \max(u, 0)$). Therefore, lemma 1.5 of [16] which permit us to apply classical Fatou lemma, gives

$$\limsup Es(x^*, F_n) \leq E(\limsup s(x^*, F_n)).$$

Hence, inequality (3.10) becomes here

$$s(x^*, w\text{-ls}\, E(F_n)) \leq s(x^*, E(\overline{co}\, F)) \quad \text{for each } x^* \in \operatorname{dom} s\,(.,\, L).$$

We finish the proof as in the second step of the proof of theorem 3.5 particularizing to the case $\mathbf{B} = \{\Omega, \phi\}$.

Q.E.D.

Remark 3.7. In theorem 3.6. the part c2) of hypothesis (H2') cannot be omitted as the following example shows. Let $\Omega = [0,\, 1]$, $\mathcal{A} = \mathbf{B}(\Omega)$ and $P =$ Lebesgue's measure on $(\Omega, \mathcal{A})$, $X = \mathbf{R}$ and $L = \mathbf{R}_+$. Moreover, for every $n \geq 1$, we define $A_n = [0,\, 1/n]$ and, for every $\omega \in \Omega$,

$$F_n(\omega) = L \text{ if } \omega \in A_n\,,\; F_n(\omega) = \{0\} \text{ if } \omega \in A_n.$$

For each $n \geq 1$, we see that $E(F_n) = L$, hence is $E(F_n) = L$. All the hypothesis of theorem 3·6 are satisfied except c2) since, by the definition of the r.s. F_n, we have for any $\omega \neq 0$

$$\operatorname{As}(F_n(\omega)) = F_n(\omega) = \{0\} \qquad \text{for each } n \geq 1/\omega$$

At last, since $\operatorname{ls} F_n(\omega) = \{0\}$ a.s., we deduce $E(\overline{co}\, \operatorname{ls} F_n) = \{0\}$ which contradicts conclusion ii) of theorem 3.6.

Remark 3.8. An inspection of the proof of theorem 3.6 shows that is possible to give a «non convex» version of Fatou's lemma for the weak upper limit. For this purpose, it suffices to consider a sequence (E_n) of r.s. with closed (possibly non convex) values in X, satisfying the two following hypotheses

$$(\alpha)\; E(\overline{co}\, F) \in \mathcal{L}_c$$

(β) for each $X^* \in$ int dom $s(.\,, E(\overline{co}\, F))$, the sequence $(s(x^*, F_n)^+)_{n \geqq 1}$ is uniformly integrable.

Moreover, it is not difficult to show that

$$\text{As}\,(\overline{co}\, F(\omega)) \subset \text{As}\, E(\overline{co}\, F)) \qquad \text{a.s.}$$

and, in particular, that hypothesis (α) implies $\overline{co}\, F(\omega) \in \mathcal{L}_c$ a.s.

§ 4 — DOMINATED CONVERGENCE AND MONOTONE CONVERGENCE THEOREMS FOR UNBOUNDED RANDOM SETS

Before giving the dominated convergence theorem, a simple lemma is useful. The proof is similar to that of lemma 3.3.

LEMMA 4. 1. *Let $D, C \in \mathbf{C}_C$ and (C_n) a sequence in $\mathbf{C}_C$. Assume the three following conditions hold*

i) $C = W\text{-}ls\ C_n$

ii) $C_n \subset D$ *for every* $n \geqq 1$.

iii) $As(D) = w\text{-}ls\ \ As(C_n)$.

Then $As(C) = As(D)$

THEOREM 4. 2. *(Dominated convergence theorem)*

Let $(F_n)_{n \geqq 1}$ a sequence of integrable r. s, with values in $\mathcal{L}_C$ and a r.s. such that

$$F(\omega) = lim\ F_n(\omega) \qquad a.s. \qquad (4.1)$$

Suppose the following hypotheses hold

(K1) the function $sup\ [m(F_n)/n \geqq 1] \in L^1$

(K2) there exist

a) an $\mathcal{A}$-measurable and integrably bounded r.s. G, with values in $\mathcal{K}_{CW}$

b) an $\mathcal{A}$-measurable bounded positive function r(.)

