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ON A LINEAR PROGRAM IN THE SPACE OF BOREL MEASURES AND THE PROBLEM OF EXACT INEQUALITIES FOR DERIVATIVES

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1. INTRODUCTION

Let C be a subset of the Euclidean space R^n. Denote by $V(C)$ the normed space of regular Borel measures μ on C with the norm $\|\mu\| = \text{var}\ \mu$ and by $V_+(C)$ the cone of nonnegative measures of $V(C)$. Let $\varphi_0, \varphi_1, \dots, \varphi_m$ be positive continuous functions on C and $\gamma_1, \gamma_2, \dots, \gamma_m$ real numbers. We will study the following linear program

$$\text{minimize} - \int_C \varphi_0 \, d\mu \tag{1}$$

subject to

$$\int_C \varphi_j \, d\mu \leqslant \gamma_j \, , \, j = 1, 2, \dots, m, \ \mu \in V_+(C) \tag{2}$$

If at least one γ_j , $j = 1, 2, \dots, m$ is negative, then the set of measures satisfying condition (2) is empty. In this case we set $\inf \varnothing = +\infty$.

When studying the generalization of the Chebyshev problem in compact sets Golstein [1] considered an analogous problem on the space of functions with bounded variation and proved a theorem of Haar type for the extremal polynomial.

In this paper we shall be concerned with the duality for the problem (1)—(2), the sufficient conditions of finiteness of its extremal value and the existence of solutions of the primal and the dual problems. The obtained results will be applied to solve a special case of the general problem of exact inequalities for derivatives which can be formulated as follows: Let $\alpha^j \in R^n$, $1 \leqslant p^j \leqslant \infty$, $j = 0,1,\dots, m$ and let γ_j, $j = 1,2,\dots, m$ be real numbers, E be the

n-dimensional torus or a subset of R^n. It is required to find the supremum of the norm $\| x^{(\alpha^0)} \|_{L_{p_0}(E)}$ over all functions x for which the norms $\| x^{(\alpha^j)} \|_{L_{p^j}(E)}$ are bounded by γ_j, $j = 1,2 ...,m$.

The first results concerning this problem were obtained by Landau [2], Hadamard [3] and Kolmogorov [4]. At present there are several works on this problem (for details see [5]). The problem of exact inequalities for derivatives in the metric L_2 was completely solved by Dinh Zung and Tihomirov [6] for the cases $E = R^n$ and $E = T^1$. The special cases when $n = 1$, $m = 2$ and $m = n + 2$ have been earlier by Hardy-Littlewood-Polya and Subbotin, respectively (see [5]). The case $E = T^n$, $n > 1$, is still open.

With the help of the Parseval–Plancherel equality the problem of exact inequalities for derivatives in the L_2 -metric can be reduce to the form (1) — (2). Therefore, the methods of linear programming and convex analysis and, the above mentioned can be obtained to this problem. To illustrate this idea we shall examine in detail the case $E = T^1$ and prove the results announced in [6].

2. THE LINEAR PROGRAM (1) – (2)

Although the dimension of the space $V(C)$ is infinite, the problem (1) — (2) has a rather simple duality. Its dual problem is a convex program in the space R^n. This allows us to study the finiteness of its extremal value and the existence of solutions of the primal problem and to find this value and these solutions in some concrete situations.

Let R^m_+, R^m_+ and $\overset{o}{R}{}^m_-$ be the positive, nonnegative and nonpositive cones of R^m, respectively, α_j the j-th coordinate of a vector $\alpha \in R^m$, and $\langle \alpha, \beta \rangle = \alpha_1 \beta_1 + ... + \alpha_m \beta_m$ for $\alpha, \beta \in R^m$. given $\varphi_0, \varphi_1, ..., \varphi_m$ we denote by $S(\Upsilon)$, $\gamma = (\gamma_1, ..., \gamma_m) \in R^m$, the extremal value of the problem (1) — (2). For terminology and notation of convex analysis see [7]. We have the following duality theorem.

