ACTA MATHEMATICA VIETNAMICA

Vol 13 No 2 (1988)

STEENROD OPERATIONS ON MOD 2 HOMOLOGY OF THE ITERATED LOOP SPACE

NGUYEN NAM HAI and NGUYEN HƯU VIET HƯNG

INTRODUCTION

In [5], [6] the second author introdu ed the Dickson characteristic classes derived from modular invariants and used them to determine the algebra structure of $H^*(\Omega_0^q S^q, Z/2)$ for $0 < q \leqslant \infty$. Here $\Omega_0^q S^q$ denotes the component of the base point of the iterated loop space $\Omega^q S^q$.

The purpose of this paper is to study the action of the opposite mod 2 Steenrod algebra $A_* = A_*(2)$ on the homology $H_*(\Omega_0^q S^q; Z/2)$ also by means of modular invariants and the Dickson classes. Roughly speaking, we will reduce this action to the usual action of the mod 2 Steenrod algebra $A = A(2)$ on the Dickson algebra $Z/2\,[x_1, ..., x_n]^{GL\,(n,\,Z/2)}$.

The paper is divided into 3 sections. Section 1 deals with recalling some needed informations on the invariants of the group $GL(n, Z/2)$ and the Dickson classes given by Dickson [2], Huỳnh Mùi [3] and the second author [5], [6]. In Section 2 the action of the Steenrod algebra on the Dickson algebra is described explicitly. Finally, in Section 3 we compute Ann $PH_*(\Omega_0^q S^q; Z/2)$, the submodule of $H_*(\Omega_0^q S^q; Z/2)$ consisting of all primitive elements annihilated by any Steenrod operations of positive degrees, for q small.

It should be mentioned that one could theoretically compute this module by means of Dyer — Lashop operations. However, such a computation does not seem to be effective (see Wellington [11]).

Using the similar results given in [7], [8] on the mod p Dickson classes and the algebra $H^*(\Omega_o^q S^q; Z/p)$ for p an old prime we will apply our method here with some minor changes to determine the action of the opposite mod p Steenrod algebra $A_*(p)$ on $H_*(\Omega_o^q S^q; Z/p)$ in a subsequent paper.

§1. PRELIMINARIES

Throughout this paper the coefficients are always taken in the field of 2 elements $Z_2 = Z/2$.

Let Σ_m be the symmetric group of all permutations on m letters. Set $H_*(\Sigma_\infty) = \lim\limits_{\overrightarrow{m}} H_*(\Sigma_m)$. The well-known Barratt – Priddy — Quillen's result

$$H_*(\Omega_o^\infty S^\infty) \simeq H_*(\Sigma_\infty)$$

(see e.g. [9]) explains why we pay attention to the homology of symmetric groups.

Let us think of Σ_{2^n} as the symmetric group on (the point set of) the vector space Z_2^n. Let E^n be the subgroup of all translations on Z_2^n. Let $x_i \in H^1(E^n) = \mathrm{Hom}(E^n, Z_2)$, $1 \leqslant i \leqslant n$, denote the dual element of the translation defined by the i-th unit vector in Z_2^n. So one obtains

$$H^*(E^n) = Z_2[x_1, \ldots, x_n]. \tag{1.1}$$

Since $GL_n = GL(n, Z_2)$ is the Weyl group of E^n in Σ_{2^n}, then GL_n acts on $H^*(E^n)$ by adjoint isomorphisms. This is the usual action of GL_n on $Z_2[x_1, \ldots, x_n]$ under the identity 1.1.

A well-known result asserts that the image of the restriction homomorphism $\mathrm{Res}(E^n, \Sigma_{2^n}) : H^*(\Sigma_{2^n}) \to H^*(E^n)$ is a subalgebra of the invariant algebra $Z_2[x_1, \ldots, x_n]^{GL_n}$. Furthermore, Huỳnh Mùi showed in [3; II. 6.2] that

$$\mathrm{ImRes}(E^n, \Sigma_{2^n}) = Z_2\left[x_1, \ldots, x_n\right]^{GL_n}. \tag{1.2}$$

According to L. E. Dickson [2] we have

$$Z_2\left[x_1, \ldots, x_n\right]^{GL_n} = Z_2\left[Q_{n,0}, \ldots, Q_{n,n-1}\right]. \tag{1.3}$$

Here $Q_{n,s}$, for $0 \leqslant s < n$, is the Dickson invariant of dimension $2^n - 2^s$ defined by the following equation in $Z_2[x_1, \ldots, x_n][X]$

$$\prod_x (X - x) = X^{2^n} + \sum_{s=0}^{n-1} Q_{n,s} X^{2^s},$$

where X is an indeterminate and the product in the left hand side runs over all $x \in H^1(E^n) = Z_2 x_1 \oplus \ldots \oplus Z_2 x_n$.