*c) a **B**-measurable integrable r.s. H, with values in $\mathcal{L}_C$, verifying*

c1) $F_n(\omega) \subset G(\omega) + r(\omega)H(\omega)$ *a.s. for every* $n \geqq 1$

c2) $As(H(\omega)) = w\text{-}ls\ \ As(F_n(\omega))$ *a.s.*

Under the foregoing hypotheses, we have

i) the r.s. F is integrable

ii) $E(F \mid \mathbf{B})(\omega) = lim\ E(F_n \mid \mathbf{B})(\omega)$ *a.s.*

Proof i) By proposition 6. 4. 8. of [6] we have

$$m(F(\omega)) = \lim\ m(F_n(\omega)) \qquad \text{a.s.}$$

From this equality and from (K1), we deduce that the r.s. is integrable.

ii) Hypothesis (K1) also allows us to apply theorem 3.2 of Hiai [8] which gives here

$$E(F \mid \mathbf{B})(\omega) \subset s\text{-li } E(F_n \mid \mathbf{B})(\omega) \quad \text{a.s.}$$

This multivalued version of Fatou's lemma for strong lower limit is valid for closed (possibly non convex) valued r.s. Now, we shall prove

$$w\text{-}ls\ E(F_n \mid \mathbf{B})(\omega) \subset E(F \mid \mathbf{B})(\omega) \quad \text{a.s.}$$

To this end, we first proceed as in the first step of the proof of theorem 3.5. Next, we note that relations (4.1.), c1) and c2) permit us to apply lemma 4.1 with $D = H(\omega)$, $C = F(\omega)$ and $C_n = E_n(\omega)$ for all $n \geq 1$, which gives

$$As(F(\omega)) = As(H(\omega)) \quad \text{a.s.}$$

This is the relation (3.13) of the second step of the proof of theorem 3.5. The end of the proof is similar to that of theorem 3.5.

Q.E.D.

At this point, it would be easy to give a version of dominated covergence theorem for integrals, i.e. in the case where $\mathbf{B} = \{\Omega, \emptyset\}$. We leave it to the reader and we pass on to monotone convergence theorem. First, two simple remarks will be useful.

Remark 4.3. a) If $(C_n)_{n \geq 1}$ is a sequence in C_C and if $C = \bigcap C_n$ is non void, then we have the equality $As(C) = \bigcap As(C_n)$.

b) It is easy to check that a closed convex set C is the Mosco limit of a non increasing sequence (C_n) of closed convex sets if and only if $C = \bigcap C_n$.

In [8] Hiai, proved a monotone convergence theorem for the conditional expectations of a non decreasing sequence of r.s. The next result concerns the case of a non increasing sequence.

THEOREM 4.4 *Let $(F_n)_{n \geq 1}$ a non increasing sequence of integrable r.s. with values in $\mathcal{L}_c$ and define the r. s. F by $F(\omega) = \bigcap_n F_n(\omega)$. Suppose the following hypotheses hold*

(K1) the function $\sup [m(F_n) / n \geq 1]$ is in L^1,

(L2) there exist

a) an $\mathcal{A}$-measurable and integrably bounded r.s. G. With values in K_{Cw}.

b) an $\mathcal{A}$-measurable bounded function $r(.)$

c) a $\mathbf{B}$-measurable integrable r.s. H. with values in $\mathcal{L}_c$, verifying

$$F_1(\omega) \subset G(\omega) + r(\omega)\, H(\omega) \quad a.s.$$

Then, the r.s. F is integrable and we have

$$E(F \mid \mathbf{B})(\omega) = \bigcap_{n \geq 1} E(F_n \mid \mathbf{B})(\omega) \quad a.s. \tag{4.2}$$

Proof. Hypothesis (K1) and equality

$$m(F(\omega)) = \sup [m(F_n(\omega)) / n \geq 1] \quad \text{a.s.}$$

show the integrability of F.

in order to prove relation (4.2), we first note that by remark 4.3 b). $F(\omega)$ is the limit of the sequence $(F_n(\omega))$, for each ω. Considering again remark 4.3, b) and theorem 4.2. it only remains to verify that the part c2) of hypothesis (K2) is satisfied, namely

$$As(H(\omega)) = w\text{-ls}\, As(F_n(\omega)) \qquad \text{a.s.}$$

But this equality is easily obtained by remark 4.3 a) and b).

Q.E.D.

§ 5. THE CASE OF INTEGRANDS

The results of the two previous sections being valid for unbounded random sets, they apply in the special case where the r.s. are epigraphs of integrands. In this section, we shall reformulate theorem 4.2 and 4.4 in term of integrands. First, we recall some definitions and known facts.

