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CONFORMALLY AND SUPERCONFORMALLY COVARIANT EQUATIONS FOR NON-VANISHING MASS FIELDS

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1. INTRODUCTION

The conformal covariant theory was studied as early as in 1956 but it began attracting much attention only since 1969 with the discovery of the scaling invariant law in the process of inelastic interactions between leptons and hadrons. In recent time, conformally covariant equations have also been studied in the works [1,2].

In the conformal covariant theory, one assumes that, under conformal transformations, the physical quantities are transformed according to their dimensions. Within the pure mathematical formalism this fact can be explained as follows: a conformal transformation is considered as a global or local transformation of measuring units, therefore measured values have to be transformed in an inverse proportion to the changes of the associated measuring units in order to ensure the invariance of the physical quantities [3]. We shall use this interpretation for the transformation laws of the mass of fields under the conformal transformations.

In Section 2 of this paper, necessary and sufficient conditions for the non-zero mass free field equations

$$(L_\mu \partial^\mu - m(x))\, \varphi(x) = 0 \tag{1.1}$$

$$(\square(x) - m^2(x))\, \varphi(x) = 0 \tag{1.2}$$

to be conformally covariant are derived by the R-inversion method. This method is simpler than that used previously in the works [1, 2].

In Section 3, extending the conformal symmetry to the superconformal one, we consider the superconformal covariance of motion equations for the non-zero mass free scalar superfield.

As is known, after extending the Poincare group to the conformal group, there exist equations of the forms (1, 1), (1, 2), which are covariant with respect to the conformal group [1, 2]. In extending the Poincare group to the supersymmetry group, we still have supersymmetrically covariant equations for the scalar superfield:

$$\left(\frac{1}{2}\bar{D}^{\alpha}D_{\alpha} - M\right)\Phi(x, \theta) = 0, \tag{1.3}$$

$$(\square - M^2)\,\Phi(x, \theta) = 0 \tag{1.4}$$

where, the scalar superfield $\Phi(x, \theta)$ contains, as its components, almost the fields considered in the equations (1,1), (1,2). Finally, since the superconformal group is the extension of the Poincare group in both above manners, the question naturally arises as to whether there exitst superconformally covariant equations for the non-zero mass free scalar superfield. Using the representations of the superconformal algebra studied by Dao Vong Duc in [8], we show tha do not exist such equations.

2. THE CONFORMALLY COVARIANT FIELD EQUATIONS WITH NON-ZERO MASS

Let us consider the equation:

$$(L_\mu \partial^\mu + m(x))\,\varphi(x) = 0 \tag{2.1}$$

where L_μ are some operators which will be determined later, $m(x)$ is the mass function of the field. Components of the field $\varphi(x)$ — considered as a relativistic field—are transformed under fully reducible representation $M^{\varphi}_{\mu\nu}$ of $SL(2, C)$, which is decomposed into the irreducible representations

$$\varphi(x) = \begin{vmatrix} \varphi^{\tau_1}(x) \\ \vdots \\ \varphi^{\tau_n}(x) \end{vmatrix} \xrightarrow{SL(2,\,C)} \begin{vmatrix} \varphi'^{\tau_1}(x') \\ \vdots \\ \varphi'^{\tau_n}(x') \end{vmatrix} = \begin{pmatrix} M^{\tau_1}_{\mu\nu} & & \\ & \ddots & \\ & & M^{\tau_n}_{\mu\nu} \end{pmatrix} \begin{pmatrix} \varphi^{\tau_1}(x) \\ \vdots \\ \varphi^{\tau_n}(x) \end{pmatrix}$$

$$M^{\varphi}_{\mu\nu} = M^{\tau_1}_{\mu\nu} \oplus \dots \oplus M^{\tau_n}_{\mu\nu}.$$

Every irreducible representation τ_i of SL (2, C) is labelled by $\tau_i = (l_{0i}, l_{1i})$ according to [4].

Under a conformal transformation t, the coordinates x_μ, the interval element $ds(x)$, the mass function $m(x)$, and the field functions $\varphi(x)$ are transformed as follows (see [1]):

$$x_\mu \to x'_\mu$$

$$ds^2(x) \to ds'^2(x') = \sigma^2(t, x)\, ds^2(x)$$

$$m(x) \to m'(x') = \sigma^{-1}(t, x)\, m(x)$$

$$\varphi(x) \to \varphi'(x') = S^{\varphi}(t, x)\,\varphi(x).$$

and equation (2.1) becomes

$$(L_\mu \partial'^\mu + m'(x'))\, \varphi'(x') = 0 \tag{2.2}$$

The function $\sigma(l,x)$ and the cocycle $S^\varphi(l,x)$ are expressible in terms of the basis $\{M^\varphi_{\mu\nu}, D^\varphi, K^\varphi_\mu\}$ of the Weyl group (see [1]):

$$S^\varphi(e^{iaP}, x) = 1$$

$$S^\varphi\left(e^{-\frac{i}{2} M_{\mu\nu} w^{\mu\nu}}; x\right) = 1 - \frac{i}{2} w_{\mu\nu} M^{\varphi\mu\nu}$$

$$S^\varphi(e^{-ibD}; x) = 1 - ibD^\varphi$$

$$S^\varphi(e^{icK}; x) = 1 + iC^\mu \left\{K^\varphi_\mu - 2x_\mu D^\varphi + 2x^\nu M^\varphi_{\mu\nu}\right\}$$

