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ON A FIXED POINT THEOREM OF KRASNOSELSKII AND ITS APPLICATIONS

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INTRODUCTION

Let X be a Banach space and let K be a bounded closed convex subset of X. A well-known theorem of Krasnoselskii [8] states that if U is a contraction on K (i. e. $\| Ux - Uy \| \leqslant k \| x - y \|$ for $0 < k < 1$) and F is a compact operator on K such that

$$Ux + Fy \in K \text{ for every } x, y \text{ in } K \qquad (*)$$

then $U + F$ has a fixed point. Krasnoselskii's theorem has been extended by Nashed and Wong [9] to the case U is either a φ—contraction or a bounded linear operator such that U^p is a φ—contraction for some p.

Our aim in this paper is to present another variant to Krasnoselskii's theorem and its applications to the existence of periodic solutions of differential equations on a Banach space. In our version, K is the closure of an unbounded open convex subset of X, F is a bounded-compact operator and the condition (*) is to be replaced by the following weaker one

$$Ux + Fx \in K \text{ for every } x \text{ in } K \qquad (**)$$

We shall consider operators that are quasibounded in the following sense (Gramas [5])

$$\limsup_{\| x \| \to \infty} \| Tx \| / \| x \| < \infty.$$

If T is a quasibounded operator, we put

$$| T | = \limsup_{\|x\| \to \infty} \| Tx \| / \| x \| .$$

Then $| T |$ is called the quasinorm of T. If T is bounded linear then T is precisely equal to the norm of T as a bounded linear operator.

The remainder of the paper is divided into two sections. Section 1 is devoted to a fixed point theorem of the Krasnoselskii type. Section 2 is devoted to its applications to the existence of periodic solutions of differential equations in Banach spaces.

1. A FIXED POINT THEOREM OF KRASNOSELSKII TYPE

Throughout this paper. X denotes a real Banach space, G denoted an unbounded convex open set in X and $Cl(G)$ its closure.

DEFINITION 1. 1. Let φ be a continuous real-valued function on the positive real number such that

$$0 < \varphi(r) < r \quad \text{for} \quad r > 0.$$

An operator $U : Cl(G) \to X$ is said to be a φ-*contraction* (Boyd and Wong [2]) if

$$\| Ux - Uy \| \leqslant \varphi(\| x - y \|) \text{ for every } x, y \text{ in } Cl(G).$$

DEFINITION 1.2. A continuous operator $F : Cl(G) \to X$ is said to be *bounded-compact* if subset $F(A)$ is relatively compact for each bounded subset A of $Cl(G)$ such that $F(A)$ is bounded.

We have the following theorem

THEOREM 1. *Let G be an unbounded convex open set in X and let $O \in G$. Let*

$$U : Cl(G) \to X$$

be either a φ-contraction or the restriction to $Cl(G)$ of a bounded linear operator U' on X such that $(U')^p$ is a φ-contraction for some $p \geqslant 1$. Let

$$F : Cl(G) \to X$$

be a bounded-compact operator. Put $T = U + F$ and suppose that T maps $Cl(G)$ into itself. If $| T | < 1$ then T has a fixed point.

Remarks. 1) Note that if in addition G is bounded and T maps $Cl(G)$ into itself then F is compact. This case has been studied by Ang-Hoa (see Theorem 1 [1]).

2. This theorem is to be compared with Theorem 4 of Browder – Nussbaum [3] and with Corollary 6 of Hale and Lopes [6], and it is a generalization of Theorem 1 part (ii) of Ang-Hoa (loc. cit.).

For the proof of Theorem 1, we shall need some properties of the Browder-Nussbaum degree [3]. Let G be a domain in X and H, F be mappings of Cl(G) into X satisfying the following conditions:

a) For each fixed v in $Cl(G)$, the mapping

$$S_v : Cl(G) \to X$$

defined by $S_v u = Hu + Fv$ is a homeomorphism G onto an open subset G_v of X, mapping $Cl(G)$ homeomorphically onto $Cl(G_v)$.

b) The mapping $v \to S_v$ is a locally compact mapping of $Cl(G)$ into the space of homeomorphisms of $Cl(G)$ into X with the topology of uniform convergence on $Cl(G)$.