Let $q_{n,s}$ be the Dickson coinvariant dual to $Q_{n,s}$ with respect to the basis of all monomials of the Dickson invariants. Then, taking the dual of Dickson's result we get

$$H_*(E^n)_{GL_n} = \Gamma(q_{n,0}, \ldots, q_{n,n-1}),$$

the divided polynomial algebra generated by $q_{n,0}, \ldots, q_{n,n-1}$ with the divided powers γ_i, $i \geqslant 0$ (see [1]).

Now for every $K = (k_0, \ldots, k_{n-1})$, $k_i \geqslant 0$, we define the Dickson homology class D_K by

$$D_K = i(\Sigma_{2^n}, E^n)(q_K), \tag{1.4}$$

where $i(\Sigma_{2^n}, E^n): H_*(E^n)_{GL_n} \to H_*(\Sigma_{2^n})$, denotes the homomorphism induced by the inclusion $E^n \subset \Sigma_{2^n}$, and

$$q_K = \gamma_{k_0}(q_{n,0}) \ldots \gamma_{k_{n-1}}(q_{n,n-1}).$$

For convenience, we define the length and the height of $K = (k_0, \ldots, n_{n-1})$ to be $l(K) = n$, $|K| = k_0 + \ldots + k_{n-1}$ respectively. Further, we equip the index set

$$J = \{K = (k_0, \ldots, k_{n-1});\ n > 0, k_i \geqslant 0\}$$

with a partial summation defined only for elements of the same length in terms of the coordinates.

The regular embedding $\Omega_0^q S^q \to \Omega_0^\infty S^\infty$ induces the monomorphism

$$H_*(\Omega_0^q S^q) \to H_*(\Omega_0^\infty S^\infty) \cong H_*(\Sigma^\infty). \tag{1.5}$$

Indentifying $H_*(\Omega_0^q S^q)$ with its image under this monomorphism we noted in [6] that

$$D_K \in H_*(\Omega_0^q S^q) \Leftrightarrow |K| = k_0 + \ldots + k_{n-1} < q$$

for $K = (k_0, \ldots, k_{n-1})$.

By means of the weak homotopy approximation of $\Omega_0^q S^q$ by the space of Σ_∞ — orbits in the configuration space $F(R^q, \infty)$ (see May [4]) the structure of the Hopf algebra $H_*(\Omega_0^q S^q)$ is described as follows.

THEOREM 1.1 [5], [6] (i) *As algebras*

$$H_*(\Omega_0^q S^q) = Z_2\,[D_K;\ K \in J^+(q)]$$

for $1 < q \leqslant \infty$, where

$$J^+(q) = \{K = (k_0, \ldots, k_{n-1}) \in J\,;\ n > 0,\ |K| < q,\ k_0 > 0\}.$$

(ii) *The comultiplication is given by*

$$\Delta D_K = \sum_{L+M=K} D_L \otimes D_M \tag{1.6}$$

for $K, L, M \in J$.

(iii)

$$D_{(\underbrace{0,..0}_{s}, k_0, \ldots, k_{n-1})} = D^{2^s}_{(k_0, \ldots, k_{n-1})}.$$

Given $H \in J$ let $W_H \in H^*(\Sigma_\infty) = H^*(\Omega_0^\infty S^\infty)$ denote the dual element of D_H with respect to the basis consisting of all monomials in the right hand side of (1.6) for $q = \infty$ Passing to the dual of Theorem 1.1 we obtained in [5] the isomorphism of algebras

$$H^*(\Sigma_\infty) = Z_2\,[W_H\,;\ H \in J_{od}], \tag{1.7}$$

where $J_{od} = \{(h_0, \ldots, h_{n-1}) \in J\,;\ n > 0,\ h_i \text{ is odd for some } i\}$.
So W_H is called the universal Dickson class of the type H for $H \in J_{od}$.

Moreover, the kernel of the epimorphism $H^*(\Omega_0^\infty S^\infty) \to H^*(\Omega_0^q S^q)$ can be computed by means of (1.5) and Theorem 1.1. Hence we obtained in [6] the algebra isomorphism

$$H^*(\Omega_0^q S^q) \cong Z^2\,[W_H\,;\ H \in J_{od}(q)] \Big/ \left(W_H^{2^{h(q,H)}};\ H \in J_{od}(q)\right) \tag{1.8}$$

where

$$h(q, H) = \min\{h \in N\ \ 2^h\,|H| \geqslant q\},$$
$$J_{od}(q) = \{H \in J_{od} : |H| < q\}.$$

§2. STEENROD OPERATIONS ON THE DICKSON ALGEBRA

This section is devoted to computing the action of the Steenrod algebra A on the Dickson algebra

$$Z_2\,[x_1, \ldots, x_n]^{GL_n} = Z_2\,[Q_{n,0}, \ldots, Q_{n,n-1}].$$

This action is derived from the usual action of A on the cohomology $H^*(E^n) = Z^2\,[x_1, \ldots, x_n]$.