THEOREM 1. *For any $\gamma \in R^m_+$ the value $S(\gamma)$ is equal to the extremal value of the following convex programming problem in the space R^m:*

$$\text{maximize } \langle \Upsilon, \beta \rangle \tag{3}$$

subject to

$$\langle \Phi(t), \beta\rangle + \varphi_0(t) \leqslant 0,\ t \in C;\ \beta \in R^m_- \tag{4}$$

where

$$\Phi(t) = (\varphi_1(t), \dots, \varphi_m(t)).$$

Proof. Consider $S(\Upsilon)$ as a function on R^m with values in $R \cup \{\pm\infty\}$. This function is convex in R^m. It is obvious that the domain of definition of S, $\operatorname{dom} S$, is R^m_+. Denote by S^* the conjugate function of S. From the definition of S^* we derive

$$S^*(\beta) = \begin{cases} J(\beta), & \beta \in R^m_- \\ +\infty, & \beta \in R^m_- \end{cases}$$

where

$$J(\beta) = \sup_{\mu V_+(C)} \int_C \{\varphi_0 + \langle \Phi(t), \beta\rangle\} d\mu,\ \beta \in R^m_-.$$

If $\varphi_0(t) + \langle \Phi(t), \beta\rangle \leqslant 0,\ \forall t \in C$, then $J(\beta) = 0$. If $\varphi_0(\theta) + \langle \Phi(\theta), \beta\rangle > 0$ for some point $\theta \in C$, then taking the measures μ_ξ with $d\mu = \xi\,\delta(t-\theta)dt$ ($\delta(t)$ being the Dirac function) we obtain

$$\int_C \{\varphi(t) + \langle \Phi(t), \beta\rangle\} d\mu_\xi = \xi$$

and, consequently, $J(\beta) = +\infty$. Hence

$$S^*(\beta) = \begin{cases} 0, & \text{if } \beta \in R^m_- \text{ and } \varphi_0(t) + \langle \Phi(t), \beta\rangle \leqslant 0,\ t \in C \\ +\infty & \text{otherwise} \end{cases} \tag{5}$$

According to Fenchel-Moreau's theorem we have

$$S^{**}(\Upsilon) = S(\gamma),\ \forall \gamma \in \operatorname{int} \operatorname{dom} S$$

Since $\operatorname{int} \operatorname{dom} S = \overset{0}{R}{}^m_+$ by (5) and by the formula

$$S^{**}(\gamma) = \sup_{\beta \in R^m} \{\langle \gamma, \beta\rangle - S^*(\gamma)\}$$

we see that the proof of the theorem is complete.

THEOREM 2. *In order that the extremal value $S(\gamma)$ be finite for any $\gamma \in R^m_+$, i. e.*

$$S(\gamma) > -\infty,\ \forall \gamma \in R^m_+,$$

it is necessary and sufficient that

$$\inf_{t \in C} \{\varphi(t)/\varphi_0(t)\} > 0 \tag{6}$$

where

$$\varphi(t) = \max_{1\leqslant j\leqslant m} \varphi_j(t)$$

Proof. According to Theorem 1 we have $S(\gamma) > -\infty$, for all $\gamma \in R^m_+$ if and only if there exists a point $\beta^* \in R^m_-$ such that

$$\langle \Phi(t), \beta^* \rangle + \varphi_0(t) \leqslant 0, \ \forall t \in C \tag{7}$$

If relation (7) holds, then, obviously, $\delta = -(\beta^*_1 + ... + \beta^*_m) > 0$ and

$$1 \leqslant -\langle \Phi(t), \beta^* \rangle / \varphi_0(t) \leqslant \delta\varphi(t)/\varphi_0(t), \ \forall t \in C$$

Consequently,

$$\varphi(t)/\varphi_0(t) > 1/\delta, \ \forall t \in C.$$

Conversely, if

$$\inf_{t\in C} \{ \varphi(t)/\varphi_0(t) \} = a > 0,$$

then the point $\beta^* = -(1/a, ..., 1/a)$ satisfies (7).

We now consider the question of existence of solutions of the problem (1) — (2) and examine how to find these solutions. In general, the finiteness of its extremal value does not imply the existence of its solutions. In the other hand there always exist solutions of the dual problem. If C is a compact set, then the problem (1) — (2) has solutions as shown by the following

THEOREM 3. *Let C be a compact set. Then for any $\gamma \in R^m_+$ the extremanl value of problem (1)—(2) $S(\gamma)$ is finite and there exists a soiuion μ of the form*

$$d\mu = \sum_{k=1}^{s} a_k \delta(t - t^k)\, dt \tag{8}$$

where $a_k > 0$, $t^k \in C$, $k = 1, 2, ..., s$, $1 \leqslant s \leqslant m + 1$.

