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REGULARITY AND SUFFICIENT OPTIMALITY CONDITIONS FOR SOME CLASSES OF MATHEMATICAL PROGAMMING PROBLEMS

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§ 1. INTRODUCTION

Let f be a functional defined on a real Banach space X, C a non-empty closed convex subset of X, F a map from X into a real Banach space Y and K a closed convex cone in Y with vertex at the origin. Let us consider the mathematical programming problem

$$(\mathrm{I})\ \begin{cases} \text{minimize } f(x), \\ \text{subject to } F(x) \in K,\ x \in C. \end{cases}$$

Necessary optimality condition for Problem (I) is well-known (see [1]). Sufficient optimality conditions for Problem (I) have been given in [2], [3] [4], [5], [6]. In [2], the problem is solved under the assumption that $C = X$ and $K = \{ 0 \}$. The papers [3 — 6] deal with the case where $C = X$ or K is the non-positive orthant of a finite-dimensional space.

In this paper we shall be concerned with sufficient optimality conditions for Problem (I). The paper is organized as follows. In Section 2, using the generalized open mapping theorem in [1], we shall show that under the regularity assumption, introduced by Zowe and Kurcyusz [1], the feasible set of Problem (I) can be approximated by the linearizing cone. In Section 3, some first and second-order sufficient optimality conditions for Problem (I) will be obtained. Section 4 is devoted to the discussion of a case which was studied in [3], [4].

§ 2. AN APPROXIMATION PROPERTY FOR THE FEASIBLE SET OF PROBLEM (I)

The set of all feasible points for Problem (I) is denoted by M, i. e.,

$$M = C \cap F^{-1}(K)$$

For fixed $x \in X$ and $y \in Y$ let $C(x)$ and $K(y)$ denote the conical hull of $C-\{x\}$ and $K-\{y\}$, respectively, i. e.,

$$C(x) = \{ \lambda(c - x) \mid c \in C,\ \lambda \geqslant 0 \},$$
$$K(y) = \{ k - \lambda y \mid k \in K,\ \lambda \geqslant 0 \}.$$

Throughout this paper we assume that the maps f, F are continuously Fréchet differentiable at $\overline{x} \in M$.

We recall the regularity concept of Zowe and Kurcyusz [1]:

The feasible point $\overline{x}$ is called regular for Problem (I) if

$$F'(\overline{x})C(\overline{x}) - K(F(\overline{x})) = Y. \tag{2.1}$$

It was pointed out in [1] that (2.1) is equivalent to either of the following conditions:

$$0 \in \text{int}\,(F(\overline{x}) + F'(\overline{x})C(\overline{x}) - K), \tag{2.2}$$

$$0 \in \text{int}\,(F'(\overline{x})C(\overline{x}) - K(F(\overline{x}))). \tag{2.3}$$

It is worth noticing [1] that either of the following two conditions implies the regularity of $\overline{x}$;

(!) $\overline{x} \in \text{int}\, C$ and $F'(\overline{x})$ is surjective ;

(!!) There is some $\widehat{x} \in C(\overline{x})$ such that

$$F'(\overline{x})\widehat{x} \in \text{int}\; K(F(\overline{x})).$$

We recall the generalized open mapping theorem in [1] which is needed later.

THEOREM 2. 1 ([1]). *Let $\overline{x} \in C$, $\overline{y} \in K$ and T be a continuous linear operator between the Banach spaces X and Y. Then, the following statements are equivalent:*

(i) $Y = TC(\overline{x}) - K(\overline{y})$,

(ii) $B_Y(0, \rho) \subset T(C - \overline{x})_1 - (K - \overline{y})_1$, *for some* $\rho > 0$, *where*:

$$(C - \overline{x})_1 = (C - \{\overline{x}\}) \cap B_X(0, 1)\,,$$

$$(K - \overline{y})_1 = (K - \{\overline{y}\}) \cap B_Y(0, 1)\,,$$

and $B_Y(0, \rho)$ stands for the closed norm ball around zero with radius $\rho > 0$ in Y.

