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SOME APPLICATIONS OF P-ADIC NEVALINNA THEORY

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INTRODUCTION

Classical Nevanlinna theory is so beautiful that one would naturally be interested in determining how such a theory would look in the p-adic case. There are two « fundamental theorems » which occupy a central place in Nevalinna theory. In [2] Ha Huy Khoai proved an analogue of the first theorem. However in the general case he did not obtain the same type of theorem as in Nevalinna theory. In [3] using a minor modification of the definition of the characteristic function in [2] we proved p-adic analogues of the two « fundamental theorems » of Nevalinna theory. In the present paper we show some applications of the results of [3] in the study of p-adic meromorphic functions. We first recall some facts from [3].

Let p be a prime number, let Q_p be the field of p-adic numbers, and let C_p be the p-adic completion of the algebraic closure of Q_p. Let D be the unit disc in C_p : $D = \{z \in C_p \ ; \ | z | < 1\}$. The absolute value in C_p is normalized so that $| p | = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p.

Let $f(z)$ be a p-adic analytic function on D represented by a convergent power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z)$. This graph is a straight line with slope n. Since we have

$$\lim_{n\to\infty} v(a_n) + nt = \infty$$

for all $t > 0$, it follows that for every t there exists an n for which $v(a_n) + nt$ is minimal. Let $v(f, t)$ denote the boundary of the intersection of all half-planes lying under the lines Γ_n. Then in any finite segment $[r, s]$, $0 < r < s < \infty$ there are only finitely many Γ_n which appear in $v(f, t)$. Thus, $v(f, t)$ is a polygon line.

$$T(\varphi, a, t) = m\left(\frac{1}{\varphi - a}, t\right) + N\left(\frac{1}{\varphi - a}, t\right)$$

$$\Theta(a, \varphi) = 1 - \overline{\lim_{t \to 0}} \frac{\overline{N}\left(\frac{1}{\varphi - a}, t\right)}{T(\varphi, t)}$$

In [3] the following results are proved

I. $\log_p |\varphi(0)| + T\left(\frac{1}{\varphi}, t\right) = T(\varphi, t).$ (0. 1)

II. A p-adic analogue of Nevanlinna's first fundamental Theorem:

$$T(\varphi, a, t) = T(\varphi, t) + 0(1) \quad (0.2)$$

III. The basic inequality:

Let $\varphi(z)$ be a non-constant meromorphic function on $\{z \in C_p ; |z| < r\}$, $a_1, \dots, a_q$ be distinct numbers of C_p, $|a_i - a_k| \geqslant \delta > 0$ for $i \neq k$, $t = -\log_p r$. Then we have

$$m(\varphi, t) + \sum_{i \pm 1}^{q} m\left(\frac{1}{\varphi - a_i}, t\right) \leqslant 2T(\varphi, t) - N_1(t) + 0(1) \quad (0.3)$$

where $N_1(t)$ is a non-negative function given by the formula

$$N_1(t) = N\left(\frac{1}{\varphi'}, t\right) + 2N(\varphi, t) - N(\varphi' t).$$

IV. A p-adic analogue of Nevanlinna's second fundamental Theorem.

Let $\varphi(z)$ be a meromorphic function on D. Then the set of values $a \in C_p$ such that $\Theta(a, \varphi) > 0$ is finite or countable and we have $\sum_{a \in C_p \cup \infty} \Theta(a, \varphi) \leqslant 2.$

§ 1. ON MEROMORPHIC FUNCTIONS

1.1 — *Determining a meromorphic function from its distribution of values.*

We now consider the problem of determining a p-adic meromorphic function from knowledge of the sets on which it takes certain values. In the classical case, there is a well-known theorem of Nevanlinna which states that, if $f_1(z)$ and $f_2(z)$ are two meromorphic functions on the complex plane, and if the two equations $f_1(z) = a$ and $f_2(z) = a$ have the same set of roots for five different values of a, then the functions $f_1(z)$ and $f_2(z)$ coincide. We shall obtain an analogous result for p-adic meromorphic function on the disc D.

