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IMBEDDING THEOREMS OF SOBOLEV SPACES OF INFINITE ORDER

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INTRODUCTION

Let $\{a_\alpha\}$, $\{b_\alpha\}$ be arbitrary sequences of nonnegative numbers, G be an arbitrary domain in R^n or torus T^n and $1 \leqslant p \leqslant \infty$, $1 \leqslant r < \infty$. Then

$$W^\infty \{a_\alpha, p, r\} (G) = \{f : \| f \|_a = (\sum_{\alpha \geqslant 0} a_\alpha \| D^\alpha f \|^r_{L_p(G)})^{1/r} < \infty\}$$

is called the Sobolev space of infinite order in G.

Analogously we define the space $W^\infty \{b_\alpha, p, r\} (G)$ and consider the following imbedding

$$W^\infty \{a_\alpha, p, r\} (G) \hookrightarrow W^\infty \{b_\alpha, p, r\} (G) \qquad (0)$$

which frequently arises from the study of nonlinear differential equations of infinite order.

In [4] the necessary and sufficient conditions for imbedding and compactly imbedding (0) as a special case of imbedding of the abstract limit spaces were established. These conditions were given in terms of the asymptotic behaviour of the norms of the imbedding operators $i_{k,m} : X_k \to Y_m$. Hence, the study of the imbedding (0) is led to the difficult and unknown (at this moment) problem of the exact estimates of the norms of the imbedding of anisotropic Sobolev spaces of finite order, as their orders tend to infinity. Thus, together with the general functional criteria, the imbedding conditions which are expressed in «algebraic» terms, become very important. In particular, it is useful to give conditions in terms of the parameters a_α, b_α, p, r of the spaces $W^\infty \{a_\alpha, p, r\} (G)$ and $W^\infty \{b_\alpha, p, r\} (G)$, of the characteristic functions of these spaces etc. Most of these conditions are only sufficient, but they can be easily verified. The problem of the algebraic imbedding conditions of Sobolev spaces of infinite order was studied mainly in the one-dimensional case $(n = 1)$ (see [4], [7], [13], [14]).

In this paper we also consider the case $n = 1$. Our aim is to establish some algebraic imbedding conditions of Sobolev spaces of infinite order for the case $G = R$ by using a method different from that of Dubinsky and Balasova. Some of these results are better than the corresponding ones in [4], [6], [7], [14]. The cases $G = T$, $G = (c, \infty)$, $G = (c, d)$ were considered in [7], [13], [14].

It should be noted that the theory of Sobolev spaces of infinite order and related problems such as the nontriviality, the boundary value problems, the imbedding theory, the trace theory, the geometrical characteristics etc have been studied by Dubinsky and others (see, for example, [1]—[14]).

Imbedding conditions.

We shall assume that $a_0 > 0$, $b_0 > 0$ because in the contrary case, $W^\infty\{a_n, p, r\}(R)$ and $W^\infty\{b_n, p, r\}(R)$ become factor-spaces. Further, it is well known that $W^\infty\{a_n, p, r\}(R)$ is nontrivial if and only if $R_a > 0$, where R_a is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ (see [2]). Hence, we shall study the imbedding

$$W^\infty\{a_n, p, r\}(R) \to W^\infty\{b_n, p, r\}(R) \tag{1}$$

with the assumption $R_a > 0$.

For simplicity of notation we put

$$\|\cdot\|_p = \|\cdot\|_{L_p(R)}, W_a^\infty = W^\infty\{a_n, p, r\}(R).$$

It is clear that the inequalities

$$b_n \leqslant K a_n,\ n \geqslant 0, \tag{2}$$

where K is some constant, guarantee the imbedding (1). Of course, the condition (2) may be a criterion for imbedding (1) only in the case when the power of functions of W_a^∞ is rich enough. We have the following theorem ([5]).

THEOREM 1. *Let $a_{n+1} \leqslant a_n^2 < 1$; $n \geqslant 0$. Then the imbedding* (1) *is valid if and only if* (2) *is satisfied.*

However, in general, this condition is very limited. At least since the equalities $a_n = 0$ immediately imply $b_n = 0$ for the corresponding n. This shortcoming will be disappeared in the theorems proved below.