Denote by L the linearizing cone of M at $\overline{x}$, i. e. ,

$$L = \{x \in X \mid x \in C(\overline{x}) \text{ and } F'(\overline{x})\, x \in K(F(\overline{x}))\}. \tag{2.4}$$

DEFINITION 2. 1. *The feasible set M is said to be approximated at $\overline{x} \in M$ by L, if there exists a map $h : M \to L$ such that*

$$\| h(x) - (x - \overline{x}) \| = 0(\| x - \overline{x} \|) \text{ for } x \in M, \tag{2.5}$$

where

$$0(\| x - \overline{x} \|)/\| x - \overline{x} \| \to 0 \qquad (\text{when } \| x - \overline{x} \| \to 0).$$

THEOREM 2. 2 *Let $\overline{x}$ be a regular point of Problem (I). Then the feasible set M is approximated at $\overline{x}$ by L.*

Proof. Let $x \in M$. Since F is differentiable at $\overline{x}$, we get

$$F(x) - F(\overline{x}) = F'(\overline{x})(x - \overline{x}) + r(x, \overline{x}), \tag{2.6}$$

where

$$\| r(x, \overline{x}) \| = 0(\| x - \overline{x} \|).$$

Noting that $r(x, \overline{x}) \in \| r(x, \overline{x}) \| B_Y(0, 1)$, and applying Theorem 2. 1 to the map $F'(\overline{x})$ we can find a number $\rho > 0$ such that

$$\rho r(x, \overline{x}) \in F'(\overline{x})(\| r(x, \overline{x}) \| (C - \{\overline{x}\}) \cap B_X(0, 1)) - \\ - \| r(x, \overline{x}) \| (K - \{F(\overline{x}\}) \cap B_Y(0, 1). \tag{2.7}$$

From this it follows that there exist $\xi \in (C - \{\overline{x}\}) \cap B_X(0, 1)$ and $\eta \in (K - \{F(\overline{x})\} \cap B_Y(0, 1)$ such that

$$\rho r(x, \overline{x}) = F'(\overline{x})(\| r(x, \overline{x}) \| \xi) - \| r(x, \overline{x}) \| \eta. \tag{2.8}$$

For each $x \in M$ we put

$$h(x) = x - \overline{x} + \frac{\| r(x, \overline{x}) \|}{\rho} \xi. \tag{2.9}$$

Then, because $\| \xi \| \leqslant 1$ we have

$$\| h(x) - (x - \overline{x}) \| \leqslant \frac{\| r(x, \overline{x} \|}{\rho},$$

which implies that

$$\| h(x) - (x - \overline{x}) \| = 0(\| x - \overline{x} \|).$$

Now we show that h is a map from M into L, in other words, that for every $x \in M$, $h(x) \in L$, i. e., $h(x) \in C(\overline{x})$ and $F'(\overline{x})h(x) \in K(F(\overline{x}))$.

In view of (2.9), there is $\tilde{x} \in C$ such that

$$\begin{aligned} h(x) &= x - \overline{x} + \frac{\| r(x, \overline{x}) \|}{\rho}(\tilde{x} - \overline{x}) \\ &= x + \frac{\| r(x,\overline{x}) \|}{\rho}\tilde{x} - \left(1 + \frac{\| r(x,\overline{x}) \|}{\rho}\right)\overline{x} \\ &= \frac{\rho + \| r(x,\overline{x}) \|}{\rho}\left(\frac{\rho}{\rho + \| r(x,\overline{x}) \|} x + \frac{\| r(x,\overline{x}) \|}{\rho + \| r(x,\overline{x}) \|} \tilde{x} - \overline{x}\right) \end{aligned}$$

Since C is a convex set, it follows that $\frac{\rho}{\rho + \| r(x,\overline{x}) \|} x + \frac{\| r(x,\overline{x}) \|}{\rho + \| r(x,\overline{x}\|} \tilde{x} \in C$, hence $h(x) \in C(\overline{x})$.