Then, the necessary and sufficient conditions for the equation (2.2) to have the same form as (2.1) are:

$$\left[M^\varphi_{\mu\nu}, L_\rho\right] = i(g_{\nu\rho} L_\mu - g_{\mu\rho} L_\nu) \tag{2.3}$$

$$[D^\varphi; L_\mu] = 0 \tag{2.4}$$

$$L_\mu(M^{\varphi\nu\mu} - D^\varphi g^{\nu\mu}) = 0 \tag{2.5}$$

$$[K^\varphi_\mu, L_\mu] = 0 \tag{2.6}$$

According to [4], the form of the operators L_k ; $(k = 1,2,3)$, satisfying equation (2.3), which ensures that equation (2,1) will be relativistically covariant, can be found if one knows the form of the operator L_0 in the canonical basis $\{\xi^\tau_{lm}\}$, with $\tau = \tau_1, \ldots, \tau_n$. In this basis, the explicit form of L_0 is given by

$$L_0\, \xi^\tau_{lm} = \sum_{\tau' l' m'} C^{\tau\tau'}_{lm,l'm'}\, \xi^{\tau'}_{l',m'}, (\tau,\tau' \in \tau_1,\ldots,\tau_n) \tag{2.7}$$

$$C^{\tau\tau'}_{lm,l'm'} = C^{\tau\tau'}_l \delta_{ll'} \delta_{mm'}.$$

The equation (2.1) will have the non trivial operators L_μ (i.e. at least one of the operators L_μ is non-vanishing) if and only if a least one of the numbers $C^{\tau\tau'}_l$ is non-zero. The factor $C^{\tau\tau'}_l$ can be non-vanishing only if $\tau = (l_0, l_1)$ and $\tau' = (l'_0, l'_1)$ are interlocked, . e. if either

$$\Delta l_0 = 0 \quad \text{and} \quad \Delta l_1 = \pm 1$$

or

$$\Delta l_0 = \pm 1 \text{ and } \Delta l_1 = 0. \tag{2.8}$$

According to [1], equation (2.1) is covariant with respect to the dilatation transformation D only if $D^{\varphi} = il\varphi$ ($l\varphi$ is called the scale dimension of the field $\varphi(x)$), and the scale dimension of the operators L_μ equals zero. These conditions hold for equation (2.4).

Let us consider the conditions for equation (2.1) to be covariant with respect to the special conformal transformation. They correspond to the two equations (2.5), (2.6). We suppose that equation (2.1) is covariant with respect to the Poincaré and dilatation transformations. Since

$$K_\mu = RP_\mu R \; ; \; M_{\mu\nu} = RM_{\mu\nu}R \; ; \; D = -RDR,$$

equation (2.1) will be covariant with respect to the special conformal transformation if it is also covariant with respect to the R-transformation. Therefore, we shall consider the R-covariance of equation (2.1). According to [5], under the R-transformation, the quantities of equation (2.1) are transformed as follows:

$$x_\mu \to x'_\mu = - x_\mu / x^2$$

$$dS^2(x) \to dS'^2(x') = dS^2(x) / x^4$$

$$m(x) \to m'(x') = x^2 m(x)$$

$$\varphi(x) \to \varphi'(x') = (x^2)^{-l\varphi} R^{\varphi}\left(-\frac{x}{x^2}\right)\varphi(x).$$

Necessary and sufficient conditions for the form of equation (2,1) to be preserved under the R-transformation are:

$$R^{\varphi}(x)\, L_\mu \partial'^{\mu} (x^2)^{-l\varphi} R^{\varphi}\left(-\frac{x}{x^2}\right) + R^{\varphi}(x)\, L_\mu (x^2)^{-l\varphi}\, \partial'^{\mu} R^{\varphi}\left(-\frac{x}{x^2}\right) = 0 \qquad (2.9)$$

$$R^{\varphi}(x)\, L_\mu\, R^{\varphi}\left(-\frac{x}{x^2}\right) = \left(-\delta^{\nu}_{\mu} + 2\frac{x^{\nu} x_\mu}{x^2}\right) L_\nu \qquad (2.10)$$

Using the follwing relations for the function $R^{\varphi}(x)$ studied in [6]

$$\partial_\mu R^{\varphi}\left(-\frac{x}{x^2}\right) = -2ix^{\nu} R^{\varphi}\left(-\frac{x}{x^2}\right) M^{\varphi}_{\mu\nu}$$

$$x^{\nu}\left[R^{\varphi}(x), M^{\varphi}_{\mu\nu}\right]_+ = 0$$

and solving the system of equations (2.9) and (2, 10), we find

$$l\varphi = -\frac{3}{2} \qquad (2.11)$$

This result coincides with that obtained by another method in the work [1].