The resulting formula is more explicit in comparison with that given in Wilkerron [12].

THEOREM (2.1)

$$Sq^k Q_{n,s} = \begin{cases} Q_{n,t}\, Q_{n,r} & k = 2^n - 2^r + 2^s - s^t,\ n \geqslant r \geqslant s \geqslant t, \\ & \textit{where } r = \textit{simplies } s = t, \\ 0 & \textit{otherwise,} \end{cases}$$

where $Q_{n,n} = 1$ *by convention.*

Proof. Since dim $Q_{n,s} = 2^n - 2^s$ so we need only to compute

$$Sq^k Q_{n,s} \text{ for } 0 < k < 2^n.$$

From private discussion with Huỳnh Mùi we know the following formula (see also Wilkerson [12] for the proof).

$$Sq^k Q_{n,s} = Sq^{k-2^{s-1}} Q_{n,s-1} + (Sq^{k-2^{n-1}} Q_{n,n-1}). Q_{n,s} \tag{2.2}$$

for $0 < k < 2^n$. Here Sq^i and $Q_{n,i}$ are interpreted as 0

for $i < 0$. Applying this formula we have

$$Sq^{k-2^{n-1}} Q_{n,n-1} = Sq^{k-2^{n-1}-2^{n-2}} Q_{n,n-2} + (Sq^{k-2^n} Q_{n,n-1})\, Q_{n,n-1} =$$

$$= Sq^{k-2^{n-1}-2^{n-2}} Q_{n,n-2}$$

(by information on the dimension)

. . . .

$$= Sq^{k-(2^n-2^{n-t})} Q_{n,n-t}.$$

If there exists a number t such that $k = 2^n - 2^{n-t}$ then

$$Sq^{k-(2^n-2^{n-t})}\, Q_{n,n-t} = Q_{n,n-t}.$$

Otherwise, we can choose t large enough so that $k - (2^n - 2^{n-t}) < 0$.

This means $Sq^{k-(2^n-2^{n-t})} Q_{n,n-t} = 0.$

Hence, for $o > k < 2^n$ we get

$$Sq^{k-2^{n-1}} Q_{n,n-1} = \begin{cases} Q_{n,r} & k = 2^n - 2^r \\ 0 & \text{otherwise.} \end{cases}$$

Now formula (2,2) becomes

$$Sq^k Q_{n,s} = \begin{cases} Sq^{k-2^{s-1}} Q_{n,s-1} + Q_{n,r}\, Q_{n,s} & k = 2^n - 2^r,\ 0 \leqslant r < n, \\ Sq^{k-2^{s-1}} Q_{n,s-1} & \text{otherwise,} \end{cases} \tag{2.3}$$

for $0 < k < 2^n$.

We consider the following two cases.

Case where $k = 2^n - 2^r$ $(0 \leqslant r < n)$.

If $r \geqslant s$ then using formula 2.3 we have:

$$Sq^{2^n-2^r} Q_{n,s} = Sq^{2^n-2^{s-1}} Q_{n,s-1} + Q_{n,r} \cdot Q_{n,s}$$

$$= Sq^{2^n-2^r-2^{s-1}-2^{s-2}} Q_{n,s-2} + Q_{n,r} \cdot Q_{n,s}$$

$$\cdots\cdots\cdots\cdots$$

$$= Q_{n,t} \cdot Q_{n,s},$$

because $2^n - 2^r - 2^{s-1} - \ldots - 2^{s-t} \neq 0$ for any t.

If $r < s$, set $r = s - t$ $(t > 0)$. Then using (2.3) again we get

$$Sq^{2^n-2^{s-t}} Q_{n,s} = Sq^{2^n-2^{s-t}-2^{s-1}} Q_{n,s-1} + Q_{n,r} \cdot Q_{n,s}$$

$$\cdots\cdots\cdots\cdots$$

$$= Sq^{2^n-2^{s-t}-2^{s-1}-\ldots-2^{s-t}} Q_{n,s} + Q_{n,r} \cdot Q_{n,s}$$

$$= Sq^{2^n-2^s} Q_{n,s-t} + Q_{n,r} \cdot Q_{n,s}$$

$$= Q_{n,s} \cdot Q_{n,r} + Q_{n,r} \cdot Q_{n,s}$$

$$= Q_{n,s} \cdot Q_{n,r} + Q_{n,r} \cdot Q_{n,s} = 0.$$

Finally, we obtain

$$Sq^k Q_{n,s} = \begin{cases} Q_{n,s} \cdot Q_{n,s} & k = 2^n - 2^r,\ r \geqslant s, \\ 0 & k = 2^n - 2^r,\ r < s. \end{cases} \qquad (2.4)$$

Case where $k \neq 2^n - 2^r$ *for every* r *with* $0 \leq r < n$.