Combining (2.6), (2.8) and (2.9) we get

$$\begin{aligned} F'(\overline{x}) h(x) &= F'(\overline{x})(x - \overline{x}) + F'(\overline{x})\left(\frac{\| r(x,\overline{x}) \|}{\rho} \xi\right) \\ &= F'(\overline{x})(x - \overline{x}) + r(x, \overline{x}) + \frac{\| r(x,\overline{x}) \|}{\rho} \eta \\ &= F(x) - F(\overline{x}) + \frac{\| r(x,\overline{x}) \|}{\rho} \eta. \end{aligned} \tag{2.10}$$

But, noting that $\frac{\| r(x,\bar{x}) \|}{\rho} \eta \in (K - \{F(\bar{x})\})$ we can write $\frac{\| r(x,\bar{x}) \|}{\rho} \eta =$
$= \eta_1 - F(\bar{x})$ for some $\eta_1 \in K$. (2.11)

Since $F(x) \in K$, substituting (2.11) into (2.10) yields:

$$F'(\bar{x})h(x) = F(x) + \eta_1 - 2F(\bar{x}) \in K - 2F(\bar{x}).$$

Therefore, $F'(\bar{x})h(x) \in K(F(\bar{x}))$, which completes the proof.

§ 3. FIRST AND SECOND-ORDER SUFFICIENT OPTIMALITY CONDITION FOR PROBLEM (I)

The Lagrangian of Problem (I) is defined to be:

$$\mathcal{L}(x, \Lambda) = f(x) - \langle \Lambda, F(x) \rangle.$$

A second-order sufficient optimality condition for Problem (I) can then be stated as follows

THEOREM 3.1. *Let $\bar{x}$ be a regular point of Problem (I). Supppose that the maps f, F are twice continuously Fréchet differentiable at $\bar{x}$ and*

a) There exists a Lagrange multiplier $\Lambda \in K^*$ such that

$$\mathcal{L}'_x(\bar{x},\Lambda) = f'(\bar{x}) - \Lambda^*.F'(\bar{x}) \in (C(\bar{x}))^*, \tag{3.1}$$

$$\langle \Lambda, F(\bar{x}) \rangle = 0, \tag{3.2}$$

b) There is a number $\sigma > 0$ such that

$$\mathcal{L}''_{xx}(\bar{x}, \Lambda)(\xi, \xi) \geqslant \sigma \, \| \xi \|^2 \text{ for all } \xi \in L. \tag{3.3}$$

Then $\bar{x}$ is a local solution of Problem (I).

Remark 3. 1. Theorem 3. 1 is more general than Theorem 2. 2 in [5] not only by the presence of the constraint $x \in C$, but also because the uniform positivity of the Lagrange multiplier Λ is not assumed.

Proof of Theorem 3. 1. Let x be an arbitrary feasible point of Problem (I), i. e. $x \in C$ and $F(x) \in K$ ($x \in M$).

We first observe that $x - \bar{x} \in C(\bar{x})$ and $\langle \Lambda, F(\bar{x}) \rangle \geqslant 0$. The twice differentiability of f and F yields

$$f(x) \geqslant f(x) - \langle \Lambda, F(x) \rangle = f(\bar{x}) - \langle \Lambda, F(\bar{x}) \rangle +$$

$$+ \mathcal{L}'_x(\bar{x}, \Lambda)(x - \bar{x}) + \frac{1}{2} \mathcal{L}''_{xx}(\bar{x}, \Lambda)(x - \bar{x}, x - \bar{x}) + r_1(x, \bar{x}), \tag{3.4}$$

where $|r_1(x - \bar{x})| = 0(\| x - \bar{x}\|^2)$.