On the other hand, from equations (2.3), (2.5), we can derive the following equation (see [2])

$$(\Gamma L_0 - L_0 \Gamma) = \left(\frac{3}{2} + l\varphi\right) L_0 \qquad (2.12)$$

where Γ is the Casimir operator of the Lorentz group

$$\Gamma = \frac{1}{4} M_{\mu\nu} M^{\mu\nu} = \Gamma^{\tau_1} \oplus \dots \oplus \Gamma^{\tau_n}$$

$$\Gamma^{\tau_i} = \frac{1}{4} M^{\tau_i}_{\mu\nu} M^{\tau_i \, \mu\nu} = l_{0i}^2 + l_{1i}^2 - 1$$

Substituting the value of $l\varphi$ into equation (2, 12), we have

$$\Gamma L_0 - L_0 \Gamma = 0$$

Using the definition (2, 7) of operator L_0 for the above equation, we deduce that $\Gamma^{\tau'} = \Gamma^{\tau}$, (i. e. $l_0'^2 + l_1'^2 = l_0^2 + l_1^2$ if $C_l^{\tau\tau'} \neq 0$. Thus, if the irreducible representations $\tau = (l_0, l_1)$ and $\tau' = (l'_0, l'_1)$ are interlocked, they are also strongly interlocked.

Finally, from the commutator

$$[D, K_\mu] = -iK\mu$$

and
$$D^\varphi = il\varphi$$

we have $K^\varphi_\mu = 0$, and therefore (2.6) is automatically fulfilled. Thus, we obtain:

THEOREM 1. *Equation (2, 1) is conformally covariant if and only if the following conditions hold: the field function has* $l_\varphi = -\frac{3}{2}$, $K^\varphi_\mu = 0$, *and when restricted to the Lorentz group, it realizes the fully reducible representation* $M^\varphi = M^{\tau_1} \oplus \dots \oplus M^{\tau_n}$ *of* $SL(2, C)$, *in which every pair of really interlocked representation* τ *and* τ' *is also strongly interlocked.*

Note that the conformal covariance of the second order field equations can also be studied on the basis of the R-method by using analogous arguments.

3. THE SUPERCONFORMALLY COVARIANT EQUATIONS FOR THE FREE SCALAR SUPERFIELD WITH NON-VANISHING MASS

According to the arguments in the introduction, we consider the superconformal covariance of the two equations

$$(D^{(1)}(x, \theta) - m(x, \theta)) \; \Phi(x, \theta) = 0 \tag{3.1}$$

$$(D^{(2)}(x, \theta) - m^2(x, \theta)) \; \Phi(x, \theta) = 0 \tag{3.2}$$

We express the changes of the field function $\varphi(x, \theta)$ in (3. 3) by a matrix $S^{\Phi}(t,(x, \theta))$ expanded for the group parameters

$$\Phi'_A(x', \theta') = (S^{\Phi}(t,(x, \theta))\ \Phi(x, \theta))_A = \{[1 + i\varepsilon_I S^{\Phi}(F_I,(x, \theta))]\ \Phi(x, \theta)\}_A \qquad (3.10)$$

From the relation (3. 3), we have

$$\Phi'_A(x' \theta') = \Phi_A(x, \theta) - i\varepsilon_I [f(F_I), \Phi_A(x, \theta)]_{-(I, A)} + \\ + \left\{ \delta x_\mu \frac{\partial \Phi_A(x, \theta)}{\partial x_\mu} + \delta\bar{\theta}^\alpha \frac{\partial \Phi_A(x, \theta)}{\partial \bar{\theta}^\alpha} \right\}$$

where the commutators $[f(F_I), \Phi_A(x, \theta)]_{-(I, A)}$ are the graded ones defined in [7]. Using (3.8) and (3.9) we obtain:

$$\Phi'_A(x', \theta') = \Phi_A(x, \theta) - i\varepsilon_I \{[f(F_I); \Phi_A(x, \theta)]_{-(I, A)} + l(F_I)\Phi_A(x, \theta)\}. \qquad (3.11)$$

The representation $l(F_I)$ of the superconformal algebra, and the commutators $[f(F_I), \Phi_A(x, \theta)]_{-(I, A)}$ can be found in [8]. Substituting their explicit forms into (3.11) and comparing the latter with the expansion (3. 10), we find the transforming matrices $S^{\Phi}(t, (x, \theta))$, corresponding to each of the elementary transformations:

$$S^{\Phi}(e^{iaP}; (x, \theta)) = 1 \qquad (3.12)$$

$$S^{\Phi}(e^{-\frac{i}{2}M_{\mu\nu}\omega^{\mu\nu}}; (x, \theta)) = 1 - \frac{i}{2}\omega^{\mu\nu} M^{\Phi}_{\nu\mu} \qquad (3.13)$$