We have the following lemma ([15]).

LEMMA 1. *Let $1 \leqslant p \leqslant \infty$ and function $f(x) \in C^\infty(R)$ such that $D^n f(x) \in L_p(R)$, $n \geqslant 0$. Then there exists the limit*

$$d_f = \lim_{n\to\infty} \|D^n f\|_p^{1/n},$$

moreover

$$d_f = \sigma_f = \sup\{|\zeta| : \zeta \in \operatorname{supp} \tilde{f}(\zeta)\}.$$

where $\tilde{f}(\zeta) = Ff(x)$ is the Fourier transform of the function $f(x)$.

LEMMA 2. *The following equality is valid:*

$$\{d_f : f \in W_a^\infty\} = [0, R_a^{1/r}].$$

Proof. By Lemma 1 it follows that

$$\{d_f : f \in W_a^\infty\} \subset [0, R_a^{1/r}]. \tag{3}$$

To prove the inverse inclusion, assume that $g(x) \in W_a^\infty$, $d_g < R_a^{1/r}$ and $g(x) \neq$ const. The existence of such a function follows from the inclusion $F^{-1} C_0^\infty(-R_a^{1/r}, R_a^{1/r}) \subset W_a^\infty$ which is valid by the well-known Paley-Wiener-Schwartz theorem and the Bernstein-Nikolsky inequality (see [16], p. 115). Then $g(x)$ differs from polynomials. Therefore, from (3) and Lemma 1 we get $0 < \sigma_g = d_g < R_a^{1/r}$. Consequently, all the functions $f(x) = g(\lambda x)$, $0 < \lambda d_g < R_a^{1/r}$ belong to W_a^∞ because $d_f = \lambda d_g$. This means

$$[0, R_a^{1/r}] \subset \{d_f : f \in W_a^\infty\}.$$

Therefore, we can choose functions $f_n(x) \in W_a^\infty$ so that $\sigma_{f_n} < \sigma_{f_{n+1}} < R_a^{1/r}$, $\lim_{n\to\infty} \sigma_{f_n} = R_a^{1/r}$ and supp $\tilde{f}_n(\xi) \cap$ supp $\tilde{f}_m(\xi)$ are empty if $m \neq n$.

Put

$$f(x) = \sum_{n=1}^{\infty} \gamma_n f_n(x), \quad \sum_{n=1}^{\infty} |\gamma_n| \, |||f_n|||_a < \infty.$$

Then $f(x) \in W_a^\infty$ and $\sigma_f = R_a^{1/r} = d_f$. This completes the proof.

From Lemma 2 and the Bernstein-Nikolsky inequality we derive

THEOREM 2. *We have*

1. If $R_a > R_b$ *then* $W_a^\infty \subset\!\!\!/\!\!\!\sqsubset W_b^\infty$;

2. If $R_a < R_b$ *then* $W_a^\infty \hookrightarrow W_b^\infty$;

3. If $R_a = R_b$ *and* $\sum_{n=0}^{\infty} b_n R_a^n < \infty$ *then* $W_a^\infty \hookrightarrow W_b^\infty$.

LEMMA 3. *Let* $\lambda > 0$. *Then* $T_\lambda(f) = \lambda^{-1/p} f(\lambda^{-1} x)$ *is an one-to-one isometric mapping of* $W^\infty\{a_n, p, r\}(R)$ *onto* $W^\infty\{a_n \lambda^{nr}, p, r\}(R)$.

Proof. Since

$$\| D^n T_\lambda(f) \|_p = \lambda^{-n} \| D^n f \|_p, \quad n \geqslant 0,$$

the proof is straight-forward.

THEOREM 3. *Let* $R_a < \infty$ *and suppose that there exists a number* $R_a \leqslant \xi < \infty$ *so that*

$$M = \sup_{n \geqslant 0} \left(\sum_{k=0}^{n} b_k \xi^k\right) \left(\sum_{k=0}^{n} a_k \xi^k\right)^{-1} < \infty. \tag{4}$$

Then the imbedding (1) holds.