It follows from Assumption a) and (3. 4) that

$$f(x) \geqslant f(\bar{x}) + \frac{1}{2} \mathcal{L}''_{xx}(\bar{x}, \Lambda)(x - \bar{x}, x - \bar{x}) + r_1(x, \bar{x}). \tag{3.5}$$

Moreover, according to Theorem 2. 2, for $x \in M$, $x - \bar{x}$ can be represented as a sum of two elements: $x - \bar{x} = x_1 + x_2$, where $x_1 \in L$, $\|x_2\| = 0(\|x-\bar{x}\|)$.

Since $\mathcal{L}''_{xx}(\bar{x}, \Lambda)(y, y)$ is a bilinear form, (3. 5) implies the existence of a number $\delta_1 > 0$ for $\varepsilon > 0$ such that for every $x \in M \cap B_X(x, \delta_1)$,

$$f(x) \geqslant f(\bar{x}) + \frac{1}{2} \mathcal{L}''_{xx}(\bar{x}, \Lambda)(x_1, x_1) + \mathcal{L}''_{xx}(\bar{x}, \Lambda)(x_1, x_2) + \\ + \frac{1}{2} \mathcal{L}''_{xx}(\bar{x}, \Lambda)(x_2, x_2) - \frac{\varepsilon}{2} \|x - \bar{x}\|^2. \tag{3.6}$$

We observe that for $\varepsilon > 0$, there exists $\delta > 0$ ($\delta \leqslant \delta_1$) such that for every $x \in M \cap B_X(\bar{x}, \delta)$,

$$\|x_2\| \leqslant \varepsilon \|x - \bar{x}\|, \tag{3.7}$$

hence,

$$\|x_1\| = \|x - \bar{x} - x_2\| \geqslant \|x - \bar{x}\| - \|x_2\| \geqslant (1 - \varepsilon)\|x - \bar{x}\| \tag{3.8}$$

Combining (3.7) and (3.8) we get

$$\|x_2\| \leqslant \frac{\varepsilon}{1-\varepsilon} \|x_1\| \tag{3.9}$$

It follows from (3.6) and Assumption b) that

$$f(x) \geqslant f(\bar{x}) + \frac{1}{2} \{ \sigma \|x_1\|^2 - 2 \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| \|x_1\| \|x_2\| - \\ - \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| \|x_2\|^2 - \varepsilon \|x - \bar{x}\|^2 \}. \tag{3.10}$$

From (3.8), (3.9) and (3.10) we see that for every $x \in M \cap B_X(\bar{x}, \delta)$,

$$f(x) \geqslant f(\bar{x}) + \frac{1}{2} \{ \sigma + \frac{2\varepsilon}{1-\varepsilon} \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| - \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| \} \|x_1\|^2 - \\ - \frac{\varepsilon}{2} \|x - \bar{x}\|^2$$

$$\geqslant f(\bar{x}) + \frac{(1-\varepsilon)^2}{2} \Big\{ \sigma - \frac{2\varepsilon}{1-\varepsilon} \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| \\ - \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| - \frac{\varepsilon}{(1-\varepsilon)^2} \Big\} \|x - \bar{x}\|^2 \tag{3.11}$$

If we choose $\varepsilon > 0$ so small that

$$\sigma - \frac{2\varepsilon}{1-\varepsilon} \|\mathcal{L}''_{xx}(\bar{x}, \Lambda)\| - \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 = \mathcal{L}''_{xx}(\bar{x}, \Lambda)\| - \frac{\varepsilon}{(1-\varepsilon)^2} > 0,$$

then, it follows from (3.11) that for every $x \in M \cap B_X(\bar{x}, \delta)$, $f(x) \geqslant f(\bar{x})$. The proof is thus complete.

Remak 3.2. From Theorem 3.1 we can derive a corollary which is stronger than Theorem 2.2 in [5] because it does not assume the uniform positivity of the Lagrange multiplier.

Now we establish some first-order **sufficient** optimality condition for Problem (I).