Let $Tu = Hu + Fu$ for u in $Cl(G)$. Suppose $T^{-1}(O)$ is a compact subset of G. Then $\deg(T, G, O)$ is defined. (In fact, the Browder – Nussbaum degree is defined for more general operators, but this simplified version is all that we shall need).

The following proposition is implicitly contained in the Browder – Nussbaum paper (loc. cit).

PROPOSITION 1.1. *(i) If $deg(T, G, 0) \neq 0$, then there exists an x in G such that $Tx = 0$.*

(ii) Let A, B be continuous mappings of $Cl(G) \times [0, 1]$ into X such that $A(.,t)$ and $B(.,t)$ are continuous uniformly with respect to t in $[0, 1]$, and for each $0 \leqslant t \leqslant 1$, the map $A_t(.) \equiv A(.,t)$ is a homeomorphism of G onto an open set G_t of X, taking $Cl(G)$ homeomorphically onto $Cl(G_t)$ and the map $B_t(.) = B(.,t)$ is a locally compact operator of $Cl(G)$ into X (i. e. each point x of $Cl(G)$ has a neighborhood N such that $B_t(N)$ is relatively compact). Suppose that for each $0 \leqslant t \leqslant 1$, the pair A_t, B_t satisfies condition (b) above. Suppose further that for each t, the set $(A_t + B_t)^{-1}(0)$ is compact and

$$(A_t + B_t)^{-1}(0) \cap \partial G = \phi$$

where G denotes the boundary of G. Then

$$deg(A_0 + B_0, G, O) = deg(A_1 + B_1, G, O).$$

We shall also need the following lemmas:

LEMMA 1.1. *Let U, F satisfy the conditions of Theorem 1. Then for each $0 \leqslant t \leqslant 1$ the map $H_t = I - tU$ is a homeomorphism of G onto open subset of X, taking $Cl(G)$ homeomorphically onto $Cl(H_t(G))$.*

Proof. We have

$$\|x - y\| - \varphi(\|x - y\|) \leqslant \|H_t(x) - H_t(y)\| \leqslant \|x - y\| + \varphi(\|x - y\|)$$ which show that H_t is a homeomorphism of $Cl(G)$ onto a closed subset of X. We shall show that $H_t(G)$ is an open subset of X. Let $x_0 \in G$, and let $r > 0$ be such that the closed ball $B'(x_0, r)$ is contained in G. Put $\rho = \sup\{\varphi(s) \colon 0 \leqslant s \leqslant r\}$. Then $\rho < r$. For v with $\|v\| < r - \rho$ define the map V on the closed ball $B'(0, r)$ as follows.

$$Vh = tU(x_0 + h) - y_0 + v$$

where $y_0 = tU(x_0)$. We shall show that V maps $B'(0, r)$ into itself. Indeed

$$\|Vh\| \leqslant \|tU(x_0 + h) - tU(x_0)\| + \|v\| + t\varphi(\|h\|) + \|v\| \leqslant \rho + r - \rho = r.$$

Since, it is clear that V is a φ-contraction, V has a fixed point h by a theorem of Boyd and Wong (loc. cit.) i.e.

$$h = tU(x_0 + h) - y_0 + v$$

or

$$x_0 + h - tU(x_0 + h) = x_0 - y_0 + v.$$

We have proved that the open ball $B(x_0 - y_0, r - \rho)$ is contained in the image of $B'(x_0, r)$ under H_t. It follows that G has an open image under H_t as claimed.

The case U is the restriction to $Cl(G)$ of a bounded linear operator U' such that $(U')^p$ is a φ-contraction for some $p < 1$ is handled in a similar way.

LEMMA 1.2. *Let U, F, T satisfy the conditions of Theorem 1. Then for each $0 \leqslant t \leqslant 1$, the set $(I - tT)^{-1}(0)$ is compact.*

Remark. Note that T can map some bounded sets onto unbounded sets.

Proof. Since $|T| < 1$, there exists for each k with $|T| < k < 1$ an r such that $\|Tx\| < k\|x\|$ forall x in Cl(G) with $\|x\| \geqslant r$. This follows that for all x satisfying

$$x = tT(x) \text{ with } 0 \leqslant t \leqslant 1$$

we must have $\|x\| \leqslant r$. Put

$$A_t = (I - tT)^{-1}(0)$$

then A_t is closed and bounded. We have furthermore

$$tF(A_t) = (I - tU)A_t$$

which shows that $F(A_t)$ is bounded for each $0 \leqslant t \leqslant 1$ and $A_0 = \{0\}$. It follows from the bounded-compactness of F that $F(A_t)$ is relatively compact.