$$S^{\Phi}(e^{-ibD}; (x, \theta)) = 1 - ibD^{\Phi} \qquad (3.14)$$

$$S^{\Phi}(e^{icK}; (x, \theta)) = 1 + ic^\mu \{K^{\Phi}_\mu - 2x_\mu D^{\Phi} + 2x^\nu M^{\Phi}_{\mu\nu} + \\ + \frac{3i}{4}\bar{\theta}\gamma_\rho \gamma_\mu \theta E^{\Phi} + \bar{i\theta}\gamma_\mu Q^{\Phi} + \frac{i}{4}\varepsilon_{\mu\nu\lambda\rho}\bar{\theta}\gamma_\rho \gamma^\nu \theta M^{\Phi\lambda\rho}\} \qquad (3.15)$$

$$S^{\Phi}(e^{-i\bar{\zeta}S}; (x, \theta)) = 1. \qquad (3.16)$$

$$S^{\Phi}(e^{-i\bar{\xi}Q}; (x, \theta)) = 1 - i\bar{\xi}^\alpha \{ Q^{\Phi}_\alpha - \frac{i}{2}(\sigma_{\mu\nu}\theta)_\alpha M^{\Phi\mu\nu} - \\ - \theta_\alpha D^{\Phi} - \frac{3}{2}(\gamma_5 \theta)_\alpha E^{\Phi}\} \qquad (3.17)$$

$$S^{\Phi}(e^{-idE}; (x, \theta)) = 1 - idE^{\Phi} \qquad (3.18)$$

where $M^{\Phi}_{\mu\nu}$, D^{Φ}, K^{Φ}_μ, Q^{Φ}_α, E^{Φ} are the matrices acting on the indices of the superfields and a_μ, b, c_μ, $\omega_{\mu\nu}$, $\bar{\zeta}^\alpha$, $\bar{\xi}^\alpha$, d are the group parameters associated with the group generators P_μ, D, K_μ, $M_{\mu\nu}$, S_α, Q_α, E respectively.

where $\mathcal{D}^{(1)}(x,\theta)$ and $\mathcal{D}^{(2)}(x,\theta)$ are some differential operators to be determined later, $m\,(x,\theta)$ is the mass function of the superfield. These equations are generalizations of equations (1,1) – (1,4). In the general case $\Phi\,(x,\theta)$ is a column matrix with components $\Phi_A(x,\theta)$, the operators $\mathcal{D}^{(1,2)}(x,\theta)$, $m\,(x,\theta)$, and $m^2\,(x,\theta)$ are square matrices.

In this section, we use the following notation: $\mathcal{C}$ denotes the superconformal group t its elements, $\mathcal{M}$ the superspace with supercoordinates (x,θ), $\mathcal{H}$ a vector space of state vectors $|\,\alpha\rangle$, $l(t)$ a representation of $\mathcal{C}$ on $\mathcal{M}$, and $f(t)$ a representation of $\mathcal{C}$ on $\mathcal{H}$.

Under a transformation t, the state vectors $|\,\alpha\rangle$ and the supercoordinates (x,θ) are changed as follows:

$$t:|\,\alpha\rangle \rightarrow f(t)\,|\,\alpha\rangle$$
$$t:(x,\theta)\rightarrow(x',\theta')$$

Conversely, if we keep these states $|\,\alpha\rangle$ unchanged, then the field functions $\Phi(x,\theta)$ and the operators $\mathcal{D}^{(1)}(x,\theta)$ and $\mathcal{D}^{(2)}(x,\theta)$ must be transformed alternatively for the $|\,\alpha\rangle$.

$$t:\Phi_A\,(x,\theta)\rightarrow\Phi'_A\,(x',\theta')=f(t^{-1})\,\Phi_A\,(x',\theta')\,f(t) \tag{3.3}$$

$$t:\ \hat{\mathcal{D}}^{(1,2)}\rightarrow\hat{\mathcal{D}}'^{(1,2)}=f(t^{-1})\,\hat{\mathcal{D}}^{(1,2)}\,f(t) \tag{3.4}$$

In the supercoordinate representation of the superconformal algebra, the expression (3,4) takes the form:

$$\hat{\mathcal{D}}'^{(1,2)}\,(x',\theta')=l(t^{-1})\,\hat{\mathcal{D}}^{(1,2)}(x,\theta)\;l(t) \tag{3.5}$$

Since the supercoordinates (x,θ) are operators too, they are transformed as $\hat{\mathcal{D}}(x,\theta)$, and in the supercoordinate representation, they take the form:

$$(x',\theta')=l(t^{-1})\,(x,\theta)\,l(t)=l(t^{-1})\,(x,\theta) \tag{3.6}$$

In order to prove the nonexistence of the superconformally covariant equations, it is enough to retain only the first order terms in the infinitesimal expansion of the transformations t for group parameters. Therefore, in this section, we shall only expand the transformations t as follows:

$$t=1+i\,\varepsilon_I\,F_I \tag{3.7}$$

where F_I are the generators of the superconformal group $\mathcal{C}$ and ε_I are group parameters associated with F_I.