THEOREM 3.2. *Let $\bar{x}$ be a regular point of Problem (I). Assume, in addition, that there is a number $\beta > 0$ such that*

$$\langle f'(\bar{x}), x \rangle \geqslant \beta \| x \| \text{ for all } x \in L, \tag{3.12}$$

Then, there are $\alpha \in (0, \beta)$ and $\rho > 0$ such that for all $x \in M$ with $\| x - \bar{x} \| \leqslant \rho$:

$$f(x) \geqslant f(\bar{x}) + \alpha \| x - \bar{x} \|,$$

That is, $\bar{x}$ is a strictly local solution of Problem (I).

Proof. According to Theorem 2.2, any $x \in M$ can be represented as a sum $x = \bar{x} + x_1 + x_2$, where $x_1 \in L$ and $\| x_2 \| = 0(\| x - \bar{x} \|)$.

Because f is differentiable, we get

$$f(x) = f(\bar{x}) + \langle f'(\bar{x}), x - \bar{x} \rangle + r_2 (x, \bar{x}), \tag{3.13}$$

where

$$| r_2(x, \bar{x}) | = 0 (\| x - \bar{x} \|).$$

It follows from (3.12) that for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every $x \in M \cap B_X(\bar{x}, \delta_1)$,

$$\begin{aligned} f(x) &= f(\bar{x}) + < f'(\bar{x}), x_1 > + < f'(\bar{x}), x_2 > + r_2(x, \bar{x}) \\ &\geqslant f(\bar{x}) + \beta \| x_1 \| - \varepsilon \| f'(\bar{x}) \| \| x - \bar{x} \| - \varepsilon \| x - \bar{x} \|. \end{aligned} \tag{3.14}$$

On the other hand, arguing as in the proof of Theorem 3.1, we can find $\delta_2 > 0$ such that

$$\| x_1 \| \geqslant (1 - \varepsilon) \| x - \bar{x} \| \text{ for all } x \in M \cap B_X(\bar{x}, \delta_2). \tag{3.15}$$

Denote $\delta = \min \{\delta_1, \delta_2\}$. Then the inequalities (3.14) and (3.15) imply that for every $x \in M \cap B_X(\bar{x}, \delta)$,

$$f(x) \geqslant f(\bar{x}) + (\beta - \varepsilon (1 + \beta + \| f'(\bar{x}) \|)) \| x - \bar{x} \|. \tag{3.16}$$

Given any $\alpha \in (0, \beta)$, we can choose $\varepsilon < \dfrac{\beta - \alpha}{1 - \beta + \| f'(\bar{x}) \|}$ so that $\beta - \varepsilon(1 + \beta + \| f'(\bar{x}) \|) > \alpha$. Then, it follows from (3.16) that for every $x \in M \cap B_X(\bar{x}, \delta)$, $f(x) \geqslant f(\bar{x}) + \alpha \| x - \bar{x} \|$. The proof is thus complete.

§ 4. FIRST-ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR MATHEMATICAL PROGRAMMING

This section deals with the approximation property and first-order sufficient optimality conditions for the mathematical programming problem. A second-order sufficient condition for this problem can be found in [3]. It should be emphasized that in this section no regularity condition is assumed.

We shall use the Hoffman lemma (see [3]), as the main tool for deriving our result.

LEMMA 4.1 ([3]). *Let X, Y be Banach spaces, $\wedge$ a linear operator from X onto Y, i.e., $\wedge X = Y$, $x_1^*, \ldots, x_m^*$ elements of the conjugate space X^* and*

$$L_3 = \{x \in X \mid < x_i^*, x > \leqslant 0, \ i = 1, \ldots, m, \ \wedge x = 0\}.$$