If $k = 2^n - 2^r + 2^s - 2^{s-t}$ for certain r, t with $0 \leqslant r < n$, $t \leqslant s$, then by information on the dimensions we see easily

$$Sq^k Q_{n,s} = 0 \text{ for } r \leqslant s.$$

Suppose $r > s$. By virtue of (2. 3) and (2.4) we have

$$Sq^k Q_{n,s} = Sq^{2^n-2^r+2^{s-1}+\ldots+2^{s-t}} Q_{n,s}$$

$$= Sq^{2^n-2^r+2^{s-2}+\ldots+2^{s-t}} Q_{n,s-1}$$

$$\cdots\cdots\cdots\cdots$$

$$= Sq^{2^n-2^r} Q_{n,s-t}\, Q_{n,s-i} \cdot Q_{n,r}.$$

If $k \neq 2^n - 2^r + 2^s - 2^{s-t}$ for any r, t with $0 \leqslant r < n$, $t \leqslant s$, then using 2.3. again we get

$$Sq^k \; Q_{n,s} = Sq^{k-2^{s-1}} \; Q_{n,\, s-1}$$

$$\dots\dots\dots\dots\dots$$

$$= Sq^{k-2^{s-1}} - \dots - 2^{s-u} \; Q_{n,\, s-u}$$

$$= \begin{cases} Q_{n,\, s-u} & k = 2^s - 2^{s-u},\; u \leqslant s, \\ 0 & \text{otherwise} \end{cases}$$

In short, if $k \neq 2^n - 2^r$ for every r with $0 \leqslant r < n$, we obtain

$$Sq^k \; Q_{n,s} = \begin{cases} Q_{n,t}\, Q_{n,r} & k = 2^n - 2^r + 2^s - 2^t,\; t \leqslant s < r < n, \\ Q_{n,t} & k = 2^s - 2^t, \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

Combining (2.4). and (2.5) completes the proof.

§3. ON THE MODULE ANN $PH_*(\Omega_0^q S^q)$

For any space X let $PH_*(X)$ be the submodule of $H_*(X)$ consisting of all primitive elements. That means

$$PH_*(X) = \{x \in H_*(X) : \Delta x = 1 \otimes x + x \otimes 1\},$$

where $\triangle$ denotes the coproduct in the homology.

Furthermore, suppose M is an A_*-module. Let $Ann\; M$ be the submodule of M consisting of all elements annihilated by any Steenrod operation of positive degree.

For various targets, it is important to compute $Ann\; PH_*\,(\Omega_0^q S^q)$. In [11] aided by computer, Wellington computed that module with $q = \infty$ up to the dimension 200.

Now we first determine $PH_*\,(\Omega_0^q S^q)$. Set

$$J(q) = \{K \in J;\; |K| < q\}.$$

For any $K \in J(q)$, let d_K be the dual element of W_K with respect to the basis of $H^*(\Omega_0^q S^q)$ represented by the monomials in the right hand side of (1.8). With the index set $J_{0d}(q)$ mentioned in (1.8) we have

PROPOSITION 3.1 *$PH_*(\Omega_0^q S^q) = Span\{d_K ; K \in J_{od}(q)\}$, the submodule spanned by $\{d_K ; K \in J_{od}(q)\}$. Moreover, for $K \in J_{od}(q)$, d_K is the unique primitive element satisfying*

$$\langle d_K , W_H \rangle = \delta_{KH}$$

for any $H \in J_{od}(q)$.

Proof. Let $IH^*(X)$ denote the ideal of decomposable elements in $H^*(X)$ for any space X. As is well-known, $PH_*(X)$ is dual to the module of undecomposable elements

$$QH^*(X) = H^*(X)/IH^*(X).$$

From (1.8) we derive immediately

$$QH^*(\Omega_0^q S^q) = \text{span}\{[W_H] ; H \in J_{od}(q)\},$$

where $[W_H]$ denotes the image of W_H in the quotient module $QH^*(\Omega_0^q S^q)$. By the definition of d^K we obtain $PH_*(\Omega_0^q S^q) = \text{Hom}(\text{Span}\{[W_H] ; H \in J_{od}(q)\}, Z_2) = \text{Span}\{d_K ; K \in J_{od}(q)\}$.