Corresponding to (3.7), the differentials δx_μ, $\delta\bar{\theta}^\alpha$ are expanded for the group parameters in the first order:

$$x'_\mu=x_\mu+\delta x_\mu=l(t^{-1})x_\mu=x_\mu-i\varepsilon_I\,l(F_I)x_\mu \tag{3.8}$$

$$\bar{\theta}'^\alpha=\bar{\theta}^\alpha+\delta\bar{\theta}^\alpha=l(t^{-1})\bar{\theta}^\alpha=\bar{\theta}^\alpha-i\varepsilon_I\,l(F_I)\bar{\theta}^\alpha. \tag{3.9}$$

In this approximation, we caculate the changes of the quantities in the equations (3. 1), (3. 2) under the transformation t.

Now, we consider the changes of the mass function $m(x, \theta)$ under the superconformal transformations t. Following [3], we suppose that under the superconformal transformations, the mass function $m'(x, \theta)$ is changed in the inverse proportion to the 4-dimensional line element:

$$t: dS^2(x, \theta) \to dS'^2(x', \theta') = \sigma^2(t, (x, \theta))\, dS^2(x, \theta), \tag{3.19}$$

$$t: m(x, \theta) \to m'(x', \theta') = \sigma^{-1}(t, (x, \theta))\, m(x, \theta). \tag{3.20}$$

Thus, in order to calculate the changes of the mass function, we must calculate the changes of the line element under the transformations t. There are several ways to define the line element. In [9], the line element of the superspace was chosen as follows:

$$dw_\mu = \delta x_\mu + \frac{i}{2}\delta\bar{\theta}\gamma_\mu\theta. \tag{3.21}$$

But this line element is not transformed locally under the superconformal transformations. For this reason, in this section, we choose the Minkowski conventional 4-dimensional interval to be the line element:

$$dw_\mu = \delta x_\mu. \tag{3.22}$$

Here, in order that dw_μ be transformed locally, this line element must be defined only for two points of superspace with the same value of the Grassman coordinates θ.

Let us calculate the dilatation coefficient $\sigma(t, (x, \theta))$ defined in (3.19) corresponding to the line element difined by (3.22). We denote the change of a coordinate x_μ on a fibre θ, under the transformation t, as follows:

$$\begin{aligned} t:&\quad x_\mu(\theta) &&\to x'_\mu(\theta') \equiv y_\mu(x_\mu, \theta') \\ t:&\quad (x_\mu + \delta x_\mu)(\theta) &&\to (x_\mu + \delta x_\mu)'(\theta') \equiv y_\mu(x_\mu + \delta x_\mu, \theta') \\ t:&\quad dW_\mu(\theta) &&\to dW'_\mu(\theta') = y_\mu(x_\mu + \delta x_\mu, \theta') - y_\mu(x_\mu, \theta') \end{aligned} \tag{3.23}$$

From (3.19) and (3.23) we have the equation for $\sigma(t, x, \theta))$:

$$dS'^2 = dW'_\mu dW'^\mu = \sigma^2(t, (x, \theta))\, dx_\nu dx^\nu. \tag{3.24}$$

With the approximation (3.7) — (3.9), the solution of this equation is

$$\sigma(t, (x, \theta)) = 1 + i\varepsilon_I, \sigma(F_I, (x, \theta)) = 1 - \frac{i}{4}\varepsilon_I\{\partial_\lambda[l(F_I)x^\lambda]\}. \tag{3.25}$$

Substituting the explicit forms of the representation $l(F_I)$ into (3.25), we find the dilatation coefficients $\sigma(F_I, (x, \theta))$ corresponding to each of the elementary transformations of $\mathcal{C}$:

$$\sigma(e^{iaP}; (x, \theta)) = \sigma(e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}; (x, \theta)) =$$

$$= \sigma(e^{-idE}; (x, \theta)) = \sigma(e^{-i\bar{\zeta}s}; (x, \theta)) = 1 \tag{3.26}$$

$$\sigma(e^{icK}; (x, \theta)) = 1 + 2cx; \tag{3.27}$$

$$\sigma(e^{-ibD}; (x, \theta)) = 1 + b, \tag{3.28}$$

$$\sigma(e^{-i\bar{\xi}Q}; (x, \theta)) = 1 - \frac{1}{2}\bar{\xi}\theta. \tag{3.29}$$

Now, we consider necessary and suffficient conditions for the equations (3.1) and (3.2) to be superconformally covariant. Under the transformation t, the equation (3.1) takes the form :

$$(\widehat{\mathcal{D}}'^{(1)}(x', \theta') - m'(x', \theta'))\, \Phi'(x', \theta') = 0. \tag{3. 1'}$$

Using the expressions (3.5), (3.10), (3.20), we can write equation (3.1') as

$$[l(t^{-1})\, \mathcal{D}^{(1)}(x, \theta)\; l(t) - б^{-1}(t, (x, \theta))\; m(x, \theta))]\; S^{\Phi}(t, (x, \theta))\; \Phi(x, \theta) = 0. \tag{3. 30}$$

With the approximation (3.7), the inverse matrix $(S^{\Phi}(t, (x, \theta))^{-1}$ is given by $(S^{\Phi}(t, (x, \theta)))^{-1} = S^{\Phi}(t^{-1}, t(x, \theta)) \simeq S^{\Phi}(t^{-1}, (x, \theta))$.