Then

$$\rho(x, L_3) \leqslant C \{ \sum_{i=1}^{m} < x_i^*, x >_+ + \| \wedge x \| \},$$

where $\rho(.)$ stands for the distance function, the constant C is independent of x and

$$< x_i^*, x >_+ = \begin{cases} < x_i^*, x >, & \text{if } < x_i^*, x > \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that if L_3 is of the following form:

$$L_3' = \{ x \in X \mid < x_i^*, x > \leqslant 0, \ i \neq i_0, < x_{i_0}^*, x > 0, \ \wedge x = 0\}, \ (1 \leqslant i_0 \leqslant m),$$

then, we have

$$\rho(x, L_3') \leqslant C \{ \sum_{i \neq i_0} < x_i^*, x >_+ + | < x_{i_0}, x > | + \| \wedge x \| \}.$$

Consider the problem

$$\text{(II)} \begin{cases} \text{minimize } f_0(x), \\ \text{subject to } F(x) = 0, \\ \text{and } f_i(x) \leqslant 0 \ (i = 1, \ldots, m), \end{cases}$$

where $f_0, \ldots, f_m$ are functionals defined on X, $F : X \to Y$, X and Y are Banach spaces.

As in Section 3, denote by M_1 the feasible set of Problem (II). We consider a point $\bar{x} \in M_1$ such that $f_i(\bar{x}) = 0$ $(i = 1, \ldots, m)$. The linearizing cone of M_1 at x is denoted by L_2:

$$L_2 = \{ x \in X \mid < f_i'(\bar{x}), x > \leqslant 0 \ (i = 1, \ldots, m), \ F'(\bar{x}) x = 0 \}.$$

Before stating the first-order sufficient conditions we shall study an approximation property for the feasible set of Problem (II). It is known that M_1 can be approximated by L_2 at $\bar{x}$ if $\dim X < +\infty$ (see [6]). Here, we shall show how this fact can be extended to the infinite-dimensional case.

THEOREM 4.1. *Suppose that the maps $F, f_1, \ldots, f_m$ are Fréchet differentiable at $\bar{x}$ and $F'(\bar{x}) X = Y$. Then, the feasible set M_1 can be approximated at $\bar{x}$ by L_2.*

Proof. By virtue of Hoffman's lemma any feasible point x ($x \in M_1$) can be represented as the sum of $x_1 \in L_2$ and x_2 such that

$$\| x_2 \| \leqslant C \{ \sum_{i=l}^{m} < f_i'(\overline{x}), x >_+ + \| F'(\overline{x})(x - \overline{x}) \| \}. \qquad (4.1)$$

By setting then $h(x) = x_1$, we defined a map h from M_1 into L_2. We shall prove that (2. 5) is also satisfied.

In view of the differentiability of F and f_i $(i = 1, ..., m)$ at $\overline{x}$, we get

$$F(x) = F(\overline{x}) + F'(\overline{x})(x - \overline{x}) + r(x, \overline{x}), \qquad (4.2)$$

$$f_i(x) = f_i(\overline{x}) + < f_i'(\overline{x}), x - \overline{x} > + r_i(x, \overline{x}), \qquad (4.3)$$

where $\| r(x, \overline{x}) \| = 0(\| x - \overline{x} \|)$, $| r_i(x, \overline{x}) | = 0(\| x - \overline{x} \|)$.

Since $x \in M_1$, $F(\overline{x}) = 0$ and $f_i(\overline{x}) = 0$ $(i = 1, ..., m)$, it follows from (4. 2), (4. 3) that

$$\| F'(\overline{x})(x - \overline{x}) \| = 0(\| x - \overline{x} \|), \qquad (4.4)$$

$$< f_i'(\overline{x}), x - \overline{x} > \leqslant | r_i(x, \overline{x}) | . \qquad (4.5)$$

Consequently, substituting (4. 4) and (4. 5) into (4. 1) we get the relation

$$\| x_2 \| = 0(\| x - \overline{x} \|),$$

from which (2.5) follows. Therefore, M_1 is approximated at $\overline{x}$ by L_2.