Finally, suppose $x \in H_*(\Omega_0^q S^q)$ is primitive and

$$\langle x, W_H \rangle = \delta_{KH}$$

for any $H \in J_{od}(q)$. Since x is primitive, x is annihilated by every monomial of $\{W_H ; H \in J_{od}(q)\}$ with at least 2 factors. By the definition of d_K this implies $x = d_K$. The proposition is proved.

Remark 3.1 The Hopf algebra $H_*(\Omega_0^q S^q)$ can be described by the generators d_K 's as follows.

(i) $$H_*(\Omega_0^q S^q = Z_2 [d_K ; K \in J^+(q)]$$

(ii) $$d^2_{(k_0, \ldots, k_{n-1})} = d_{(0, k_0, \ldots, k_{n-1})}$$

for any $(k_0, \ldots, k_{-1}) \in J(q)$

(iii) If K is in $J_{ev}(q) = J(q) / J_{od}(q)$. Then there exists a unique expansion $K = 2^k . H$, where k is a positive integer and $H \in J_{od}(q)$. We get

$$\Delta d_{2^k H} = \sum_{\substack{l+m=k \\ l, m \geq 0}} d_{2^l H} \otimes d_{2^m H}$$

To compute Ann $PH_*(\Omega_0^q S^q)$ we need the following proposition.

Suppose I is a finite subset of $J_{od}(q)$. Denote by I_0 the subset of I consisting of all sequences K with the shortest length. We have

PROPOSITION 3.2 *If* $\sum_{K\in I} d_K \in \operatorname{Ann} PH_*(\Omega_0^q S^q)$

then $$\sum_{K\in I} D_K \in \operatorname{Ann} H_*(\Omega_0^q S^q) \qquad (3.3)$$

Remark 3.2 In general, the inverse implication is not true.

To prove the proposition we first define homogenuous multiplicity for monomials of Dickson elements D_K by putting

$$\mu(D_K) = 2^{l(K)},$$

$$\mu(x.y) = \mu(x) + \mu(y).$$

For convenience, zero is considered as having any multiplicity. Using the result of Section 2 and the Cartan formula we note that the module spanned by monomials of Dickson elements of the same multiplicity is an A_*-module.

LEMMA 3.1.

$$d_K = D_K \text{ (elements of higher mutiplicities) for any } K \in J_{od}(q). \qquad (3.4)$$

Proof. Given $K, L \in J(q)$. We define $L \leqslant K$ if and only if

$$l(K) = l(L) = n,\ l_i \leqslant k_i \qquad (0 \leqslant i < n)$$

where $$K = (k_0, \dots, k_{n-1}),\ L = (l_0, \dots, l_{n-1}).$$

Let C be the subalgebra of $H_*(\Omega_0^q S^q)$ generated by D_L with $L \leqslant K$. It is easy to see that

$$C = Z_2[D_L ; L \leqslant K].$$

Moreover, according to Theorem 1.1, C is a Hopf subalge ra of $H_*(\Omega_0^q S^q)$.

Let C^* be the Hopf algebra dual to C. We obtain the canonical epimorphisms

$$i^* : H^*(\Omega_0^q S^q) \to C^*,$$

$$\bar{i}^* : QH^*(\Omega_0^q S^q) \to QC^*,$$

where QC^* is the module of undecomposiole elements of C^*. Denote by $w_K \in C^*$ the dual element of D_K with respect to the basis of C consisting of all monomials of D_L' s. We get immediately

$$i^*(W_K) = w_K,\ [w_K] = \overline{i^*}[W_K] \neq 0 \text{ in } QC^*.$$

Let e_K be the unique primitive ele ent in C characterized by the condition

$$\langle e_H, \overline{i^*}[W_H]\rangle = \delta_{KH}$$

for $H \in J_{od}(q)$. Combining this condition with the definition of C yields

$$e_K = D_K + \text{(elements of higher multiplicities)}.$$

Finally, the definition of d_K and e_K implies

$$e_K \equiv i(e_K) = d_K ,$$

where $i: \ C \subset H_* \ (\Omega_0^q S^q)$ denotes the canonical inclusion. The lemma is proved.

Proof of proposion 3.2 Using Lemma 3.1 we get

$\sum_{K \in I} d_K = \sum_{K \in I_0} D_K$ (+ elements of higher multiplicities). Recall that the action of the Steenrod algebra A preserves multiplicity, Hence, if $\sum_{K \in I} d_K$ is annihilated by all Steenrod operations of positive degrees, then so is $\sum_{K \in I_0} D_K$. The proposition follows.