Therefore, the equation (3. 30) becomes :

$$\{б(t, (x, \theta))\, S^{\Phi}(t^{-1}, (x, \theta))\, l(t^{-1}), \mathcal{D}^{(1)}(x, \theta) \cdot l(t)\, S^{\Phi}(t, (x, \theta)) - m(x, \theta)\}\, \Phi(x, \theta) = 0. \tag{3.31}$$

Comparing (3.31) with (3, 1), we obtain :

THEOREM 2. *The first order equation (3.1) for a non-zero mass free superfield* $\Phi(x, \theta)$ *is superconformally covariant if and only if there is an operator* $\mathcal{D}^{(1)}(x, \theta)$ *satisfying the following condition*:

$$\Big\{[\mathcal{D}^{(1)}(x, \theta);\, l(F_I)] + [\mathcal{D}^{(1)}(x, \theta);\, S\Phi\, (F_I, (x, \theta)] + б + (F_I(x, \theta))\, \mathcal{D}^{(1)}(x, \theta)\Big\}\, \Phi(x, \theta) = 0 \tag{3.32}$$

In a similar way, we derive the following theorem for the second order equation (3. 2):

THEOREM 3. *The second order equation (3.2) for a non-zero mass free superfield* $\Phi(x, \theta)$ *is superconformally covariant if and only if there is an operator* $\mathcal{D}^{(2)}(x, \theta)$ *satisfying the following condition:*

$$\Big\{[\mathcal{D}^{(2)}(x, \theta),\, l(F_I)] + [\mathcal{D}^{(2)}(x, \theta);\, S^{(\Phi)}(F_I\, (x, \theta)] + 2б(F_I(x, \theta))\; \mathcal{D}^{(2)}(x, \theta)\Big\}\, \Phi(x, \theta) = 0. \tag{3.33}$$

In the general case, the field function $\Phi(x, \theta)$ has many components, therefore each of the equations (3.32) and (3.33) is a matrix equation, i.e. there is a system of equations for each of the operators $\mathcal{D}^{(1)}(x, \theta)$ and $\mathcal{D}^{(2)}(x, \theta)$. We shall apply these equations only for the case when the superfield $\Phi(x,\theta)$ is the scalar superfield. This will be sufficient for us, because the scalar superfield contains almost all the physical fields.

For the scalar superfield, we have several restrictive conditions:

$$S^{\Phi}(M_{\mu\nu}, (x,\theta) = M^{\Phi}_{\mu\nu} = 0\,;\;\; S^{\Phi}(E;\; x,\theta)) = E^{\Phi} = 0.$$

and therefore, with respect to each of the generators $M_{\mu\nu}$, P_μ, S_α, E, D, K_μ, Q_α of $\mathcal{C}$, the equation (3.32) for operator $\mathcal{D}^{(1)}(x,\theta)$ takes the following concrete form :

$$[\mathcal{D}^{(1)}(x,\theta),\ l(M_{\mu\nu})]\ \Phi\ (x,\theta) = 0, \tag{3.34}$$

$$[\mathcal{D}^{(1)}(x,\theta),\ l(P_\mu)]\ \Phi\ (x,\theta) = 0, \tag{3.35}$$

$$[\mathcal{D}^{(1)}(x,\theta),\ l(S_\alpha)]\ \Phi\ (x,\theta) = 0, \tag{3.36}$$

$$[\mathcal{D}^{(1)}\ (x,\theta),\ l(E)]\ \Phi(x,\theta) = 0, \tag{3.37}$$

$$\{[\mathcal{D}^{(1)}\ (x,\theta),\ l(D)]\ -\ i\mathcal{D}^{(1)}(x,\theta)\}\ \Phi(x,\theta) = 0, \tag{3.38}$$

$$\{[\mathcal{D}^{(1)}(x,\theta), l(K_\mu)] + [\mathcal{D}^{(1)}\ (x,\theta);\ (-2x_\mu D^\Phi + i\bar{\theta}\gamma_\mu\ Q^\Phi)] - $$

$$- 2ix_\mu\ \mathcal{D}^{(1)}(x,\theta)\}\ \Phi(x,\theta) = 0, \tag{3.39}$$

$$\{[\mathcal{D}^{(1)}(x,\theta);\ l(Q_\alpha)] - [\mathcal{D}^{(1)}\ (x,\theta);\ \theta_\alpha D^\Phi\ ] +$$

$$+ \frac{i}{2}\ \theta_\alpha \mathcal{D}^{(1)}(x,\theta)\Big\}\ \Phi(x,\theta) = 0. \tag{3.40}$$

Solving this system of equations, we can find the explicit form of the operator $\mathcal{D}^{(1)}(x,\theta)$.