Proof. Since C is a compact set and the functions $\varphi_0, \varphi_1, ..., \varphi_m$ are continuous and positive, it is easy to see that the condition (6) holds. Therefore, according to Theorem 2 $S(\gamma)$ is finite. Let M be the set of all measures satisfying condition (2). Obviously

$$b = \min_{t\in C} \varphi(t) > 0$$

For any $\mu \in M$ we can write

$$\sum_{k=1}^{m} \gamma_k \geqslant \sum_{k=1}^{m} \int_C \varphi_k(t)\, d\mu \geqslant \int_C \varphi(t)\, d\mu \geqslant b\, \| \mu \|$$

This means that M is contained in the sphere $Q = \{\mu \in V(C) \mid //\mu// \leqslant r\}$ where $r = (\gamma_1 + .. + \gamma_m)/b$, i.e. M is bounded. Moreover, M being closed, M is weakly compact. From this it follows that the continuous linear functional

$$-\int_C \varphi_o(t)\, d\mu$$

attains its minimum on M. Thus the existence of solutions of the problem (1)—(2) is proved. Moreover, the set of its solutions is a extreme subset of the weakly compact convex set M. Hence according to the well-known Kreĭn-Milman theorem (see [8]) there exists at least an extreme point of M which is a solution of (1)—(2). Denote by $\widehat{\mu}$ such a solution. Obviously, $\widehat{\mu}$ is not equal to zero. Since $\widehat{\mu}$ is an extreme point of the intersection of the weakly compact convex set $Q_+ = Q \cap V_+(C)$ with m closed semispaces of $V(C)$, it is not difficult to prove that μ is a convex combination of no more than $m+1$ extreme points of Q_+. On the other hand a positive extreme point μ of Q_+ has the form $d\mu = a\delta(t-\theta)dt$, $\theta \in C$, $a > 0$ (see [8]). From this (8) follows.

THEOREM 4. *Assume that the condition of finiteness (6) holds. Then for any $\gamma \in \mathring{R}^m_+$ the set of solutions of the dual program (3)—(4) is a nonempty compact convex set.*

Proof. For a given $\Upsilon \in R^m_+$ let B denote the set of points of R^m satisfying (4). From (6) — (7) we infer that B is not empty. In addition it is plain that B is closed and convex. Observe that in the problem (3) — (4) the maximum may be taken only over the nonempty compact convex subset

$$\{\beta \in B / \langle \Upsilon, \beta\rangle \geqslant \langle \gamma, \beta^*\rangle \text{ where } \beta^* = -(1/a, \dots, 1/a),$$

$$a = \inf_{t \in C} \{\varphi(t)/\varphi_0(t)\}.$$

The theorem is thus proven.

3. EXACT INEQUALITIES FOR DERIVATIVES

Let L be the space of periodic functions x for which the following norm is finite

$$\| x \|_2 = \Big(\int_{-\pi}^{\pi} | x(t) |^2 \ dt\Big)^{1/2}.$$

For $\alpha \in R$ and a function x denote by $x^{(\alpha)}$ the Weil fractional derivative of rder α of x.

Let $\alpha^0, \alpha^1, \ldots, \alpha^m$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ be real numbers. The problem of ine-ualities for derivatives of periodic functions may be formulated as follows:

$$\text{maximize} \ \| x^{(\alpha^0)} \|_2^2 \tag{9}$$

ubject to

$$\| x^{(\alpha^j)} \|_2^2 \leqslant \gamma_j, \ j = 1, \ldots, m. \tag{10}$$

For simplicity of presentation we shall consider only functions x with ero-mean, i.e.

$$\int_{-\pi}^{\pi} x(t) dt = 0.$$

We shall show that the problem (9) — (10) may be reduced to the orm (1) — (2).

If x is a periodic function and $x^{(\alpha)} \in L_2$, $\alpha \in R$, then by Parseval equality ve have

$$\| x^{(\alpha)} \|_2^2 = \sum_{k=1}^{\infty} k^{2\alpha} p_k^{\,2}$$

vhere $p_k = (2\pi)^{1/2}(x_k^{\,2} + x^2)$, x_k denotes the k-th coefficient of the Fourier eries of x. Hence the problem (9) — (10) is equivalent to the following problem

$$\text{maximize} \ \sum_{k=1}^{\infty} k^{2\alpha^0} p_k$$

ubject to

$$\sum_{k=1} k^{2\alpha^j} p_k \leqslant \gamma_j, \ j = 1, \ldots, m, \ p_k \geqslant 0, \ k \in N.$$

This problem may be rewritten as

$$\text{minimize} \ -\int_N t^{2\alpha^0} d\mu \tag{11}$$

ubject to

$$\int_N t^{2\alpha^j} d\mu \leqslant \gamma_j, \ j = 1, \ldots, m, \ \mu \in V_+(N) \tag{12}$$

Thus the problem (11) — (12) has been reduced to the form (1) — (2).