For each $a \in C_p$ let $E_a(\varphi)$ denote the set of points $z \in D$ for which $\varphi(z) = a$, where each point is taken as many times as its multiplicity as a root of the equation $\varphi(z) - a = 0$.

$$T(\varphi, a, t) = m\left(\frac{1}{\varphi - a}, t\right) + N\left(\frac{1}{\varphi - a}, t\right)$$

$$\Theta(a, \varphi) = 1 - \overline{\lim_{t\to 0}} \frac{\overline{N}\left(\frac{1}{\varphi - a}, t\right)}{T(\varphi, t)}$$

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For each $a \in C_p$ let $E_a(\varphi)$ denote the set of points $z \in D$ for which $\varphi(z) = a$, where each point is taken as many times as its multiplicity as a root of the equation $\varphi(z) - a = 0$.

THEOREM 1.1. *Suppose that $\varphi_1(z)$ and $\varphi_2(z)$ are two meromorphic functions on D for which there exist three distinct values $a_1, a_2, a_3 \in C_p$ such that $E_{a_i}(\varphi_1) = E_{a_i}(\varphi_2)$ $i = 1, 2, 3$. Further, suppose that at least one of them is not a ratio of two bounded analytic functions. Then $\varphi_1 \equiv \varphi_2$.*

Proof. Assume that $\varphi_1(z) = \dfrac{f_1(z)}{g_1(z)}$, $\varphi_2(z) = \dfrac{f_2(z)}{g_2(z)}$, where f_1, f_2, g_1, g_2 are analytic functions on D. For every $a \in C_p$, we have

$$N\left(\frac{1}{\varphi_1 - a}, t\right) = N\left(\frac{1}{f_1 - ag_1}, t\right) = T\left(\frac{1}{f_1 - ag_1}, t\right) - m\left(\frac{1}{f_1 - af_1}, t\right) =$$
$$= T\left(\frac{1}{f_1 - ag_1}, t\right) + 0(1) = T(f_1 - ag_1, t) + 0(1) = m(f_1 - ag_1, t) + 0(1)$$

Hence, $N\left(\dfrac{1}{\varphi_1 - a}, t\right)$ is bounded if and only if $m(f_1 - ag_1, t)$ is bounded or, equivalently, if and only if $f_1 - ag_1$ is a bounded analytic function.

Now let there exist two distinct a_i $(i = 1,2)$ such that $N_{a_i}(t) = N\left(\dfrac{1}{\varphi_1 - a_i}, t\right) = N\left(\dfrac{1}{\varphi_2 - a_i}, t\right)$ is bounded. Then, as has been previously indicated, $f_1 - a_i g_1$ and $f_2 - a_i g_2$ $(i = 1, 2)$ are bounded, consequently, f_1, f_2, g_1, g_2 are bounded functions, i.e. each of $\varphi_1(z)$ and $\varphi_2(z)$ is a ratio of two bounded analytic functions. In view of the preceding remark and the assumption of the theorem this implies that there is a_i such that $N_{a_i}(t) = N\left(\dfrac{1}{\varphi_2 - a_i}, t\right) = N\left(\dfrac{1}{\varphi_2 - a_1}, t\right)$ is unbounded.

Without loss generality we may assume that $N_{a_1}(t)$ is unbounded, $a_2 = 0$, $a_3 = \infty$. Then consider the functions

$$p(z) = \frac{f_1(z)}{f_2(z)}; \quad q(z) = \frac{g_1(z)}{g_2(z)}$$

Since $E_0(\varphi_1) = E_0(\varphi_2)$, $E_\infty(\varphi_1) = E_\infty(\varphi_2)$, we see that $p(z)$ and $q(z)$ are two analytic functions which have no zero in D. Therefore $p(z)$ and $q(z)$ are bounded in D.

Now, if $p(z) \neq q(z)$, then we have

$$T(p - q, t) \leqslant T(p,t) + T(q, t) =$$
$$m(p,t) + m(q,t) = 0(1)$$

On the other hand, $\varphi_1(z) = \varphi_2(z)$, and consequently, $p(z) = q(z)$ for all $z \in E_{a_1}(\varphi_1) = E_{a_1}(\varphi_2)$.