Since $I-tU$ is homeomorphism for each $0 \leqslant t \leqslant 1$, by Lemma 1.1, and since

$$A_t = (I - tU)^{-1} tF(A_t)$$

the set A_t is compact.

Proof of Theorem 1. Since F is bounded-compact, F is locally compact. If $I-T$ does not vanish on the boundary ∂G of G, then, since $0 \in G$ and since G is convex, $I-tT$ does not vanish on ∂G for $0 \leqslant t \leqslant 1$. Consider the homotopy $I-tT$, $0 \leqslant t \leqslant 1$. From Lemmas 1.1, 1.2 and Proposition 1.1 it follows that

$$\deg(I-T, G, 0) = \deg(I, G, 0) = 1.$$

Hence T has a fixed point in G. The proof is complete.

COROLLARY 1. *Let U be either a φ-contraction on X or a bounded linear operator on X such that some iterate U^p, $p \geqslant 1$, is a φ-contraction. Let F be a bounded—compact operator on X. Put $T = U + F$.*
If $|T| < 1$ then $R(I-T) = X$, where R denotes the range of a map.

Proof. This follows from Theorem 1 for $G = X$. Indeed, if y is any point of X, then the operator $T + y$ satisfies $|T + y| < 1$. Hence, by Theorem 1, $T + y$ has a fixed point, say x, which clearly satisfies

$$x - Tx = y.$$

2. APPLICATIONS

As an application of the previous result, we shall consider the ω-periodic equation

$$\frac{dx}{dt} = A(t)x + f(t, x), \; t \geqslant 0 \tag{1}$$

where the unknown x is a function on $[0, \infty]$ to a real Banach space X and $A(t)$ and f satisfy the conditions:

(A_1) $\{A(t)\}$ is a family of bounded linear operators on X, continuous and periodic with period ω in t.

(A_2) The map $f : [0, \infty) \times X \to X$ is continuous in (t, x), periodic with period in t and f is compact, i.e. f maps bounded subsets of $[0, \infty) \times X$ into relatively compact subsets of X.

(A_3) For any given φ in X, the equation (I) has a unique solution $x(\varphi)(t)$ on $[0, \infty)$ satisfying

$$x(\varphi)(0) = \varphi.$$

We shall begin with the following

THEOREM 2. Let X be a real Banach space with norm $|.|$. Let:

(i) $\{A(t)\}$ *be a family of bounded linear operators on X and continuous in t.*

(ii) $f : [0,\infty) \times X \to X$ *be compact,*

(iii) $\lim_{x} |f(t, x)| / |x| = 0$ *uniformly in t on each bounded interval of* $[0,\infty)$.

Consider the initial value problem

$$\begin{cases} \frac{dx}{dt} A(t) \times (t) + f(t,x), \ t \geqslant 0 \\ x(0) = \varphi. \end{cases} \qquad (II)$$

Then:

(a) The Problem (II) has a solution on $[0, a]$ *for* $a > 0$ *arbitrary.*

(b) If we suppose in addition that

(iv) For any given φ *in X, the Problem (II) has a unique solution* $x(\varphi)$ *on* $[0,a]$.

Then, the map $\varphi \to x(\varphi)$ *is continuous, and furthermore, the map*

$$T : [0, a] \times X \to X$$

defined by $T(t, \varphi) = x(\varphi)(t)$ *is continuous in* (t, φ).