Remark 3.3 **(i)** It is easy to see that if $h \in \text{Ann}\, PH_* (\Omega_0^q S^q)$ then $h^{2^m} \in \text{Ann}\, PH_* (\Omega_0^q S^q)$ for every non-negative integer m. (ii) A classical result asserts that in $H_* (E^1) = \Gamma (q_1, 0)$ one has

$\gamma_i (q_{1,0}) \in \text{Ann}\, H_* (E^1) \Leftrightarrow i = 2^k - 1$, for some k. (see e. g, Steenrod [10; chap. I]).

From the definition of D_K it follows that

$D_{(i)} \in \text{Ann}\, H_*(\Omega_0^q S^q) \Leftrightarrow i = 2^k - 1 < q$, for some k.

In this case, $D_{\underbrace{(0,\dots,0,2^k-1)}_{n}} = D^{2^{n-1}}_{(2^k-1)} \in \text{Ann}\, H_* (\Omega_0^q S^q)$.

Define

$$h_{2^k-1} = \begin{cases} d_{(2^k-1)} + d_{\underbrace{(1,0,\dots,0)}_{k}} & \text{if } k > 1, \\ d_{(1)} & \text{if } k = 1. \end{cases}$$

We can check directly that if $2^k - 1 < q$ then

$$h_{2^k-1} \in \text{Ann}\ PH_* \ (\Omega_0^q S^q).$$

In the rest of the paper we compute $\text{Ann}\, PH_* (\Omega_0^q S^q)$ for q small.

THEOREM 3.1.

(i) For $q = 2,3$

$$Ann\ PH_* \ (\Omega_0^q S^4) = Span\ \{h_1^{2^m} ;\ 0 \leqslant m < \infty\}.$$

(ii) $Ann\ PH_* \ (\Omega_0^4 S^4) = Span\ \{h_1^{2^m}, h_3^{2^m}, d^{2^m}_{(1,1,1)} ;\ 0 \leqslant m < \infty\}$. (3.6)

Proof. We will always use the simple fact that $H_*(\Omega_0^q S^q)$ is an A_* - subcoalgebra of $H_*(\Omega_0^{q'} S^{q'})$ for $q < q'$.

Fix a positive integer n. We set

$$J_{od}(q, n) = \{K \in J ; |K| < q, 1(K) = n\},$$

$$H(q, n) = Span\{D_K ; K \in J_{od}(q, n)\} \subset H_*(\Omega_0^q S^q).$$

By Proposition 3.2. to determine Ann $PH_*(\Omega_0^q S^q)$ we need first to compute Ann $H(q, n)$ for every n.

Case where $q = 2$

$J_{od}(2, n)$ consists of the following sequences

$$a(n, r) = (0, \dots, 0, \underset{r+1}{\widehat{1}}, 0, \dots, 0)$$

for $0 \leqslant r < n$. Recall that

$$\dim D_{(k_0, \dots, k_{n-1})} = \Sigma\, k_i (2^n - 2^i).$$

So $H(2, n)$ admits at most one generator in each dimension. Hence $\sum_{K \in I} D_K \in \text{Ann}\, H(2, n) \Leftrightarrow D_K \in \text{Ann}\, H(2, n)$ for any $K \in I$, where I denotes an arbitrary subset of $J_{od}(2, n)$.

Taking the dual of Theorem 2.1 we get

$$\text{(a)}\quad S^k_{q*} D_{a(n,r)} = \begin{cases} D_{a(n, r+1)} & \text{for } r < n-1,\ k = 2^r \\ 0 & \text{for } r = n-1, \text{ any } k > 0. \end{cases}$$

This implies

$$\text{Ann}\, H(2, n) = Z_2 \cdot D_{a(n, n-1)}.$$

From Theorem 1.1, $D_{a(n, n-1)}$ is obviously primitive. So using Proposition 3.2 we obtain

$$\text{Ann}\ PH_*(\Omega_0^2 S^2) = \text{Span}\{D_{a(n, n-1)} ;\ 0 < n < \infty\}.$$

By the definition and Theorem 1.1.

$$D_{a(1, 0)} = h_1, \quad D_{a(n, n-1)} = h_1^{2^{n-1}}.$$

So the theorem is proved for $q = 2$.

Case where $q = 3$.

$J_{od}(3, q)$ consists of $a(n, r)$ $(o \leqslant r < n)$ mentioned above and

$$b(n, r, s) = (0, \dots, 0, \underset{r+1}{\widehat{1}}, 0, \dots, 0, \underset{r+1}{\widehat{1}}, 0, \dots, 0)$$

for $0 \leqslant r < s < n$. Again, $H(3, n)$ has at most one generator in each dimension.