Similarly, with respect to each of the generators $M_{\mu\nu}$, P_μ, S_α, E, D, K_μ, Q_α of $\mathcal{C}$, the equation (3.33) for $\mathcal{D}^{(2)}(x,\theta)$ takes the form:

$$[\mathcal{D}^{(2)}(x,\theta);\ l\ (M_{\mu\nu})]\ \Phi\ (x,\theta) = 0 \tag{3.41}$$

$$[\mathcal{D}^{(2)}\ (x,\theta);\ l(P_\mu)]\ \Phi\ (x,\theta) = 0. \tag{3.42}$$

$$[\mathcal{D}^{(2)}\ (x,\ \theta);\ l\ (S_\alpha)]\ \Phi\ (x,\theta) = 0. \tag{3.43}$$

$$[\mathcal{D}^{(2)}\ (x,\theta);\ \ l(E)]\ \ \Phi\ (x,\ \theta) = 0. \tag{3.44}$$

$$\{[\mathcal{D}^{(2)}\ (x,\theta),\ l(D)] - 2i\ \mathcal{D}^{(2)}(x,\theta)\}\ \Phi\ (x,\theta) = 0 \tag{3.45}$$

$$\{[\mathcal{D}^{(2)}(x,\theta):\ l(K_\mu)] + [\mathcal{D}^{(2)}\ (x,\theta);\ (-2x_\mu\ D^\Phi\ + i\bar{\theta}\ \gamma_\mu\ Q^\Phi)] -$$

$$-\ 4\ i\ x_\mu\ \mathcal{D}^{(2)}\ (x,\theta)\}\ \Phi\ (x,\theta) = 0. \tag{3.46}$$

$$\{\ [\ \mathcal{D}^{(2)}\ (x,\theta);\ l(Q_\alpha)] - [\mathcal{D}^{(2)}(x,\theta);\ \theta_\alpha\ D^\Phi\ ] + i\ \theta_\alpha\ \mathcal{D}^{(2)}\ (x,\theta)\}\ \Phi(x,\theta) = 0. \tag{3.47}$$

Solving this system of equations, we can find the explicit form of the operator $\mathcal{D}^{(2)}(x,\theta)$.

Solving $\mathcal{D}^{(1)}(x,\theta)$:

In order to solve the above derived equation, we expand operator $\mathcal{D}^{(1)}(x,\theta)$ for the anticommuting Majorana spinor θ analogously to the expansion of the superfields:

$$\mathcal{D}^{(1)}(x,\theta) = A(x) + \bar{\theta}^{\alpha} B_{\alpha}(x) + (\bar{\theta}\theta) C(x) + (\bar{\theta}\gamma_5\theta) E(x) +$$
$$+ (\bar{\theta}\gamma_{\nu}\gamma_5\theta) G^{\nu}(x) + (\bar{\theta}\theta)\bar{\theta}^{\alpha} H_{\alpha}(x) + (\bar{\theta}\theta)^2 K(x). \qquad (3.48)$$

where $A(x)$, $B(x)$..., $K(x)$ are some coefficient functions to be determined later. They are functions of the variables x_μ, $\frac{\partial}{\partial x_\mu}$; $\frac{\partial}{\partial \theta_\alpha}$; $\frac{\partial}{\partial \bar{\theta}^\alpha}$, and the Dirac matrices γ_μ, $\gamma_5\gamma_\mu$, γ_5, ∂_μ, with all possible combinations, so that the operator $\mathcal{D}^{(1)}(x,\theta)$ is the most general possible. These seven terms are independent of each other and sufficient to expand $\mathcal{D}^{(1)}(x,\theta)$ for parameter θ. Now, the system of equations (3.34) — 3.40) serves as a system of conditions imposed on the unknown coefficient functions of the operator $\mathcal{D}^{(1)}(x,\theta)$.

Let us consider successively these conditions:

From the condition (3.34), we deduce that $\mathcal{D}^{(1)}(x,\theta)$ is a scalar.

From the condition (3.35), we obtain the following expression

$$\left(\frac{\partial\, \mathcal{D}^{(1)}(x,\theta)}{\partial x_\mu}\right) \Phi(x,\theta) = 0.$$

Because of that, $\mathcal{D}^{(1)}(x,\theta)$ contains the variable x_μ not in the explicit form, but only in the form $\frac{\partial}{\partial x_\mu}$.

From condition (3.38), we see that the scale dimension of $\mathcal{D}^{(1)}(x,\theta)$ equals -1.

These three conditions are imposed on the coefficient functions A, B, C,..., K in (3.48). From $\frac{\partial}{\partial x_\mu}$, $\frac{\partial}{\partial \theta_\alpha}$, $\frac{\partial}{\partial \bar{\theta}^\alpha}$, and the Dirac matrices, we find all the possible coefficient functions A, B, C, ..., K in (3.48) satisfying these three conditions. The most general operator $\mathcal{D}^{(1)}(x,\theta)$ satisfying the above conditions is

$$\mathcal{D}^{(1)}(x,\theta) = a_0\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\right) + b_0\left(\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\right) + b_1\left(\bar{\theta}\hat{\partial}\frac{\partial}{\partial\bar{\theta}}\right) +$$
$$C_0(\bar{\theta}\theta)\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\right)^2 + C_2(\bar{\theta}\theta)\,\partial_\mu\partial^\mu + e_1(\bar{\theta}\gamma_s\theta)\left(\frac{\partial}{\partial\theta}\gamma_s\hat{\partial}\frac{\partial}{\partial\bar{\theta}}\right) +$$
$$+ g_1\,\overline{(\theta}\gamma_s\hat{\partial}\theta)\left(\frac{\partial}{\partial\theta}\gamma_s\frac{\partial}{\partial\bar{\theta}}\right) + h_1(\bar{\theta}\theta)\left(\bar{\theta}\hat{\partial}\frac{\partial}{\partial\bar{\theta}}\right)\left(\frac{\partial}{\partial_\alpha}\frac{\partial}{\partial\theta}\right) +$$
$$+ h_2(\bar{\theta}\theta)\left(\bar{\theta}\frac{\partial}{\partial\bar{\theta}}\right)\partial_\mu\partial^\mu + h_2(\bar{\theta}\theta)^2\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\theta}}\right)\partial_\mu\partial^\mu. \qquad (3.49)$$