Hence:

$$T(p-q,t)=T\left(\frac{1}{p-q},t\right)+0(1)\geqslant$$

$$N\left(\frac{1}{p-q},t\right)+0(1)\geqslant N_{a_1}(t)\to\infty \text{ when } t\to 0.$$

This shows that $p(z)\equiv q(z)$, and consequently, $\varphi_1(z)\equiv\varphi_2(z)$. Theorem is proved.

Consequence (*p*-adic analogue of Picard's theorem).

If a meromorphic function φ is not a ratio of two analytic bounded functions on D, then $\varphi(z)$ takes all values $a\in C_p$, except possibly one.

Proof. If φ has two excluded values, then there exist two values $a_i\in C_p$ $(i=1,2)$, such that $N\left(\frac{1}{\varphi-a_i},t\right)$ is bounded. From this it follows that φ is a ratio of two bounded analytic functions, as was shown above.

Remark. The following example shows that in theorem 1.1. the words «three values» can not be replaced by «two values». Let

$$\varphi_1(z)=\frac{2\log(1+z)+\arcsin z}{\log(1+z)+\arcsin z}$$

$$\varphi_2(z)=\frac{2\log(1+z)+2\arcsin z}{\log(1+z)+2\arcsin z}$$

$a_1=1$, $a_2=2$. *Then we have*

$$E_{a_1}(\varphi_1)=E_{a_1}(\varphi_2)=\{z;\ \log(1+z)=0\}=$$

$$=\{\theta-1;\ \theta^{p^n}=1,\ n=1,2,\ldots$$

$$E_{a_2}(\varphi_1)=E_{a_2}(\varphi_2)=\{z;\ \arcsin z=0\}$$

$$=\left\{\frac{\theta-\theta^{-1}}{2i};\ \theta^{p^n}=1,\ n=1,2\ldots\right\}$$

However $\varphi_1\not\equiv\varphi_2$.

1. 2. *Rational and transcendent meromorphic functions.*

We note that the results established for meromorphic functions on D are true for meromorphic functions on the disc $\{|z|<R;\ R\leqslant\infty\}$.

THEOREM 1. 2. *Let $R(u)$ be a rational function of degree d, $f(z)$ be a meromorphic function on $\{z\in C_p;\ z<R\}$, $R\leqslant\infty$. Then*

$$T(R(f),t)=d\,T(f,t)+0(1) \text{ when } t\to-\log_p R$$

Proof. First of all we prove that if p is a polynomial of degree k, then

$$N(p(f),t)=k\,N(f,t), \tag{1. 1}$$

$$m(p(f),t)=k\,m(f,t)+0(1), \tag{1. 2}$$

$$T(p(f,t)=k\,T(f,t)+0(1). \tag{1. 3}$$

Indeed (1.1) follows from the fact that if $f(z)$ has a pole of order λ ($\lambda > 0$) at a point z_0, then $p(f)$ has a pole of order λk at z_0. If $M(f,t) = \max_{v(z)\geqslant t} |f(z)|$ is buunded when $t \to = \log_p R$ then the equality (1.2) is trivial, because both sides are bounded. Suppose that $M(f,t)$ is unbounded.

Let $p(z) = a_k z^k + \dots + a_0$.

Then for every t sufficiently close to $-\log_p R$,

$$v(p(f), t) = v(a_k f^k, t).$$

Consequently, $m(p(f), t) = km(f, t) + 0(1)$. Relation (1.3) follows from (1.1) and (1.2), Turning to the case of Rational function $R(u)$ of degree d: $R(u) = \frac{P(u)}{Q(u)}$; max $\{\deg P, \deg Q\} = d$. Since $T\left(\frac{P}{Q}, t\right) = T\left(\frac{Q}{P}, t\right) + 0(1)$. We can assume that $\deg P \leqslant \deg Q$. If $\deg P = \deg Q$ then for a suitable constant c, deg $(P - cQ) <$ deg Q and $T\left(\frac{P}{Q}, t\right) = T\left(\frac{P}{Q} c, t\right) + 0(1) = T\left(\frac{P-cQ}{Q}, t\right) + 0(1)$.