Using the dual statement of Theorem 2.1 we have

$$\text{(b)}\qquad Sq_*^{2^n - 2^s} D_{b(n, r, s)} = D_{a(n, r)}.$$

By the same argument as in the previous case we get

$$\text{Ann } PH_* (\Omega_0^3 S^3) = \text{Span} \left\{ D_{a(n,n-1)} ; 0 < n < \infty \right\}.$$

The theorem follows for $q = 3$.

Case where $q = 4$

$J_{od}(4,n)$ consists of the above listed sequences $a(n,r)$ for $0 \leqslant r < n$, $b(n,r,s)$ for $0 \leqslant r < s < n$ and additively the following

$$c(n,r) = (0,\ldots, 0, \underset{r+1}{\widehat{3}}, 0,\ldots,0), \quad 0 \leqslant r < n$$

$$d(n,r,s) = (0,\ldots, 0, \underset{r+1}{\widehat{1}}, 0,\ldots, 0, \underset{s+1}{\widehat{2}}, 0,\ldots,0), \ 0 \leqslant r < s < n,$$

$$e(n,r,s) = (0,\ldots, 0, \underset{r+1}{\widehat{2}}, 0,\ldots 0, \underset{s+1}{\widehat{1}}, 0,\ldots,0), \quad 0 \leqslant r < s < n,$$

$$f(n,r,s,t) = (0,\ldots, 0, \underset{r+1}{\widehat{1}}, 0,\ldots, 0, \underset{s+1}{\widehat{1}}, 0,\ldots, 0, \underset{t+1}{\widehat{1}}, 0\ldots0), 0 \quad r \leqslant s < t < n$$

A simple computation shows that $H(4,n)$ has at most one generator in all dimensions except the dimensions $3(2^n - 2^r)$ and $3 \cdot 2^n - 2^r - 2^{s+1}$, where it has two generators. More explicitly, $\dim D_{c(n,r)} = \dim D_{c(n,r-1,r+1)} = 3(2^n - 2^r)$, $\dim D_{d(n,r,s)} = \dim D_{e(n,r-1,s+1)} = 3 \cdot 2^n - 2^r - 2^{s+1}$.

Passing Theorem 2.1 to the dual we obtain

(c) $Sq_*^k D_{c(n,r)} = \begin{cases} D_{c(n,r,r+1)} & r < n-1,\ k = 2^r, \\ 0 & r = n-1, \text{ any } k > 0, \end{cases}$

(c') $Sq_*^{2^n - 2^{r-1}} D_{c(n,r)} = 0,$

(d) $Sq_*^{2^s - 2^r} D_{d(n,r,s)} = D_{c(n,s)},$

(d') $Sq_*^{2^n - 2^{r-1}} D_{d(n,r,s)} = 0,$

(e) $Sq_*^{2^n - 2^r} D_{e(n,r,s)} = D_{b(n,r,s)},$

(f) $Sq_*^k D_{f(n,r,s,t)} = \begin{cases} D_{f(n,r+1,s,t)} & r+1 < s,\ k = 2^r, \\ D_{f(n,r,s+1,t)} & s+1 < t,\ k = 2^s, \\ D_{f(n,r,s,t+1)} & t+1 < n,\ k = 2t, \\ 0 & r+3 = s+2 = t+1 = n, \\ & \text{any } k > 0. \end{cases}$

Combining (c'), (d') (e) shows that the elements

$$D_{c(n,r)} + D_{e(n,r-1,r+1)} \text{ and } D_{d(n,r,s)} + D_{e(n,r-1,s+1)}$$

do not belong to Ann $H(4, n)$.

Furthermore, according to (a), (b), (c), (d), (e), (f) and individual element D_K for $K \in J_{od}(4, n)$ belongs to Ann $H(4, n)$ if and only if D_K is one of the following elements:

$$D_{a(n, n-1)}, D_{c(n, n-1)}, D_{f(n, n-3), n-2, n-1)}.$$

As a consequence

(g) Ann $H(4, n) = \text{Span}\{D_{a(n, n-1)}, D_{c(n, n-1)},$

$$D_{f(n, n-3, n-2, n-1)}\}.$$

Note that

$$D_{a(n,n-1)} = D^{2^{n-1}}_{a(1,0)} = S^{2^{n-1}}_{(1)},$$

$$D_{c(n,n-1)} = D^{2^{n-1}}_{c(1,0)} = D^{2^{n-1}}_{(3)},$$

$$D_{f(n,n-3,n-2,n-1)} = D^{2^{n-1}}_{f(3,0,1,2)} = D_{(1,1,1)}.$$

So we now restrict our attention to $D_{(1)}, D_{(3)}$ and $D_{(1,1,1)}$ (See Remark 3.3 again). Let us consider the primitive elements associated to them. Using Proposition 3.1 we can check easily