Proof. The Problem (II) is equivalent to the integral equation

$$x(t) = \varphi + \int_0^t [A(s) \ x(s) + f(s, x(s))] \ ds, \qquad t \geqslant 0$$

Let E be the Banach space of continuous functions on $[0, a]$ to X with norm

$$\| x \| = \sup \{ |x(t)|, t \in [0, a] \}.$$

We define the operators U, $F: E \to E$ by

$$\begin{cases} U(x)(t) = \int_0^t A(s)x(s)ds, \\ F(x)(t) = \int_0^t f(s,x(s))ds + \varphi, \text{ for all } t \in [0, a]. \end{cases} \qquad (4)$$

Then U is a bounded linear operator and

$$\| U^n \| \leqslant (ma)^n/n! \text{ for all } n \in N \qquad (5)$$

where

$$m = \sup \{ \| A(t) \|, t \in [0, a] \}. \qquad (5')$$

For the proof of Theorem 2, we need some lemmas

LEMMA 2.1. *Let E be a Banach space with norm* $|.|$ *and let U be a bounded linear operator on E such that*

$$\limsup_{n \to \infty} (\| U^n \|)^{1/n} = \alpha < 1.$$

Then there exists an equivalent norm $|.|_1$ *on E such that*

$$\| U \|_1 < 1.$$

Proof. For k such that $\alpha < k < 1$, there is n_0 such that for all $n \geqslant n_0$

$$(\| U^n \|)^{1/n} < k \text{ or } \| U^n \| < k^n.$$

We put

$$| x |_1 = \sum_0^\infty | U^n x | , \ x \in E. \tag{6}$$

It is clear that $| \cdot |_1$ is a norm on E and there is $K > 1$ such that

$$| x | \leqslant | x_1 | \leqslant K | x | \qquad \text{for all } x \in E.$$

Furthermore, for $x \neq 0$, we have

$$| x |_1 = | x | + | Ux |_1$$

or

$$| Ux |_1 = | x |_1 - | x | \leqslant (1 - 1/K) \ | x |_1.$$

Hence $\| U \|_1 < 1$. The proof is complete.

Let U be defined by (4), then from (5) we have

$$\limsup_{n \to \infty} (\| U^n \|)^{1/n} \leqslant \lim_{n \to \infty} \frac{ma}{(n!)}^{1/n} = 0.$$

Hence there exists an equivalent norm on E such that

$$\| U \|_1 < 1. \tag{7}$$

LEMMA 2.2. *Let F be defined by (4). Then F is compact and $| F |_1 = 0$ where $| F |_1$ is the quasinorm of F corresponding to the norm $| . |_1$ on E defined in* (6).

Proof. We first prove that F is continuous. Let $\lim_{x \to \infty} x_n = x$ in E. We claim that:

(a) For any $\varepsilon > 0$, there exist n_0 and $\delta > 0$ such that for all $t, t \in [0, a]$, $| t - t | < \delta$ and $n \geqslant n_0$, we have

$$| x_n (t) - x_n (t') | < \varepsilon/3.$$

Indeed, since x is uniformly continuous on $[0,a]$, there is $\delta > 0$ such that for all $t, t \in [0, a]$, $| t - t' | < \delta$,

$$| x (t) - x (t') | < \varepsilon/3.$$

Since $\lim_{x \to \infty} x_n = x$, there is n_0 such that for all $n \geqslant n_0$

$$| x_n (t) - x (t) | < \varepsilon/3 \qquad \text{for all } t \in [0,a].$$

For all $n \geqslant n_0$ and $| t - t' | < \delta$ we get

$$| x_n (t) - x_n (t') | \leqslant | x_n (t) - x (t) + x (t) - x (t') | + | x (t') - x_n (t') | < \varepsilon.$$

(b) The set $B = \left\{ \begin{matrix} x_n (s) / s \in [0,a] \\ n \in N \end{matrix} \right.$ is relatively compact in X.

Indeed, consider a sequence $(x_{n_k}(s_k))_{k\in N}$ in B. We can assume that $\lim_{k\to\infty} s_k = s$. We prove that

$$\lim_{k\to\infty} x_{n_k}(s_k) = x(s).$$

For any given $\varepsilon > 0$, by (a), there are n_0 and $\delta > 0$ such that for all $|t-t'| < \delta$ and $n_k \geqslant n_o$

$$\left|x_{n_k}(t) - x_{n_k}(t')\right| < \varepsilon/3.$$

Since $\lim_{k\to\infty} s_k = s$ and $\lim_{k\to\infty} x_{n_k} = x$ there exists $k_o \geqslant n_o$ such that for all $k \geqslant k_0$ $|s_k - s| < \delta$ and $\left|x_{n_k}(t) - x(t)\right| < \varepsilon/3$ for all $t \in [0,a]$. Consequently, for $k \geqslant k_o$, we have $\left|x_{n_k}(s_k) - x(s)\right| \leqslant \left|x_{n_k}(s_k) - x_{n_k}(s)\right| + \left|x_{n_k}(s) - x(s)\right| < \varepsilon$.