These ten terms are independent of each other, and a_0, b_0, b_1.... are complex coefficients to be determined later. The operator $\overline{D}^\alpha D_\alpha$ in the equation (1,3) is a special case of the operator $\mathcal{D}^{(1)}(x,\theta)$ when $a_0 = 1$, $b_1 = i$, $c_2 = \frac{1}{4}$ and the rest equals zero.

Let us substitute the expression (3·49) of $\mathcal{D}^{(1)}(x, \theta)$ into the remaining equations (3.36) — (3.40) to determine the complex coefficients $a_0, b_0, b_1 \ldots$ of $\mathcal{D}^{(1)}(x, \theta)$. Upon simple computation we find that: in order that all these equations hold it is necessary that all coefficients in the expression (3.49) of $\mathcal{D}^{(1)}(x, \theta)$ are equal to zero.

Thus, there is no non-trivial operator $\mathcal{D}^{(1)}(x, \theta)$ satisfying the equation (3.32). Hence

COROLLARY 1. *There exists no superconformally covariant first order equation of the form (3.1) for the non-zero mass scalar superfield.*

Solving $\mathcal{D}^{(2)}(x, \theta)$

By a similar method, we find easily $\mathcal{D}^{(2)}(x, \theta)$ from the system of the equations (3.41) — (3.47).

We also expand $\mathcal{D}^{(2)}(x, \theta)$ for the parameters θ_α, $\bar{\theta}^{\dot{\alpha}}$ as (3.48).

Using conditions (3.41), (3.42), (3.45), we deduce that $\mathcal{D}^{(2)}(x, \theta)$ is a scalar operator, which contains x_μ only in the form $\frac{\partial}{\partial x_\mu}$ and has the scale dimension-2. Such an operator takes the most general form as follows

$$\mathcal{D}^{(2)}(x, \theta) = a_0 \left(\frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}}\right)^2 + a_2\, \partial_\mu \partial^\mu + b_0 \left(\bar{\theta} \frac{\partial}{\partial\bar{\theta}}\right)\left(\frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}}\right)^2 +$$

$$+ b_1 \left(\bar{\theta}\, \hat{\partial}\, \frac{\partial}{\partial\bar{\theta}}\right)\left(\frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}}\right) + b_2 \left(\bar{\theta} \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \partial^\mu + C_2(\bar{\theta}\theta)\left(\frac{\partial}{\partial\bar{\theta}} \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \partial^\mu +$$

$$+ e_2 (\bar{\theta} \gamma_5 \theta)\left(\frac{\partial}{\partial\bar{\theta}} \gamma_5 \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \varrho^\mu + g_2 (\bar{\theta} \gamma_\nu \gamma_5 \theta)\left(\frac{\partial}{\partial\bar{\theta}} \gamma^\nu \gamma_5 \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \partial^\mu +$$

$$+ h_2 (\bar{\theta}\theta)\left(\bar{\theta} \frac{\partial}{\partial\bar{\theta}}\right)\left(\frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \partial^\mu + h_3 (\bar{\theta}\theta)\left(\bar{\theta}\, \hat{\partial}\, \frac{\partial}{\partial\bar{\theta}}\right) \partial_\mu \partial^\mu +$$

$$+ k_2 (\bar{\theta}\theta)^2 \left(\frac{\partial}{\partial\theta} \frac{\partial}{\partial\bar{\theta}}\right)^2 \partial_\mu \partial^\mu + k_4 (\bar{\theta}\theta)^2 (\partial_\mu \partial^\mu)^2$$

All the terms in (3.50) are independent of each other, and the coefficients a_0, a_2, b_0 ... are determined in the same way as $\mathcal{D}^{(1)}(x, \theta)$. After substituting $\mathcal{D}^{(2)}(x, \theta)$ for (3.50) into the remaining equations (3.43), (3.44), (3.47) we deduce that every coefficient in the expression (3.50) of $\mathcal{D}^{(2)}(x, \theta)$ is equal to zero and consequently

COROLLARY 2. *There exists no superconformally covariant second order equation (3, 2) for the non-zero mass free scalar superfield.*

Thus, the superconformal symmetry of the non-zero mass free fields is merely too approximative. It is exact for only the massless free fields.

In a subsequent paper, we shall consider the superconformal covariance of the motion equations for the massless free fields, and derive conditions for these equations to be superconformally covariant.

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