Proof. Put $\lambda = \xi^{1/r}$. Then by Lemma 3 one can see that the imbedding (1) holds if and only if the following imbedding

$$W^{\infty}\{a_n \xi^n, p, r\}(R) \rightarrow W^{\infty}\{b_n \xi^n, p, r\}(R) \tag{5}$$

is valid.

To prove (5), let $f(x) \in W^{\infty}\{a_n \xi^n, p, r\}(R)$.

We notice that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n \xi^n z^n$ is not more than one. Hence, by Lemmas 1 and 2 we get $\sigma_f \leqslant 1$. Therefore, applying the Bernstein—Nikolsky inequality we have

$$\| D^{n+1} f \|_p \leqslant \sigma_f \| D^n f \|_p \leqslant \| D^n f \|_p, \ n \geqslant 0, \tag{6}$$

Further, let $N \geqslant 1$ be fixed. Taking account of (4), (6) and the Abel transform

$$\sum_{n=0}^{N} x_n y_n = y_N \sum_{n=0}^{N} x_n + \sum_{n=0}^{N-1} (y_n - y_{n+1}) \sum_{k=0}^{n} x_k$$

we get

$$\sum_{n=0}^{N} b_n \xi^n \| D^n f \|_p^r =$$

$$= \| D^N f \|_p^r \sum_{n=0}^{N} b_n \xi^n + \sum_{n=0}^{N-1} (\| D^n f \|_p^r - D^{n+1} f \|_p^r) \sum_{k=0}^{n} b_k \xi^k \leqslant$$

$$\leqslant M[\| D^N f \|_p^r \sum_{n=0}^{N} a_n \xi^n + \sum_{n=0}^{N-1} (\| D^n f \|_p^r - \| D^{n+1} f \|_p^r) \sum_{k=0}^{n} a_k \xi^k] =$$

$$= M \sum_{n=0}^{N} a_k \xi^k \| D^n f \|_p^r .$$

Consequently, by letting $N \to \infty$ we obtain

$$\sum_{n=0}^{\infty} b_n \xi^n \| D^n f \|_p^r \leqslant M \sum_{n=0}^{\infty} a_n \xi^n \| D^n f \|_p^r$$

The imbedding (5) has been proved. The proof is complete.

COROLLARY 1. *Let $R_a < \infty$ and*

$$\sup_{n \geqslant 0} \left(\sum_{k=0}^{n} b_k R_a^k\right) \left(\sum_{k=0}^{n} a_k R_a^k\right)^{-1} < \infty. \tag{7}$$

Then the imbedding (1) holds

LEMMA 4. *Let $f(x) \in W^{\infty}\{a_n, p, r\}(R)$. Then for $n \geqslant 0$ we get*

1. $\| D^n f \|_p^r \leqslant \left(\frac{\pi}{2}\right)^r R_a^n \left(\sum_{k=0}^{n} a_k R_a^k\right)^{-1} \sum_{k=0}^{n} a_k \| D^k f \|_p^r$, *if* $R_a < \infty$;

2. $\| D^n f \|_p^r \leqslant 2\left(\frac{\pi}{2}\right)^r \sup_{\xi>0} \left[\xi^n a^{-1}(\xi) \right] \| f \|_a^r$, *if* $R_a = \infty$,

where $a(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$.

The next theorems are direct corollaries of Lemma 4.

THEOREM 4. *Let* $R_a < \infty$ *and*

$$\sum_{n=0}^{\infty} b_n R_a^n \left(\sum_{k=0}^{n} a_k R_a^k \right)^{-1} < \infty \tag{8}$$

Then the embedding (1) holds .

THEOREM 5. *Let* $R_a = \infty$ *and*

$$\sum_{n=0}^{\infty} b_n \sup_{\xi>0} [\xi^n a^{-1}(\xi)] < \infty. \tag{9}$$

Then the imbedding (1) holds.

Remark 1. Lemma 4 and Theorems 4,5 were proved in [4] under the assumption $p = r$, The case when p, r are arbitrary can be established by arguments similar to those of [4].

Let $\sigma > 0$. Denote by $M_{\sigma p}$ the space of all functions $f(x)$ of exponential type σ such that $f(x) \in L_p(R)$. Then by the density of $\bigcup_{\sigma>0} M_{\sigma p}$ in W_a^{∞} and from the proof of Theorem 3 we get the following theorem.