Hence one can assume that $\deg P < d$. In this case the sets of poles of the functions $R(f)$ and $\frac{1}{Q(f)}$ coincide. In fact, if the function $f(z)$ has a pole at a point z_0, then since $\deg Q > \deg P$, the order of pole of $Q(f)$ at z_0 is greater than that of $P(f)$ at z_0, and then $R(f(z_0)) = 0$. This implies that the function $R(f)$ has poles at the zeros of the function $Q(f)$ only (but not at the poles of the function $f(z)$). On the other hand, $Q(f) = 0$ at the points at which $f(z)$ is equal to one of the zeros of $Q(u)$. Since $P(u) \neq 0$ where $Q(u) = 0$, it follows that $N(R(f(t)) = N\left(\frac{1}{Q(f)}, t\right)$.

From the properties of the Newton polygon it follows that

$$m(R(f), t) = m\left(\frac{1}{Q(f)}, t\right) + 0(1)$$

because $\deg P < \deg Q$.

Thus we have

$$T(R(f), t) = T\left(\frac{1}{Q(f)}, t\right) + 0(1)$$
$$= T(Q(f), t) + 0(1) = dT(f,t) + 0(1).$$

This proves Theorem 1.2.

Consequence 1. *If $R(z)$ is a rational function of degree d, then*

$$T(R, t) = -dt + 0(1)$$

Indeed $T(z, t) = -t$.

Consequence 2. *A meromorphic function $f(z)$ is transcendent if and only if*

$$\lim_{t\to\infty} \frac{T(f, t)}{-t} = \infty$$

Proof. Suppose that $\lim_{t \to \infty} \frac{T(f, t)}{-t} = k < \infty$. Then the number of poles of $f(z)$ is not greater than $[k]$.

Indeed, otherwise, for sufficiently large we would have

$$N(f, t) = \sum_{\substack{b_i \text{ : poles} \\ v(b_i) > t}} (v(b_i) - t) \geqslant -([k] + 1)t + 0(1)$$

and then $T(f, t) \geqslant -([k] + 1)t + 0(1)$

$$\lim_{t \to -\infty} \frac{T(f, t)}{-t} \geqslant [k] + 1 > k$$

Similarly, the number of poles of $\frac{1}{f}$ is not greater than $[k]$. Thus $f(z)$ is a rational function. Combining this fact and consequence 1 yields the result.

1.3 — Fixed points of analytic functions.

Let $f(z)$ be a analytic function on C_p. We set $f_1(z) = f(z)$; $f_{k+1}(z) = f(f_k(z))$ for $k \geqslant 1$. A root of the equation $f_k(z) = z$ in called a fixed point of order k of the function $f(z)$. If α is a fixed point of order k, but is not a fixed point of order less than k, then α is called a fixed point of exact order k.

THEOREM 1.3. *A transcendent analytic function on C_p has infinitely many fixed points of exact order n, except possibly for one value n.*

To prove this theorem, we need two lemmas.

LEMMA 1. *Let $f(z)$ be a meromorphic function on $\{z \in C_p ; |z| < R\}$ and let $a_i(z)$ $(i = 1, 2, 3)$ be distinct meromorphic functions such that*

$$T(a_i, t) = o(T(f, t)) \text{ when } t \to -\log_p . R$$

Then $$T(f, t) \leqslant \sum_{i=1}^{3} \overline{N}\left(\frac{1}{f - a_i(z)}, t\right) + 0\ (T(f, t)) \tag{1.4}$$

Proof 1. Let $\varphi(z)$ be a meromorphic function on C_p.