A first-order sufficient condition for Problem (II) can be stated as follows

THEOREM 4.2. *Assume that the map F and the functionals f_i $(i = 0, 1, ..., m)$ are Fréchet differentiable at $\overline{x}$ and $F'(\overline{x}) X = Y$. Suppose, furthermore, that there is a number $\beta > 0$ such that*

$$< f_0'(\overline{x}), x > \geqslant \beta \| x \| \text{ for all } x \in L_2. \qquad (4.6)$$

Then, there $\alpha > 0$ and $\rho \geqslant 0$ such that for all $x \in M_1$ with $\| x - \overline{x} \| \leqslant \rho$,

$$f_0(x) \geqslant f_0(\overline{x}) + \alpha \| x - \overline{x} \|,$$

i. e., $\overline{x}$ is a strictly local minimum of Problem (II).

This theorem can be derived by using the same argument as that used in the proof of Theorem 3.2.

We now try to mitigate Condition (4.6) of Theorem 4.2 by replacing the cone L_2 by the following one:

$$L_2' = \ker F'(\overline{x}) \cap \bigcap_{\substack{i=1 \\ i \neq i_0}}^{m} \ker f_i'(\overline{x}) \cap \{x \mid < f_i'(\overline{x}), x > \leqslant 0\}, \ (1 \leqslant i_0 \leqslant m)$$

A modified first-order sufficient condition for Problem (II) can be formulated as follows.

THEOREM 3.4. *Assume that the map F and the functionals $f_0, f_1, \ldots, f_m$ are Fréchet differentiable at $\bar{x}$, and $F'(\bar{x}) X = Y$. Suppose, furthermore, that there exist Lagrange multipliers $y^* \in Y^*$, $\lambda_i > 0$ $(i = 1, \ldots, i_o - 1, i_o + 1, \ldots, m)$, $\lambda_{i_o} \geqslant 0$ $(1 \leqslant i_o \leqslant m)$ and a number $\beta > 0$ such that*

a) $L'_x(\bar{x}, \lambda_1, \ldots, \lambda_m, y^*) = 0$, *where*

$$L(x, \lambda_1, \ldots, \lambda_m, y^*) = f_o(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \langle y^*, F(x) \rangle,$$

b) $\langle f'_o(\bar{x}), x \rangle \geqslant \beta \| x \|$ *for all* $x \in L'_2$.

Then, $\bar{x}$ is a local minimum of Problem (II).

Proof. Let x be an arbitrary feasible point of Problem (II) ($x \in M_1$). In view of Theorem 4.1, there exists a map $h : M_1 \to L_2$ satisfying (2.5). Thus, x can be represented as $x = \bar{x} + x_1 + x_2$, where $x_1 = h(x) \in L_2$, and x_2 satisfies the following relation :

$$\| x_2 \| = \| h(x) - (x - \bar{x}) \| = 0(\| x - \bar{x} \|). \tag{4.7}$$

This means that for $\varepsilon > 0$ there exists $\rho_1 > 0$ such that for every $x \in M_1 \cap B(\bar{x}), \rho_1)$,

$$\| x_2 \| \leqslant \varepsilon \| x - \bar{x} \|. \tag{4.8}$$

Moreover, applying Lemma 4.1 to the cone L'_2, one can rewrite x_1 as $x_1 = x'_1 + x''_1$ where $x'_1 \in L'_2$ and x''_1 such that

$$\| x''_1 \| \leqslant C \{ \sum_{\substack{i=1 \\ i \neq i_0}}^{m} | \langle f'_i(\bar{x}), x_1 \rangle | + \langle f'_{i_0}(\bar{x}), x_1 \rangle + \}, (C > 0) \tag{4.9}$$

Since $x_1 \in L_2$, it follows from (4. 9) that

$$\| x''_1 \| \leqslant C \{ - \sum_{i \neq i_0} \langle f'_i(\bar{x}), x''_1 \rangle \}. \tag{4. 10}$$

We can choose a number $A > 0$ such that

$$\frac{A}{C} \min_{i \neq i_0} \lambda_i - \max_{1 \leqslant i \leqslant m} (\lambda_i \| f'_i(\bar{x}) \|) - 1 \geqslant 0. \tag{4.11}$$

Consider the following two cases :

a) $\| x_1 \| \leqslant A\varepsilon \| x - \bar{x} \|$.