This implies $\lim_{k\to\infty} x_{n_k}(s_k) = x(s)$ and hence the set B is relatively compact.

(c) $\lim_{n\to\infty} F(x_n) = F(x)$.

Let B be the set defined by (b). Put

$$A = \text{Cl}\ B \cup x([0,a]).$$

Then the set A is compact and f is uniformly continuous on $[0,a] \times A$. For any given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|x-y| < \delta$

$$|f(s, x) - f(s,y)| < \varepsilon/(1+a).$$

Since $\lim_{n\to\infty} x_n = x$, there exists n_o such that for all $n \geqslant n_o$

$$|x_n(s) - x(s)| < \delta \text{ for all } s \in [0,a].$$

Consequently

$$\left|F(x_n)(t) - F(x)(t)\right| \leqslant \int_0^t \left|f(s,x_n(s)) - f(s,x(s))\right| ds$$

$$< t\varepsilon/(1+a) < \varepsilon \text{ for all } t \in [0,a]$$

which implies that $\lim_{n\to\infty} F(x_n) = F(x)$. Hence F is continuous on E. Now, we prove that F is compact. Let Ω be a bounded set in E. Put

$$\mathcal{B} = \overline{\text{co}} \{ f(s,x(s)) / s \in [0, a] \}_{x\in\Omega}$$

where $\overline{\text{co}}$ denotes the closed convex hull of a set. Then by the compactness of f, the set $\mathcal{B}$ is compact in X. It follows that the family of functions $\{F(x)/x\in\Omega\}$ is equicontinuous. Furthermore, for all $t \in [0,a]$, we have

$$\{F(x)(t) / x\in\Omega\} \subset \mathcal{B}$$

which implies that the set $\{F(x)(t) / x\in\Omega\}$ is relatively compact in X for all t in $[0,a]$. Hence the set $F(\Omega)$ is relatively compact in E. Thus F is compact.

Finally, we prove $\lim_{\|x\|\to\infty} \| F(x) \| / \| x \| = 0$. From (iii) and the compactness of f, for any given $\varepsilon > 0$ there exists N such that

(8) $| f(t,\varphi) | \leqslant N + \varepsilon/(1+a) | \varphi |$ for all $\varphi \in X$ and $t \in [0,a]$.

Consequently,

$$| F(x)(t) | = | \int_0^t f(s,x(s))ds + \varphi |$$

$$\leq Na + \varepsilon \| x \| + | \varphi |, \text{ for all } t \text{ in } [0,a].$$

This implies that

$$| F | = \lim_{\|x\|\to\infty} \| F(x) \| / \| x \| = 0.$$

Since the norms $| \cdot |$ and $| \cdot |_1$ are equivalent, we have also $| F |_1 = 0$.

Proof of Theorem 2. a) We have, by (7) and Lemma 2.2

$$| U + F|_1 \leqslant \| U \|_1 + \| F \|_1 = \| U \|_1 < 1.$$

Therefore, by Theorem 1, the operator $U + F$ has a fixed point, say x, which is precisely a solution of (II) on $[0,a]$.

b) We prove that the map $\varphi \to x(\varphi)$ is continuous.

Let $\varphi_n \in X$, $\lim_{n\to\infty} \varphi_n = \varphi$, we put

$$x_n = x(\varphi_n) \text{ and } x = x(\varphi).$$

This means that x_n and x are solutions of the integral equations

$$x_n(t) = \varphi_n + \int_0^t [A(s)x_n(s) + f(s,x_n(s))]ds,$$

$$x(t) = \varphi + \int_0^t [A(s)x(s) + f(s,x(s))]\, ds.$$

The remainder of the proof is split into a number of steps as follows.

Step 1. Let Ω be a bounded set in X. Then the set

$$B = \{x(\varphi)(s) / s \in [0,a], \varphi \in \Omega\}.$$

is bounded in X. Here $x(\varphi)$ is a solution of (II) satisfying the initial condition $x(\varphi)(0) = \varphi$.