THEOREM 6. *Let* $R_a = \infty$ *and there are numbers* $\xi_m \uparrow +\infty$ *such that*

$$\sup_{m \geqslant 0} \sup_{n \geqslant 0} \left(\sum_{k=0}^{n} b_k \xi_m^k \right) \left(\sum_{k=0}^{n} a_k \xi_m^k \right)^{-1} < \infty.$$

Then the imbedding (1) is valid.

LEMMA 5. *Let* $0 \leqslant a_n$, $0 \leqslant x_n \leqslant x_{n+1}$, $n \geqslant 0$ *be arbitrary sequences of numbers. Then the following identity is valid:*

$$\sum_{n=0}^{\infty} a_n x_n = x_0 \varepsilon_0^a + \sum_{k=0}^{\infty} (x_{k+1} - x_k)\, \varepsilon_{k+1}^a, \tag{10}$$

where $\varepsilon_k^a = \sum_{j=k}^{\infty} a_j$, $k \geqslant 0$.

Proof. Let these be given $n \geqslant 0$. Then

$$x_0 \varepsilon_0^a + \sum_{k=0}^{n} (x_{k+1} - x_k)\, \varepsilon_{k+1}^a =$$

$$= x_0 (\varepsilon_0^a - \varepsilon_1^a) + x_1 (\varepsilon_1^a - \varepsilon_2^a) + \ldots + x_n (\varepsilon_n^a - \varepsilon_{n+1}^a) + x_{n+1} \varepsilon_{n+1}^a =$$

$$= a_0 x_0 + a_1 x_1 + \ldots + a_n x_n + x_{n+1} \varepsilon_{n+1}^a. \tag{11}$$

Indeed, if $\sum_{n=0}^{\infty} a_n x_n = \infty$, then (10) is an immediate consequence of (11).

Now let $\sum_{n=0}^{\infty} a_n x_n < \infty$. Because of (11) it is enough to show that $\lim_{n\to\infty} x_n \varepsilon_n^a = 0$ which follows from

$$0 \leqslant x_n \varepsilon_n^a = x_n(a_n + a_{n+1} + \ldots) \leqslant$$
$$\leqslant a_n x_n + a_{n+1} x_{n+1} + \ldots \quad .$$

The proof is thus complete.

THEOREM 7. *Suppose that there exists a number $0 < \xi < R_a \leqslant \infty$ such that*

$$M = \sup_{n \geqslant 0} \left(\sum_{k=n}^{\infty} b_k \xi^k\right) \left(\sum_{k=n}^{\infty} a_k \xi^k\right)^{-1} < \infty. \tag{12}$$

Then the imbedding (1) holds

Proof. Put $\lambda = \xi^{1/r}$. Then by Lemma 3 we have to show the validity of the following imbedding

$$W^{\infty}\{c_n, p, r\}(R) \hookrightarrow W^{\infty}\{d_n, p, r\}(R) \tag{13}$$

where $c_n = a_n \xi^n$, $d_n = b_n \xi^n$, $n \geqslant 0$.

Indeed put $f_n = \| D^n f \|_p$, $n \geqslant 0$. We denote by f_n^c the convex regularization of the sequence f_n by means of logarithms. Then (see [17])

$$f_0^c = f_0, \; f_n^c = \inf\{f_n, f_k^{(m-n)/(m-k)} f_m^{(n-k)/(m-k)}, 0 \leqslant k < n < m\}, \; n \geqslant 1$$

and

$$f_n^c \leqslant (f_k^c)^{(m-n)/(m-k)} (f_n^c)^{(n-k)/(m-k)}, \; 0 \leqslant k < n < m. \tag{14}$$

On the other hand, it follows from the Kolmogorov—Stein inequality (see[18]) that

$$\| D^n f \|_p^{m-k} \leqslant \left(\frac{\pi}{2}\right)^{m-k} \| D^k f \|_p^{m-n} \| D^m f \|_p^{n-k}, \; 0 \leqslant k < n < m.$$

Therefore,

$$f_n^c \leqslant f_n \leqslant \frac{\pi}{2} f_n^c, \qquad n \geqslant 0. \tag{15}$$

Hence, the imbedding (13) holds if and only if there is a constant $M_1 < \infty$ such that

$$\sum_{n=0}^{\infty} d_n (f_n^c)^r \leqslant M_1 \sum_{n=0}^{\infty} c_n (f_n^c)^r, \; f \in W_c^{\infty}. \tag{16}$$

It is enough to show the validity of (16) for all $f \in W_c^{\infty}$, $f_0^c = 1$.