$$d_{(1)} = D_{(1)},\ d_{(3)} = D_{(3)} + D_{(1)} D_{(2)} + D^3_{(1)}$$

$$d_{(1,1,1)} = D_{(1,1,1)} + D_{(1,0,0)} D_{(0,1,1)} + D_{(0,1,0)} D_{(1,0,1)} + D_{(0,0,1)} D_{(1,1,0)}$$

As seen before, $h_1 = h_{(1)}$ belongs to Ann $PH_*(\Omega^4_0 S^4$.

Besides, so do $h_3 = d_{(3)} + d_{(1,0)}$ and $d_{(1,1,1)}$. We can check this fact by making use of the following formulae obtained as consequence of Theorem 2.1.

$$Sq^k_* d_{(3)} = Sq^k_* d_{(1,0)} = \begin{cases} D_{(0,1)} & k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^k_* D_{(1,0,0)} = \begin{cases} D_{(0,1,0)} & k = 1, \\ D_{(0,0,1)} & k = 3, \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^k_* D_{(0,1,1)} = \begin{cases} D_{(0,1,0)} & k = 4, \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^k_* D_{(0,1,0)} = \begin{cases} D_{(0,0,1)} & k = 2 \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^k_* D_{(1,0,1)} = \begin{cases} D_{(0,1,1)} & k = 1, \\ D_{(1,0,0)} & k = 4, \\ D_{(0,1,0)} & k = 5, \\ 0 & \text{otherwise}, \end{cases}$$

$$Sq^k_* D_{(0,0,1)} = 0 \text{ any } k,\ S^k_{q*} D_{(1,1,0)} = \begin{cases} D_{(1,0,1)} & k = 2, \\ D_{(1,0,0)} & k = 6, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, we note that

$$h_1^{2^{n-1}} = d_{a(n,n-1)}, \; h_3^{2^{n-1}} = d_{c(n,n-1)} + d_{a(n+1,n-1)}$$

$$d_{(1,1,1)}^{2^{n-1}} = d_{f(n,n-3,n-2,n-1)} .$$

Combining this with (g) and Proposition 3.3 we obtain the theorem for $q = 4$. The proof is completed.

Conjecture 3.1

$$\text{Ann } PH_*(\Omega_o^q S^q) = \text{Ann } PH_*(\Omega_o^4 S^4) \quad \text{for } q = 5, 6.$$

The authors would like to extend their thanks to Huynh Mui and V. Snaith for helpful suggestions and discussions.

REFERENCES

1. H. Cartan, Séminaire H. Cartan, E.N.S., 1954/1955.
2. L. E. Dickson, *A fundemental system of invariants of the general modular linear group with a solution of the form problem*, Trans. AMS, 12 (1911), 75-98.
3. Huỳnh Mui, *Modular invariant theorg and cohomology algebras of symmetric groups*, J. Univ. Tokyo 22(1975), 319 – 369.
4. J.P. May, *The geometry of iterated loop spaces*, Springer Lectures Notes in Math., Vol. 271 (1972).
5. Nguyen H.V. Hung, *The mod 2 cohomology algebras of symmetric groups*, Acta Math. Viet nam. 6, 2 (1981, 41—48).
6. Nguyen H.V. Hung, *The mod 2 equivariant cohomology algebras of configuration spaces*. Acta Math. Viet nam. 7, 1 (1982), 96—100.
7. Nguyen H.V.Hung, *Algèbre de cohomologie du groupe symétriques infini et classes caractéristiques de Dickson*, C.R.A.S. Paris, t. 297, série I (1983), 611—614.
8. Nguyen H.V. Hung, *The mod 2 equivariant cohomology algebras of configuration spaces*, Submitted to Proceedings AMS.
9. S. Priddy, *On $\Omega^\infty S^\infty$ and the infinite symmetric group*, Proc. Symp. Pure. Math. AMS 22 (1971), 217 – 220.
10. N.E Steenrod and D.B.A Epstein, *Cohomology operations, Annals of Math. Studies 50*, Princeton (1962).
11. R. J. Wellington, *The unstable Adams spectral sequence for free iterated loop spaces*, Memoirs AMS. Vol. 36, No. 258 (1982).
12. C. Wilkreson, *A primer on the Dickson invariants*, Contemporary Math. AMS. Vol. 19 (1983) 421 –434.

Received June 9, 1987

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HANOI