Indeed, let m, N, ε be as in (5') and (8), we have

$$|x(\varphi)(t) | \leqslant | \varphi | + \int_0^t | A(s)x(\varphi)(s) | \, ds + \int_0^t | f(s\, x(\varphi)(s) | \, ds$$

$$\leqslant | \varphi | + \int_0^t m | x(\varphi)(s) | \, ds + \int_0^t [N + \varepsilon/(1+a) | x(\varphi)(s) |]ds$$

$$\leqslant (| \varphi | + Na) + (m + \varepsilon/(1+a)) \int_0^t | x(\varphi)(s) | \, ds.$$

By Gronwall's inequality we have

$$\begin{aligned} |x(\varphi)(t)| &\leqslant (|\varphi| + Na)e^{ma+\varepsilon} \quad \text{for all } t \text{ in } [0, a] \\ &\leqslant (M + Na)e^{ma+\varepsilon} \end{aligned}$$

where $M = \sup\{ |\varphi| / \varphi \in \Omega\}$. Hence B is bounded in X as claimed.

Step 2. Let $C = \{x_n / n \in N\}$. Then C is relatively compact in E.

We recall that the Kuratowskii measure of noncompactness on E is defined as follows: for any bounded set A in E with norm $|\cdot|_1$

$$\alpha(A) = \inf \left\{ d > 0 / A \text{ is covered by a finite number of sets with diameters} \leqslant d \right\}$$

The number $\alpha(A)$ satisfies:

— $\alpha(A) = 0 \Leftrightarrow A$ is relatively compact.

— If $A \subset B$, $\alpha(A) \leqslant \alpha(B)$.

— $\alpha(A + B) \leqslant \alpha(A) + \alpha(B)$

For all $n \in N$, we have

$$x_n = (U + F)(x_n) + (\varphi_n - \varphi)$$

where U, F are defined in (4) and $\varphi_n - \varphi$ are constant functions.

Now, since $\| U \|_1 < 1$ and F is compact, hence $T = U + F$ is a $\| U \|_1$—set contraction, i. e.

$$\alpha(T(\Omega)) \leqslant \| U \|_1 \ \alpha(\Omega) \text{ for all bounded set } \Omega.$$

Let $A = \{ \varphi_n - \varphi / n \in N\}$, then, by $\lim_{n\to\infty} \varphi_n = \varphi$, A is relatively compact in E.

We have $C \subset T(C) + A$
which implies that

$$\alpha(C) \leqslant \alpha(T(C)) \leqslant \| U\|_1 \ \alpha(C) \text{ with } \| U\|_1 < 1.$$

It follows that $\alpha(C) = 0$. Thus C is relatively compact in E.

Now, we claim that $\lim_{n\to\infty} x_n = x$. Indeed, from Step 2, the set $C = \{x_n / n \in N\}$ is relatively compact. Therefore there exists a convergent subsequence $(x_{n_k})_{k \in N}$. Put $\lim_{k\to\infty} x_{n_k} = y$.

Since $x_{n_k} = (U + F)x_{n_k} + (\varphi_{n_k} - \varphi)$ for all k and since $\lim_{k\to\infty} \varphi_{n_k} = \varphi$, we have

$$y = (U + F)(y).$$

This means that y is a solution of (II). Since the problem (II) has a unique solution, we have $x = y = \lim_{k\to\infty} x_{n_k}$.

Thus, every convergent subsequence $(x_{n_k})_k$ of $(x_n)_n$ has the same limit x. Since the set C is relatively compact, it follows that $\lim_{n\to\infty} x_n = x$.

Finally, we prove that T is continuous in (t, φ). Let $t_n \in [0, a]$, $\lim_{n\to\infty} t_n = t$ and $\varphi_n \in X$, $\lim_{n\to\infty} \varphi_n = \varphi$. Put $x_n = x(\varphi_n)$ and $x = x(\varphi)$: then $\lim_{n\to\infty} x_n = x$.

We have

$$| x_n(t_n) - x(t) | \leqslant | x_n(t_n) - x(t_n) | + x(t_n) - a(t) |$$

This implies that $\lim_{n\to\infty} x_n(t_n) = x(t)$. This means that

$$\lim_{n\to\infty} T(t_n, \varphi_n) = T(t, \varphi).$$

Hence T is continuous in (t, φ). The proof is complete.