According to Theorem 1, the dual problem of (11) — (12) is

$$\text{maximize } \langle \gamma, \beta \rangle \tag{13}$$

subject to

$$\sum_{j=1}^{m} \beta_j k^{2\alpha^j} + k^{2\alpha^0} \leqslant 0, \; k \in N, \; \beta \in R^m_- . \tag{14}$$

Using Theorem 2 we infer that the extremal value S(γ) of (11) — (12) is finite if and only if

$$\alpha^0 \leqslant \max_{1 \leqslant j \leqslant m} \alpha^j . \tag{15}$$

Since the extremal values of the primal problem (11) — (12) and the dual problem (13) — (14) are equal in order to find $S(\gamma)$ it is simpler to solve the dual problem than the primal one.

THEOREM 5. ([6]). *The extremal value of the problem (9) — (10) is equal to*

$$\inf \left(\inf \{ \gamma_j \mid \alpha^j \geqslant \alpha^0 \}, \inf \{ p\gamma_r + q\gamma_s \mid \alpha^s > \alpha^r, \gamma_s > \gamma_r \} \right)$$

$1 \leqslant j, r, s \leqslant m$

where (p, q) *is the solution of the linear system*

$$k^{2\alpha^r} p + k^{2\alpha^s} q = k^{2\alpha^0}; \; (k+1)^{2\alpha^r} p + (k+1)^{2\alpha^s} q = (k+1)^{2\alpha^0} \tag{16}$$

$k = k_{rs} = [(\gamma_s / \gamma_r)^{(1/2(\alpha^s - \alpha^r))}]$ *where the symbol* $[a]$ *denotes the integer part of the number* a.

Proof. Remark that the extremal value of the problem (9) — (10) is equal $- S_{(\gamma)}$. Without loss of generality we may assume that $\alpha^1 < \alpha^2 < \dots < \alpha^m$ and $\gamma_1 < \gamma_2 < \dots < \gamma_m$.

From (15) it follows that $S(\gamma)$ is finite if and only if $\alpha^0 \leqslant \alpha^m$ and $\gamma \in \mathring{R}^m_+$. It is clear that if $\alpha^0 > \alpha^m$ or $\gamma \notin \mathring{R}^m_+$ Theorem 5 is trivial. Consider now the case $\alpha^0 \leqslant \alpha^m$, $\gamma \in \mathring{R}^m_+$. Put

$$B = \{ \beta \in R^m_+ \mid \sum_{j=1}^{m} \beta_j k^{2\alpha^j} - k^{2\alpha^0} \geqslant 0, \; k \in N \}.$$

Theorem 1 gives

$$-S(\gamma) = \inf_{\beta \in B} \langle \gamma, \beta \rangle . \tag{17}$$

By Theorem 4 in the considered case the dual problem (13) — (14) has a solution. Consequently, there exists a point $\widehat{\beta} \in B$ such that

$$\langle \gamma, \widehat{\beta} \rangle = \inf_{\beta \in B} \langle \gamma, \beta \rangle . \tag{18}$$

Examine now the function

$$f(\xi) = \sum_{j=1}^{m} \beta_j e^{2(\alpha^j - \alpha^0)} - 1,$$

on the real line.