If u is a numerical function, i.e. with values in $\overline{\mathbb{R}} = [-\infty + \infty]$, defined on X, its *epigraph*, denoted by epi (u) is the subset of $X \times \mathbb{R}$ defined as

$$\text{epi}\,(u) = \{(x,\lambda) \in X \times \mathbb{R} / u(x) \leqslant \lambda\}.$$

The *conjugate* function of u is denoted by u^* and defined on X^* by

$$u^*(x^*) = \sup\,[< x^*, x > - u\,(x)/x \in X] \qquad x^* \in X^*.$$

The function u is said to be *proper* if it is not the constant $+\infty$ and if it does not take the value $-\infty$. If u is convex, lower semi-continuous and proper, the *asymptotic function* (or recession function) of u, denoted by $As(u)$, is defined by equality

$$\text{epi}\,(As\,(u)) = As\,(\text{epi}(u)).$$

The function u is said to be *inf-weakly compact for a certain slope*, if there exists $x^* \in X^*$ such that the function $x \to (u(x) - < x^*, x >)$ is inf-weakly compact. By theorem 1.14 of [5], if u is convex, lower semi continuous and proper, this is equivalent to

$$\text{epi}\,(u) \in \mathcal{L}_C\,(X \times \mathbf{R}).$$

Let u, u_n for $n \geqslant 1$, be numerical functions defined on X. The sequence $(u_n)_{n \geqslant 1}$ is said to be *Mosco-convergent* to u, if epi(u) is the Mosco limit of the sequence $(\text{epi}\,(u_n))_{n \geqslant 1}$ in $X \times \mathbf{R}$, in this section, this will be denoted by $u = M\text{-lim}\, u_n$.

This convergence may also be defined by the equality of the two functions $w\text{-}li_e\, u_n$ and $s\text{-}ls_e\, u_n$ which are the *weak-epi-lower limit* and the *strong-epi-upper limit* of the sequence $(u_n)_{n \geqslant 1}$, respectively, the following two formulas hold (see [3])

$$epi\,(w\text{-}li_e\, u_n) = w\text{-}ls\,(epi(u_n))$$
$$epi\,(s\text{-}ls_e\, u_n) = s\text{-}li\,(epi(u_n)).$$

An application R defined on $\Omega \times X$ with values in $\mathbf{R}$ will be called a *normal integrand* if it satisfies the two following properties:

a) the function $R(\omega,.)$ is convex and lower semi continuous a.s.

b) the multifunction $\omega \to \text{epi } R(\omega,.)$, with closed convex values in $X \times \mathbf{R}$, is measurable. This multifunction is called the *epigraphic multifunction* of R.

A normal integrand R is said to be *integrable* if the measurable multifunction in b) is integrable; this is equivalent to the existence of f in $L^1(\Omega; X)$ such that $R(., f(.))^+$ is integrable. Considering on $X \times \mathbf{R}$ the norm

$$\|(x, \lambda)\| = \|x\| + |\lambda| \qquad \text{with } x \in X \text{ and } \lambda \in \mathbf{R}$$

it is readily seen that R is integrable if and only if the positive function

$$\omega \to d(O, \text{epi } R(\omega,.)) = \inf [\, \|x\| + R(\omega, x)^+ / x \in X]$$

is integrable (here, O denotes the null vector of $X \times \mathbf{R}$).

The *conditional expectation* relative to $\mathbf{B}$ of an integrable normal integrand R, is the normal integrand

$$Q = E(R \mid \mathbf{B})$$

whose epigraphic multifunction $\omega \to \text{epi}(Q(\omega,.))$ is the conditional expectation of the integrable r.s $\omega \to \text{epi}(R(\omega,.))$ (see chapter VIII §9 of [5]).
At this point, it would be possible to reformulate all the theorems of the sections 3 and 4 in terms of integrands. For example, we give dominated convegence theorem for integrands.