Now, we consider the ω-periodic equation

$$\frac{dx}{dt} = A(t)x(t) + f(t, x(t)), \ t \geqslant 0 \tag{1}$$

where $A(t)$ and f satisfy the conditions (A_1), (A_2) (A_3).

Suppose further that

(A_4) $\lim_{|x|\to\infty} | f(t, x) | \ / \ | x | = 0$ uniformly on $[0, \omega]$.

From Theorem 2, we know that under conditions (A_1), (A_2), (A_4), for any φ in X the equation (I) has a solution $x(\varphi)$ on $[0, \omega]$ satisfying $x(\varphi)(0) = \varphi$, and if in addition we assume (A_3), then the map $\varphi \to x(\varphi)$ is continuous.

For $t \geqslant 0$, let $T(t) : X \longrightarrow X$ be defined by $T(t)(\varphi) = x(\varphi)(t)$.

From Theorem 2 and the condition (A_3), since $A(t)$ and f are periodic with period ω, we have that the map $(t, \varphi) \mapsto T(t)(\varphi)$ is continuous and $T(t + \omega) = T(t) . T(\omega)$ for all $t \geqslant 0$.

Let $U(t, s)$ be the evolution operator associated with (I), i. e. $U(t, s)$ is a family of bounded linear operators on X into itself defined for $0 \leqslant s \leqslant t < \infty$, strongly continuous in (t, s) and satisfying the conditions:

— $U(t, s). U(s, r) = U(t, r)$, $U(s, s) = I$ (the identity map),

— $\dfrac{\partial U(t, s)\, x}{\partial t} = A(t)U(t, s)x$,

— $\dfrac{\partial U(t, s)\, x}{\partial s} = - U(t, s)\, A(s)x$.

Then we have

$$x(\varphi)(t) = U(t, 0)\varphi + \int_0^t U(t, s) f(s, x(\varphi)(s))ds, \ t \geqslant 0. \tag{9}$$

Consequently

$$x(\varphi)(\omega) = U(\omega, 0)\varphi + \int_0^\omega U(\omega, s) f(s, T(s)\varphi)\, ds$$

or

$$T(\omega)\varphi = U(\omega, 0)\varphi + \int_0^\omega U(\omega, s) f(s, T(s)\varphi) ds$$

Note that, since $A(t)$ is periodic with period

$$U(t + \omega, 0) = U(t, 0) \,.\, U(\omega, 0) \text{ for all } t \geqslant 0.$$

Now, put $T_1 = T(\omega)$, $U_1 = U(\omega, 0)$ and

$$F_1(\varphi) = \int_0^\omega U(\omega, s) f(s, T(s)(\varphi)) ds. \tag{10}$$

Then we have $T_1 = U_1 + F_1$.

An ω-periodic solution of the equation (I) will be a solution $x(\varphi)(t)$ for which φ is a fixed point of T.

We recall that the family $\{A(t)\}$ is said to be stable if there exist constants $a \geqslant 1$ and $b > 0$ such that

$$\|U(t, 0)\| \leqslant ae^{-bt} \text{ for all } t \geqslant 0.$$

We have the following theorem

THEOREM 3. *Let the equation (I) satisfy the conditions (A_1) (A_2), (A_3), (A_4). Suppose that the family $\{A(t)\}$ is stable. Then the equation (I) has an ω-periodic solution.*

Proof. The proof is split into a number of steps:

Step 1. The operator F_1 defined by (10) is compact. Let $\varphi_n \in X$, $\lim_{n\to\infty} \varphi_n = \varphi$. Then, by Theorem 2, $\lim_{n\to\infty} x(\varphi_n) = x(\varphi)$ in $E = \{x : [0, \omega] \to X \text{ is continuous}\}$ with norm sup. This means that the sequence $(T(.)(\varphi_n))_n$ is uniformly convergent to $T(.)(\varphi)$ on $[0, \omega]$. From (b) in the proof of Lemma 2.2 the set

$$A = Cl \{T(s)(\varphi_n)/s \in [0, \omega]\}_{n\in N} U \{T(s)(\varphi)/s \in [0, \omega]\}$$

is compact which implies that f is uniformly continuous on $[0, \omega]\, x\, A$. This follows that the sequence $(f(., T(.)(\varphi_n))_{n\in N}$ converges uniformly to $f(., T(.)(\varphi))$ on $[0, \omega]$ and hence $\lim_{n\to\infty} F_1(\varphi_n) = F_1(\varphi)$. This shows that F_1 is continuous.