Adding $N(\varphi, t) + N\left(\frac{1}{\varphi - a_i}, t\right)$ to both sides of the basic inequality (0.3) we obtain

$$(q + 1)\ T(\varphi, t) \leqslant \sum_{i=1}^{q} N\left(\frac{1}{\varphi - a_i}, t\right) + N(\varphi, t) - N_1(t) + 2T(\varphi, t) + 0(1).$$

Since $N(\varphi', t) - N(\varphi, t) = \overline{N}(\varphi, t)$, we have

$$(q - 1)\ T(\varphi, t) \leqslant \sum_{i=1}^{q} N\left(\frac{1}{\varphi - a_i}, t\right) + \overline{N}(\varphi, t) - N\left(\frac{1}{\varphi'}, t\right) + 0(1).$$

Since a root of multiplicity k of the equation $\varphi(z) = a$ is a root of multiplicity $k-1$ of the equation $\varphi'(z) = 0$, we obtain

$$\sum_{i=1}^{q} N\left(\frac{1}{\varphi-a_i}, t\right) - N\left(\frac{1}{\varphi'}, t\right) \leqslant \sum_{i=1}^{q} \overline{N}\left(\frac{1}{\varphi-a_i}, t\right).$$

Consequently, for every $t \leqslant -\log_p R$

$$(q-1)\, T(\varphi,t) \leqslant \sum_{i=1}^{q} \overline{N}\left(\frac{1}{\varphi-a_i}, t\right) + \overline{N}(\varphi,t) + 0(1) \qquad (1.5)$$

Now, we set

$$\varphi(z) = \frac{f(z) - a_1(z)}{f(z) - a_2(s)} \cdot \frac{a_3(z) - a_2(z)}{a_3(z) - a_1(z)},$$

Using the inequality (1.5) for $a_1 = 0$, $a_2 = 1$, we obtain

$$T(\varphi,t) \leqslant \overline{N}(\varphi,t) + \overline{N}\left(\frac{1}{\varphi}, t\right) + \overline{N}\left(\frac{1}{\varphi-1}, t\right) + 0(1). \qquad (1.6)$$

From the construction of $\varphi(z)$ it follows that:

$$\overline{N}(\varphi,t) + \overline{N}\left(\frac{1}{\varphi}, t\right) + \overline{N}\left(\frac{1}{\varphi-1}, t\right) \leqslant$$

$$\sum_{i=1}^{3} \overline{N}\, \frac{1}{f-a_i(z)}, t\Big) + \overline{N}\left(\frac{1}{a_1-a_2}, t\right) + \overline{N}\left(\frac{1}{a_1-a_3}, t\right) + \overline{N}\left(\frac{1}{a_2-a_3}, t\right) +$$

$$+ 0(1) = \sum_{i=1}^{3} \overline{N}\left(\frac{1}{f-a_i(z)}, t\right) + 0(T(f,t)),$$

The inequality (1.4) follows from (1.6).

LEMMA 2. *Let $f(z)$ and $g(z)$ be analytic functions on C_p, $\varphi(z) = g(f(z))$. Then*
1) *If $g(z)$ is a polynomial of degree N, then*

$$\lim_{t \to -\infty} \frac{T(f,t)}{T(\varphi,t)} = \frac{1}{N}$$

2) *If $g(z)$ is a transcendence analytic function, then*

$$\lim_{t \to -\infty} \frac{T(f,t)}{T(\varphi,t)} = 0$$

The first statement being a direct consequence of Theorem 1.2, it suffices to prove the second one.

Suppose that $g(z) = a_o + a_1 z + \dots + a_n z^n + \dots$

For every N, $g_N(z) = a_o + a_1(z) + \dots + a_N z^N$ is a polynomial of degree N. We have

$$\lim_{t \to -\infty} \frac{T(f,t)}{T(\varphi,t)} \leqslant \lim_{t \to -\infty} \frac{T(f,t)}{T(g_N(f), t} = \frac{1}{N}$$

Hence, $\lim\limits_{t \to -\infty} \dfrac{T(f,t)}{T(\varphi,t)} = 0$, because N is an arbitrary large number.

Proof of Theorem 1.3. Let k be the smallest positive integer such that $f(z)$ has only finitely many fixed points of exact order k; $\alpha_1, \ldots, \alpha_p$. It is sufficient to prove that for every $n > k$, $f(z)$ has in finitely many fixed points of exact order n.