THEOREM 5.1 *Let R and R_n, for $n \geq 1$, be normal integrands defined on $\Omega \times X$ and satisfying*

$$R(\omega,.) = M\text{-}\lim R_n(\omega,.) \qquad \text{a.s.}$$

Assume the following two hypotheses

(J1) there exists a sequence $(f_n)_{n\geq 1}$ in $L^1(\Omega; X)$ such that

$$\sup [\, \|f_n(\cdot)\| + R_n(., f_n(.))^+ / n \geq 1 \,] \in L^1$$

(J2) there exists a normal integrand S which is $\mathbf{B}$-measurable integrable and such that

a) $S(\omega,.)$ is inf weakly compact for a certain slope a.s.

b) $R_n(\omega,.) \geq S(\omega,.)$ a.s., for every $n \geq 1$

c) $As(S(\omega,.)) = w - li_e As) R_n(\omega,.))$ a.s.

Under the foregoing hypotheses we have

i) R is an integrable $\mathbf{B}$-measurable normal integrand

ii) $E(R \mid \mathbf{B})(\omega,.) = M\text{-}\lim E(R_n \mid \mathbf{B})(\omega,.)$ a.s.

Proof. We define the r. s. F and F_n, for $n \geq 1$, by putting

$$F(\omega) = \text{epi } R(\omega,.) \text{ and } F_n(\omega) = \text{epi } R_n(\omega,.) \qquad \omega \in \Omega$$

Thanks to the remarks of the beginning of this section, we can see that the r.s. F and F_n, for $n \geq 1$, satisfy all the hypotheses of theorem 4.2 with

$$G(\omega) = \text{singleton } \{(0, 0)\} \text{ of } X \times \mathbf{R}\}$$

$$r(\omega) = 1$$

$$H(\omega) = \text{epi } S(\omega,.) \qquad \omega \in \Omega$$

Therefore we obtain

$$E(F \mid \mathbf{B})(\omega) = \lim E(F_n \mid \mathbf{B})(\omega) \qquad \text{a.s.}$$

which gives the desired conclusion.

Q.E.D.

Remark. In the previous theorem we only have used a simplified, hence less general, form of hypothesis (K2) of theorem 4.2. Let us indicate the exact translation of hypothesis (K2) in term of integrands. To this end, it is sufficient to replace the integrand S of hypothesis(J2) above, by the integrand S' such that, for all ω, $S'(\omega,.)$ is the greatest convex lower semi-continuous function less than or equal to

$$x \mapsto \inf [\lambda + r(\omega)\, S(\omega, (x - y)/r(\omega))/(y, \lambda) \in G(\omega)]$$

where $r(.)$ is a strictly positive, bounded measurable function and G an integrably bounded r.s with convex weakly compact values in $X \times \mathbf{R}$.
In the particular case of a non decreasing sequence of normal integrands, we have the following result which is a direct application of theorem 4.4.

THEOREM 5.2. *Let $(R_n)_{n \geqslant 1}$ a non decreasing sequence of integrable normal integrands and let the normal integrand R defined by*

$$R(\omega,.) = \sup\, [R_n(\omega,.)/n \geqslant 1] \qquad \omega \in \Omega$$

If we assume hypothesis (J1) of theorem 5.1 and that the function $R_1(\omega,.)$ is inf-weakly compact for a certain slope a.s., then R is an integrable **B** *- measurable normal integrand and*

$$E(R \mid \mathbf{B})(\omega, .) = \sup\, [E(R_n \mid \mathbf{B})(\omega, .)/n \geqslant 1] \qquad \text{a.s.}$$

Proof. We define the r.s. F and F_n, for $n \geqslant 1$, by

$$F(\omega) = \text{epi } R(\omega,.) \text{ and } F_n(\omega) = \text{epi } R_n(\omega,.) \qquad \omega \in \Omega.$$

Since the sequence $(R_n(\omega,.))_{n \geqslant 1}$ is non increasing, we have

$$F(\omega) = \bigcap_n F_n(\omega)$$

and theorem 4.4 gives the desired conclusion.

Q.E.D.