Let Ω be a bounded set in X. Then, by the proof of the part (b) of Theorem 2, the set

$$B = \{T(s)(\varphi)/s \in [0, \omega],\ \varphi \in \Omega\}$$

is bounded in X.

Since f is compact, the set $f([0, \omega] \times B)$ is relatively compact in X.

Consequently the set

$$D = U(\omega, .)\,([0, \omega] \times f([0, \omega] \times B))$$

is compact.

Since $U(\omega, s) f(s, T(s)\varphi) \in D$ for all $s \in [0,\omega]$ and $\varphi \in \Omega$ we have

$$F_1(\varphi) = \int_0^\omega U(\omega, s) f(s, T(s)\varphi)ds \in \omega D \text{ for all } \varphi \in \Omega.$$

This shows that $F_1(\Omega)$ is relatively compact and hence F_1 is compact.

Step 2. There exists an equivalent norm $|\,.\,|_1$ on X such that $\| U_1 \|_1 < 1$.

Proof. Since $\{A(t)\}$ is stable, there exist $a \geqslant 1$ and $b > 0$ such that

$$\| U(n\omega, 0) \| = \| U_1^n \| \leqslant ae^{-nb} \text{ for all } n \in N.$$

Consequently

$$\limsup_{n\to\infty} (\| U_1^n \|)^{1/n} \leqslant \lim_{n\to\infty} a^{1/n} . e^{-b\omega} = e^{-b\omega} < 1.$$

This shows that, by Lemma 2. 1, there is an equivalent norm $|\,.\,|_1$ on X defined by

$$| x |_1 = | x | + \sum_1^\infty U_1^n x.$$

For this norm, we have $\| U_1 \|_1 < 1$.

Step 3. $| F_1 |_1 = 0$.

Proof. For any $\varepsilon > 0$, from (A_3), there exists $N > 0$ such that $| f(t, \varphi) | \leqslant$ $\leqslant N + \varepsilon | \varphi |$ for all $\varphi \in X$ and $t \in [0, \omega]$

Put $m = \sup \{ \| U(t, s) \| , 0 \leqslant s \leqslant t \leqslant \omega \}$,
From (9), we have

$$| x(\varphi)(t) | < m | \varphi | + m \int_0^t [N + \varepsilon(| x(\varphi)(s) |]ds$$

$$m [| \varphi | + N] + m\varepsilon \int_0^t | x(\varphi)(s) | ds$$

which implies, by Gronwall's inequality

$$| x(\varphi)(t) | \leqslant m[| \varphi | + N\omega] e^{m\varepsilon\omega} \text{for all } t \text{ in } [0,\omega].$$

We have

$$-F_1(\varphi) | \leqslant \int_0^\omega U (\omega, s) f (s, x (\varphi) (s) ds$$

$$\leqslant m \int_0^\omega [N + \varepsilon | x (\varphi) (s) |]ds$$

$$\leqslant mN\omega + \omega m^2 \varepsilon[|\varphi| + N\omega] e^{m\varepsilon\omega} .$$

This shows that $| F_1 | \leqslant m^2 \omega\varepsilon e^{m\varepsilon\omega}$.

Since $\varepsilon > 0$ is arbitrary, we have $| F_1 | = 0$.

Since the norms $|.|$ and $|.|_1$ are equivalent, we also have $| F_1 |_1 = 0$.

Proof of Theorem 3. We have $|T_1|_1 \leqslant \|U_1\|_1 + |F_1|_1 < 1$. Hence, by Theorem 1, T_1 has a fixed point φ and the solution $x(\varphi)$ is precisely a periodic solution of the equation (I). The proof is complete.

Remark. The condition (A_3) of Theorem 3 in this paper is weaker than the condition (A_3) of Theorem 2 in [7].

Remark. The referee pointed out (rightly) in the first version of the manuscript that the uniform continuity of the solution operator with respect to the initial data as contained in (A_3) could be replaced by the weaker condition of continuity of the solution operator. In the present revised version of the paper, it is shown in the proof of Theorem 3 that even the continuity condition can be dispensed with. (The continuity of the solution operator follows directly from the conditions of the problem.).

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