If z_o is a root of the equation $f_n(z) = f_{n-k}(z)$ then $\alpha = f_{n-k}(z_o)$ is a fixed point of order k of $f(z)$, and hence $\alpha = \alpha_i$ or α is a fixed point of exact order $j > k$. We have $f_{n-k+j}(z_o) = f_{n-k}(z_o)$. It follows from Lemma 2 that

$$\overline{N}\left(\frac{1}{f_n - f_{n-k}}, t\right) \leqslant \sum_{J=1}^{k-1} \left(\frac{1}{f_{n-k+J} - f_{n-k}}, t\right) +$$

$$\sum_{i=1}^{p} N\left(\frac{1}{f_{n-k} - \alpha_i}, t\right) = 0\left(\sum_{1=1}^{n-1} T(f_1, t)\right) = 0(T(f_n, t)$$

Using Lemma 1 for $a_1(z) \equiv z$, $a_2(z) \equiv \infty$, $a_3(z) \equiv f_{n-k}(z)$, we have

$$(1 + 0(1))\, T(f_n, t) \leqslant \overline{N}\left(\frac{1}{f_n - z}, t\right)$$

$$\text{Hence,} \quad \sum_{1=1}^{n-1} \overline{N}\left(\frac{1}{f_1 - z}, t\right) = 0\left(\sum_{1=1}^{n-1} T(f_1, t)\right) =$$

$$= 0\,(T(f_n, t)) = 0\left(\overline{N}\left(\frac{1}{f_n - z}, t\right)\right)$$

Thus, the equation $f_n(z) = z$ has infinitely many roots which are not roots of the equations $f_i(z) = z$ where $i = 1, 2, \ldots, n-1$. Theorem 4 is proved.

§ 2 — RELATIONSHIP BETWEEN THE CHARACTERISTIC FUNCTION OF A MEROMORPHIC FUNCTION AND THAT OF ITS DERIVATIVE

In this section we use the symbol $f^{(l)}$ to denote the 1-th. derivative of a meromorphic function f.

We set

$$N_0\left(\frac{1}{f'}, t\right) = \sum_{s>t} n_0(f', o, s)\,(s - t)$$

where $n_0(f', o, s)$ is the number of zeros z of $f'(z)$ with $v(z) = t$ which are not zeros of multiplicity of $f(z) - 1$.

THEOREM 2.1. *For a meromorphic function $f(z)$ on D, we have*

$$\sum_{a \in C_p} \Theta(a, f^{(l)}) \leqslant 1 + \frac{1}{l+1}$$

Proof. If $f(z)$ has a pole of order $p(p \geqslant 1)$ at a point z_0, then $f^{(l)}$ has a pole of $p + l$ at z_0. Hence:

$$\overline{N}(f^{(l)}, t) \leqslant \frac{1}{l+1} N(f^{(l)}, t) \leqslant \frac{1}{l+1} T(f^{(l)}, t).$$

Consequently,

$$\Theta(\infty, f^{(l)}) \geqslant \frac{1}{l+1}.$$

The theorem now follows from the second fundamental Theorem.

THEOREM 2.2. *Let l be a natural number and $\Psi(z) = \sum_{i=0}^{l} a_i(z) f^{(i)}(z)$ where $a_i(z)$ are meromorphic functions on D, such that $T(a_i, t) = 0\,(T(f, t))$ when $t \to 0$. Then we have*

$$m\left(\frac{\Psi}{f}, t\right) = 0(T(f,)t$$

$$T(\Psi, t) = (l + 1)\, T(f, t) + 0(T(f, t))$$

Proof. For every meromorphic function $f(z)$,

$$m\left(\frac{f'}{f}, t\right) = 0(1) \text{ when } t \to 0. \text{ Hence}$$

$$m\left(\frac{f^{(l)}}{f}, t\right) \leqslant \sum_{i=1}^{l} m\left(\frac{f^{(i)}}{f^{(i-1)}}, t\right) = 0(1),$$

Consequently,

$$m\left(\frac{\Psi}{f}, t\right) \leqslant \sum_{i=1}^{l} \left(m(a_i, t) + m\left(\frac{f^{(i)}}{f}, t\right)\right) =$$

$$\sum_{i=1}^{l} o(T(f, t)) + 0(1)) = o(T(f, t)).$$

Hence, $m(\Psi, t) \leqslant m\left(\frac{\Psi}{f}, t\right) + m(f, t)$

$$= m(f, t) + o(T(f, t)). \tag{2.1}$$

On the other hand, if $f(z)$ has a pole of order p at a point z_0 and the orders of poles of $a_i(z)$ $(i = 1, 2, \ldots, 1)$ at z_0 are not greoter than q, then the order of pole of $\Psi(z)$ at z_0 is not greater than $p + l + q \leqslant (l + 1)\, p + q$.