We end this section with an application to the convergence of certain integral functionals. Before, we need a more general definition of Mosco convergence.
Let Y be a set and suppose that two topologies s and t are given on Y. A sequence $(C_n)_{n \geq 1}$ of subsets of Y is said to be *Mosco convergent*, with respect to s and t to the subset C if the two following equalities hold

$$C = s\text{-}\lim C_n = t\text{-}\lim C_n$$

which we shall denote by

$$C = M(s, t)\text{-lim } C_n$$

In the special case where s is finer than t this is equivalent to

$$t\text{-ls } C_n \subset C \subset s\text{-li } C_n$$

A sequence $(u_n)_{n \geq 1}$ of functions from Y to $\mathbf{R}$ is said to be *Mosco convergent* to the function u if

$$\text{epi}(u) = M(s, t)\text{-lim epi } (u_n).$$

If Y is a Banach space, let us recall that s stands for the strong topology and w for the weak topology; in Y^* the symbol w^* denotes the weak-star topology. When $Y = L^\infty(\Omega, X^*)$, the space of bounded strongly Q-measurable (class of) functions defined on Ω with values in X^*, we also consider the Mackey topology $\tau(L^\infty(\Omega; X^*), L^1(\Omega, X))$ which will be denoted by τ.

If R is an integrable normal integrand the *integral functional* I_R is the function defined for all f in $L^1(\Omega, X)$ by

$$I_R(f) = \int_\Omega R(\omega), f(\omega))\, dP \text{ if } R(.,f(.))^+ \in L^1 \text{ , } + \infty \text{ otherwise}$$

Similarly, if R^* is normal integrand conjugate to R, i. e.

$$R^*(\omega, x^*) = \sup\, [\langle x^*, x\rangle - R(\omega, x) \,/\, x \in X] \qquad x^* \in X^*$$

it is possible to define the integral functional I_{R^*} on $L^\infty(\Omega; X^*)$. By a result of Rockafeller [13], it is known that if X is a separable reflexive Banach space and if the functionals I_R and I_{R^*} are proper, then they are conjugate to each other.

Now, we recall a result of A. Salvadori (theorem 3.1 of [14]) on the convergence of integral functionals.

THEOREM 5. 3 *Let X be a separable reflexive Banach space and $(R_n)_{n \geq 1}$ a sequence of normal integrands on $\Omega \times X$ satisfying*

i) $R(\omega,.) = M(s, w)\text{-lim } R_n(\omega,.)$ *a.s.*

ii) there exists a sequence $(f_n)_{n \geq 1}$ in $L^1(\Omega, X)$ such that

$$\sup(.\, \| f_n(.) \| + R_n(.\,,\, f_n(.))^+ \,/\, n \geq 1] \in L^1 .$$

iii) there exists a sequence $(f_n^*)_{n \geq 1}$ in $L^\infty(\Omega; X^*)$ such that

$$\sup\, [\, \| f_n^*(.) \| + R_n^*(.,\, f_n^*(.))^+ \,/\, n \geq 1] \in L^\infty.$$

Then $I_R = M(s, w) - \lim\ I_{R_n}$ *and* $I_{R^*} = M(\tau, w^*) - \lim\ I_{R_n^*}$

Using conjointly this result and theorem 4.2 we easily get the following theorem on the convergence of the integral functionals associated with the integrands S_n, where $S_n = E(R_n / B)$ for $n \geq 1$.

THEOREM 5.4. *Let X be a separable reflexive Banach space, B a sub-σ-field of C and $(R_n)_{n \geq 1}$ a sequence of normal integrands on $\Omega \times X$ satisfying*

i) $R(\omega,.) = M(s, w)\text{-}\lim R_n(\omega,.)$ a.s.

ii) there exists a sequence $(f_n)_{n\geq 1}$ in $L^1(\Omega, B, P, X)$ such that

$$\sup [\, \| f_n(.) \| + R_n(., f_n(.))^+ / n \geq 1] \in L^1$$

iii) there exists a sequence $(f_n^*)_{n\geq 1}$ in $L^\infty(\Omega, B, P, X^*)$ such that

$$\sup [\, \| f_n^*(.) \| + R_n^*(., f_n^*(.))^+ / n \geq 1] \in L^\infty.$$

Moreover, assume that there exists a normal integrand T which is B-measurable integrable and such that

a) $T(\omega,.)$ is inf weakly compact for a certain slope a.s.

b) $R_n(\omega, .) \geq T(\omega, .)$ a.s., for every $n \geq 1$.

c) $As(T(w,.)) = w\text{-}li_e\ As(R_n(\omega, .))$ a.s.

Then, if $S = E(R / B)$ and $S_n = E(R_n / B)$ for $n \geq 1$ then we have

$$I_S = M(s, w)\text{-}\lim I_{S_n} \text{ and } I_{S^*} = M(\tau, w^*)\text{-}\lim I_{S_n^*}$$

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