Since $\widehat{\beta} \in B$, $f(\xi) \geqslant 0$ for any $\xi \in \ln N$. We will show that there exists at least one point $\xi \in \ln N$ such that $f(\xi) = 0$. Assume on the contrary that $f(\xi) > 0$ for any $\xi \in \ln N$. Then it is easy to prove that $\beta' = (1-\varepsilon)\,\widehat{\beta} \in B$ for some number ε $(0 < \varepsilon < 1)$. On the other hand, $<\Upsilon, \beta'> \, < \, <\Upsilon, \widehat{\beta}>$, which contradicts (18). Remark that the strongly convex function f has no more than two zeros on the real line. Combining the above arguments we conclude that the function

$$F(t) = \sum_{j=1}^{m} \beta_j t^{2\alpha^j} - t^{\alpha^0}$$

has on N either one zero or two consequent zero. The first case occurs only if for some number j $(1 \leqslant j \leqslant m)$ we have $\alpha^j \geqslant \alpha^0$, $\beta_j = 1$ and $\beta_k = 0$ for $k \neq j$. The latter occurs only if there exist two numbers r, s $(1 \leqslant r, s \leqslant m)$ such that $\alpha^r < \alpha^0 < \alpha^s$, $\beta_r, \beta_s > 0$ and $\beta_k = 0$ for $k \neq r, s$. Hence (18) yields

$$\inf_{\beta \in B} < \Upsilon, \beta > = \inf_{j, (r, s)} \left\{ \inf_{\beta \in B_j} < \Upsilon, \beta >, \inf_{\beta \in B_{rs}} < \Upsilon, \beta > \right\} \tag{19}$$

where the infimum is taken over all j with $\alpha^j \geqslant \alpha^0$ $(1 \leqslant j \leqslant m)$ and all r, s with $\alpha^r < \alpha^0 < \alpha^s$ $(1 \leqslant r, s \leqslant m)$

$$B_j = \{ \beta \in B \mid \beta_j = 1, \beta_k = 0, k \neq j \}$$

$$B_{rs} = \{ \beta \in B \mid \beta_r, \beta_s \; 0, \; \beta_k = 0, k \neq r, s \}.$$

It is easy to check that

$$\inf_{\beta \in B_j} < \Upsilon, \beta > = \Upsilon_j. \tag{20}$$

Set $\beta^* = (\beta_1^*, \ldots, \beta_m^*)$ where $\beta_r = p$, $\beta_s = q$ and $\beta_k^* = 0$ for $k \neq r, s$. Let θ_1, θ_2 be the solution of the linear system

$$k^{2\alpha^r} \theta_1 + (k+1)^{2\alpha^r} \theta_2 = \Upsilon_r \tag{21}$$

$$k^{2\alpha^s} \theta_1 + (k+1)^{2\alpha^s} \theta_2 = \Upsilon_s$$

where $k = k_{rs}$. It can be verified that $\beta^* \in B_{rs}$ and $\theta_1, \theta_2 > 0$. Let β be an arbitrary point of B_{rs}. From (16) an (21) we obtain

$$< \gamma, \beta^* > = \gamma_r \beta_r^* + \gamma_s \beta_s^* - \theta_1 (pk^{2\alpha^r} + qk^{2\alpha^s} - k^{2\alpha^0}) - \theta_2 (p\ (k+1)^{2\alpha^r} +$$

$$+ q\,(k+1)^{2\alpha^s} - (k+1)^{2\alpha^0}) = \theta_1 k^{2\alpha^0} + \theta_2 (k+1)^{2\alpha^0} + p(\gamma_r - \theta_1 k^{2\alpha^r} -$$

$$- \theta_2 (k+1)^{2\alpha^r}) + q(\gamma_s - \theta_1 k^{2\alpha^s} - \theta_2 (k+1)^{2\alpha^s}) = \theta_1 k^{2\alpha^0} + \theta_2 (k+1)^{2\alpha^0} +$$

$$+ \beta_r (\gamma_r - \theta_1 k^{2\alpha^r} - \theta_2 (k+1)^{2\alpha^r}) + \beta_s (\gamma_s - \theta_1 k^{2\alpha^s} - \theta_2 (k+1)^{2\alpha^s}) =$$

$$= \gamma_r \beta_r + \gamma_s \beta_s + \theta_1 (k^{2\alpha^0} - \beta_r k^{2\alpha^r} - \beta_s k^{2\alpha^s}) + \theta_2 ((k+1)^{2\alpha^0} - \beta_r (k+1)^{2\alpha^r} -$$

$$- \beta_s (k+1)^{2\alpha^s}) \leqslant \gamma_r \beta_r + \gamma_s \beta_s = < \gamma, \beta >.$$

This means that

$$\inf_{\beta \in B_{rs}} < \gamma, \beta > = < \gamma, \beta^* > = p\gamma_r + q\gamma_s . \qquad (22)$$

Combining (17), (19), (20), (22) we get the proof of the theorem.

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