Fix $f \in W_c^{\infty}$, $f_0^c = 1$, $\sigma_f > 1$. The existence of the function $f(x)$ follows from $R_c = \frac{R_a}{\xi} > 1$ and Lemma 2.

Put $n_0 = \inf\{ n : f_n^c > 1 \}$. Then by (14) and $f_0^c = 1$ we have

$$1 < (f_{n_0}^c)^{1/n_0} \leqslant (f_m^c)^{1/m}, \; m > n_0.$$

Therefore,

$$1 < (f_n^c)^{1/n} \leqslant (f_m^c)^{1/m}, \; n_0 \leqslant n < m.$$

Hence,

$$f_n^c \leqslant (f_{n+1}^c)^{n/n+1} < f_{n+1}^c, \; n_0 \leqslant n.$$

Consequently, taking account of Lemma 5 and (12), we get

$$\sum_{n=n_0}^{\infty} d_n (f_n^c)^r = (f_{n_0}^c)^r \sum_{n=n_0}^{\infty} d_n + \sum_{n=n_0}^{\infty} ((f_{n+1}^c)^r - (f_n^c)^r) \sum_{k=n+1}^{\infty} d_k \leqslant$$

$$\leqslant M \; [(f_{n_0}^c)^r \sum_{n=n_0}^{\infty} c_n + \sum_{n=n_0}^{\infty} ((f_{n+1}^c)^r - (f_n^c)^r) \sum_{k=n+1}^{\infty} c_k] =$$

$$= M \sum_{n=n_0}^{\infty} c_n (f_n^c)^r < \infty.$$

which shows that $f \in W_d^\infty$. Therefore, since $f \in W_c^\infty$, $\sigma_f > 1$ is arbitrarily chosen, we get from Lemma 2

$$[0, R_c^{1/r}] \subset [0, R_d^{1/r}]$$

i. e., $R_c \leqslant R_d$. Hence, taking the definition of n_0 into account, we obtain

$$\sum_{n=0}^{\infty} d_n (f_n^c)^r = \sum_{n=0}^{n_0-1} d_n (f_n^c)^r + \sum_{n=n_0}^{\infty} d_n (f_n^c)^r \leqslant$$

$$\leqslant \sum_{n=0}^{n_0-1} d_n + M \sum_{n=n_0}^{\infty} c_n (f_n^c)^r \leqslant M_1 \sum_{n=0}^{\infty} c_n (f_n^r)^r, \tag{17}$$

$$\text{where } M_1 = M + \frac{1}{c_0} \sum_{n=0}^{\infty} d_n .$$

At last, let $f \in W_c^\infty$, $f_0^c = 1$, $\sigma_f \leqslant 1$. Then by (14), (15), $f_0^c = 1$ and Lemma 1 we get

$$f_n^c \leqslant \lim_{m \to \infty} (f_m^c)^{n/m} = \sigma_f^n \leqslant 1, \; n \geqslant 1.$$

Therefore,

$$\sum_{n=0}^{\infty} d_n (f_n^c)^r \leqslant \sum_{n=0}^{\infty} d_n \leqslant M_1 c_0 \leqslant M_1 \sum_{n=0}^{\infty} c_n (f_n^c)^r . \tag{18}$$

Combining (17) and (18) we have

$$\sum_{n=0}^{\infty} d_n (f_n^c)^r \leqslant M_1 \sum_{n=0}^{\infty} c_n (f_n^c)^r$$

for all $f \in W_c^{\infty}$, $f_0^c = 1$. This completes the proof of (16) and the theorem follows.

COROLLARY 2. *Let $R_a = \infty$ and*

$$\sup_{n \geqslant 0} \varepsilon_n^b (\varepsilon_n^a)^{-1} < \infty . \tag{19}$$

Then the imbedding (1) holds.