Therefore,

$$N(\Psi, t) \leqslant (l + 1)\, N(f, t) + \sum_{i=0}^{1} N(a_i, t)$$

$$= (1 + 1) N(f, t) + o(T(f, t)). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$T(\Psi, t) = m(\Psi, t) + N(\Psi, t) \leqslant$$
$$m(f, t) + (l + 1) N(f,t) + o (T(f,t)) \leqslant$$
$$(l + 1) T(f, t) + o(T(f,t)).$$

Theorem 2.7 is proved.

THEOREM 2.3. *Let f (z) be a non-constant meromorphic function on D, $\Psi(z)$ be given as in Theorem 2.2. Then we have*

$$T(f, t) \leqslant \overline{N}(f, t) + N\left(\frac{1}{f}, t\right) + \overline{N}\left(\frac{1}{\Psi - 1}, t\right) - N_0\left(\frac{1}{\Psi}, t\right) + o(T(f,t)).$$

Proof. Applying the basic inequality (0.3) for $\Psi(z)$, with $q = 2$, $a_0 = 0$, $a_1 = 1$ we obtain

$$m(\Psi, t) + m\left(\frac{1}{\Psi}, t\right) + m\left(\frac{1}{\Psi - 1}, t\right) \leqslant 2\, T(\Psi, t) - N_1(t) + 0(1). \qquad (2.3)$$

Now we have

$$2\, T(\Psi, t) - N_1(t) = m(\Psi, t) + m\left(\frac{1}{\Psi - 1}, t\right) + N(\Psi, t)$$
$$+ N\left(\frac{1}{\Psi - 1}, t\right) - N\left(\frac{1}{\Psi'}, t\right) - 2N(\Psi, t) + N(\Psi', t) + 0(1). \qquad (2.4)$$

On the other hand, if $\Psi(z)$ has a pole of order l at a point z_0, then $\Psi'(z)$ has a pole of order $l + 1$ at z_0 and the poles of $\Psi(z)$ must be either the poles of $f(z)$ or $a_i(z)$, hence

$$N(\Psi', t) - N(\Psi, t) = \overline{N}(\Psi, t) \leqslant \overline{N}(f, t) + \sum_{i=1}^{1} \overline{N}\, a_i, t)$$
$$= N(f, t) + o(T(f, t)).$$

Since a root of multiplicity l of the equation $\psi(z) = 1$ is a root of multiplicity $l - 1$ of the equation $\psi'(z) = 0$, we obtain

$$N\left(\frac{1}{\psi - 1}, t\right) - N\left(\frac{1}{\psi'}, t\right) = \overline{N}\left(\frac{1}{\psi - 1}, t\right) - N_0\left(\frac{1}{\psi'}, t\right)$$

It follows from (2.3), (2.4) that

$$m\left(\frac{1}{\psi}, t\right) \leqslant \overline{N}(f, t) + \overline{N}\left(\frac{1}{\psi - 1}, t\right) - N_0\left(\frac{1}{\psi'}, t\right) + 0\,(T(f, t)). \qquad (2.5)$$

In view of Theorem 2. 2, we have

$$T(f, t) = m\left(-\frac{1}{f}, t\right) + N\left(-\frac{1}{f}, t\right) + 0\,(1) \leqslant$$
$$\leqslant m\left(\frac{1}{\psi}, t\right) + m\left(\frac{\psi}{f}, t\right) N\left(\frac{1}{f}, t\right) + 0(1) =$$
$$m\left(\frac{1}{\psi}, t\right) + N\left(\frac{1}{f}, t\right) + 0\,(T(f, t)). \qquad (2.6)$$

Now theorem follows from (2.5) and (2 6).

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