Remark 2. Condition (7) is essentially weaker than (8). Indeed, let (8) be satisfied. Then there exists a constant $C < \infty$ such that

$$\sum_{n=0}^{m} b_n R_a^n \Big(\sum_{k=0}^{n} a_k R_a^k\Big)^{-1} \leqslant C, \; m = 0, 1, \ldots$$

For fixed m, we have

$$b_n R_a^n \Big(\sum_{k=0}^{m} a_k R_a^k\Big)^{-1} \leqslant b_n R_a^n \Big(\sum_{k=0}^{n} a_k R_a^k\Big)^{-1}, \; 0 \leqslant n \leqslant m . \tag{20}$$

On adding inequalities (20) we find

$$\Big(\sum_{n=0}^{m} b_n R_a^n\Big) \Big(\sum_{n=0}^{m} a_n R_a^n\Big)^{-1} \leqslant \sum_{n=0}^{m} b_n R_a^n \Big(\sum_{k=0}^{n} a_k R_a^k\Big)^{-1} \leqslant C$$

for any $m \geqslant 0$, which immediately implies (7).

Further, put $a_0 = 1$, $a_k = 1/k$, $k \geqslant 1$ and

$$b_k = \begin{cases} \ln 2, & k = 2^n, \; n \geqslant 0 \\ 0, & k \neq 2^n, \; n \geqslant 0. \end{cases}$$

Then $R_a = 1$ and it is easy to verifyed that (7) holds but (8) does not.

Remark 3. It is easy to give an example for which (19) holds but (9) does not.

Example. Put $a_n = (n!)^{-1}$, $n \geqslant 0$; $b_0 = 1$, $b_n = e^n (2\pi n^{n+1})^{-1}$, $n \geqslant 1$. Then $R_a = \infty$ and

$$\sup_{\xi > 0} [\xi^n a^{-1}(\xi)] = \sup_{\xi > 0} \xi^n e^{-\xi} = n^n e^{-n}, \; n \geqslant 1.$$

Therefore

$$\sum_{n=1}^{\infty} b_n \sup_{\xi > 0} [\xi^n a^{-1}(\xi)] = \sum_{n=1}^{\infty} \frac{1}{2\pi n} = \infty.$$

At the same time, we have $\varepsilon_n^b (\varepsilon_n^a)^{-1} < 1$, $n \geqslant 1$ because

$$b_n a_n^{-1} = e^n n! \, (2\pi n^{n+1})^{-1} < 1, \; n = 1, 2, \ldots$$

Remark 4. Let $p=r=2$. Then

$$\sup_{0<\xi<R_a}\left(\sum_{n=0}^{\infty} b_n\xi^n\right)\left(\sum_{n=0}^{\infty} a_n\xi^n\right)^{-1}<\infty \tag{21}$$

is a criterion for imbedding (1) (see [4]). The main tool for this proof was the Parseval equality which is not suitable when $p\neq 2$.

THEOREM 8. *If the imbedding* (1) *holds then we have* (21).

Proof. We have to show that (21) is deduced from the following condition:

$$\sum_{n=0}^{\infty} b_n(f_n^c)^r \leqslant M\sum_{n=0}^{\infty} a_n(f_n^c)^r\ ,\ f\in W_a^{\infty}, \tag{22}$$

where M is some constant.

Let $0<\xi<R_a^{1/r}$ be given. Then, by Lemmas 1 and 2 there is a function $f\in W_a^{\infty}$ such that $\sigma_f=\xi$. It follows from (14) that

$$(f_n^c)^2\leqslant f_{n-1}^c f_{n+1}^c\ ,\ n\geqslant 1.$$

Hence

$$\varepsilon_n\leqslant \varepsilon_{n+1},\ n\geqslant 1, \tag{23}$$

where $\varepsilon_n=f_n^c/f_{n-1}^c$, $n\geqslant 1$. Therefore, by Lemma 1 and (15), we get

$$f_n^c=f_o^c\varepsilon_1\dots\varepsilon_n,\ n\geqslant 1,\ \lim_{n\to\infty}\varepsilon_n=\xi.$$

Hence, in view of (23) we have

$$f_n^c=\xi^n f_o^c\delta_1\delta_2\dots\delta_n,\ \varepsilon_n=\xi\delta_n,\ 0<\delta_n\leqslant\delta_{n+1},\ n\geqslant 1,\ \lim_{n\to\infty}\delta_n=1. \tag{24}$$

Denote by $N=N(f)$ the number such that

$$\sum_{n=o}^{N} b_n\xi^n\geqslant\frac{1}{2}b(\xi)=\frac{1}{2}\sum_{n=o}^{\infty} b_n\xi^n. \tag{25}$$

Since $\sigma_f<\infty$ and from the Bernstein-Nikolsky inequality we get $D^k f\in W_a^{\infty}$ for any $k\geqslant 0$. Therefore, by (22) and (24) we obtain

$$\sum_{n=o}^{\infty} b_n(\xi^{n+k}f_o^c\delta_1\dots\delta_{n+k})^r\leqslant M\sum_{n=o}^{\infty} a_n(\xi^{n+k}f_o^c\delta_1\dots\delta_{n+k})^r,\ k\geqslant 0.$$

Hence,

$$\sum_{n=o}^{\infty} b_n\xi^{nr}\delta_{k+1}^r\dots\delta_{n+k}^r\leqslant M\sum_{n=o}^{\infty} a_n\xi^{nr}\delta_{k+1}^r\dots\delta_{n+k}^r,\ k\geqslant 0.$$

Therefore, we deduce from (24) and (25) that

$$\left(\sum_{n=o}^{N} b_n\xi^{nr}\right)\delta_{k+1}^{rN}\leqslant\sum_{n=o}^{N} b_n\xi^{nr}\delta_{k+1}^r\dots\delta_{n+k}^r\leqslant$$

$$\leqslant M\sum_{n=o}^{\infty} a_n\xi^{nr}\delta_{k+1}^r\dots\delta_{n+k}^r\leqslant M\sum_{n=o}^{\infty} a_n\xi^{nr}. \tag{26}$$

Now, we choose a number $k \geqslant 1$ such that $\delta_{k+1}^{rN} \geqslant \frac{1}{2}$. Then, combining (25) and (26) yields

$$\sum_{n=0}^{\infty} b_n \xi^{nr} \leqslant 4M \sum_{n=0}^{\infty} a_n \xi^{nr}.$$

The proof is thus complete.

It is natural to ask whether the imbedding (1) is compact?

Theorem 9. Let $W_a^{\infty} \neq \{0\}$. *Then imbedding (1) is not compact.*

Proof. It is enough to show that the following imbedding

$$W_a^{\infty} \hookrightarrow L_p(R)$$

is not compact. Fix a function $f(x) \in W_a^{\infty}$ such that $\sigma_f \leqslant 1$ and $f(x) \neq$ const. Then $\lim_{|x| \to \infty} f(x) = 0$ (see [16], p. 117).

Put

$$E = \{g_n(x) = f(x+n),\ n \geqslant 0\}.$$

Then E is bounded in W_a^{∞} because $|||g_n|||_a = |||f|||_a$, $n \geqslant 0$. On the other hand, the Nikolsky inequality (see [16], p.125) shows that

$$\|D^n v\|_{\infty} \leqslant 2\sigma_v^{1/p} \|D^n v\|_p\ ,\ n \geqslant 0.$$

Therefore, all we have to prove is that we cannot find a subsequence of E which converges in $L_{\infty}(R)$. For this purpose, choose a point $x_o \in R$ so that $|f(x_o)| = \|f\|_{\infty}$. Then by

$$\| g_n(x) - g_{n+m}(x) \|_{\infty} \geqslant | f(x_o) - f(x_o+m) |\ ,\ m, n \geqslant 1$$

we get

$$\lim_{m \to \infty} \| g_n - g_{n+m} \|_{\infty} \geqslant \lim_{m \to \infty} | f(x_o) - f(x_o+m) | = | f(x_o) | > 0$$

for any $n \geqslant 1$, from which the desired conclusion follows. The proof of the theorem is thus complete.

Concluding remark. It should be noted that $1 \leqslant r < \infty$ and $r = \infty$ are quite different cases. That is why in the paper we did not consider the limiting case $r = \infty$ in the paper and shall discuss it elsewhere.

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