For $x \in M_1$, we have $x - \bar{x} = x'_1 + x''_1 + x_2$. Hence,

$$f_0(x) = f_0(\bar{x}) + \langle f'_0(\bar{x}), x'_1 \rangle + \langle f'_0(\bar{x}), x''_1 \rangle + \langle f'_0(\bar{x}), x_2 \rangle + 0_1(\| x - \bar{x} \|), \tag{4.12}$$

where $0_1(\|x-\bar{x}\|)/_{\|x-\bar{x}\|} \to 0$ (as $\|x-\bar{x}\| \to 0$)

By virtue of (4.7), one gets

$$\langle f_0'(\bar{x}), x_2\rangle = 0_2(\|x-\bar{x}\|),$$

where $0_2(\|x-\bar{x}\|)/_{\|x-\bar{x}\|} \to 0$ (when $\|x-\bar{x}\| \to 0$)

Hence, there is a number $\rho_2 > 0$ such that for all $x \in M_1 \cap B(\bar{x}, \rho_2)$,

$$\left.\begin{aligned} &\|x_2\| \leqslant \frac{1}{2}\|x-\bar{x}\|, \\ &|\langle f_0'(\bar{x}), x_2\rangle + 0_1(\|x-\bar{x}\|)| \leqslant \frac{\beta}{4}\|x-\bar{x}\|, \end{aligned}\right\} \tag{4.13}$$

from which we get

$$\|x_1' + x_1''\| = \|x-\bar{x}-x_2\| \geqslant \frac{1}{2}\|x-\bar{x}\|.$$

Therefore,

$$\|x_1'\| \geqslant \|x_1'+x_1''\| - \|x_1''\| \geqslant \frac{1}{2}\|x-\bar{x}\| - A\varepsilon\|x-\bar{x}\|. \tag{4.14}$$

Moreover,

$$|\langle f_0'(\bar{x}), x_1''\rangle| \leqslant A\varepsilon\|f_0'(\bar{x})\|\,\|x-\bar{x}\|. \tag{4.15}$$

Taking account of Assumption b), one can see from (4.12) – (4.15) that

$$f_0(x) - f_0(\bar{x}) \geqslant \beta\|x_1'\| + \langle f_0'(\bar{x}), x_1''\rangle + \langle f_0'(\bar{x}), x_2\rangle + 0_1(\|x-\bar{x}\|)$$

$$\geqslant \frac{\beta}{2}\|x-\bar{x}\| - A\beta\varepsilon\|x-\bar{x}\| - A\varepsilon\|f_0'(\bar{x})\|\,\|x-\bar{x}\| - \frac{\beta}{4}\|x-\bar{x}\|$$

$$= \left(\frac{\beta}{4} - \varepsilon\beta A - \varepsilon A\|f_0'(\bar{x})\|\right)\|x-\bar{x}\|. \tag{4.16}$$

We can choose $\varepsilon > 0$ $\left(\varepsilon \leqslant \frac{1}{2}\right)$ so small that

$$\frac{\beta}{4} - \varepsilon\beta A - \varepsilon A\|f_0'(\bar{x})\| > 0.$$

Consequently, taking $\rho = \min\{\rho_1, \rho_2\}$, we have

$$f_0(x) - f_0(\bar{x}) \geqslant 0 \text{ for every } x \in M_1 \cap B(\bar{x}, \rho).$$

b) $\|x_1''\| > A\varepsilon\|x-\bar{x}\|$.

By (4. 10) we get

$$A\varepsilon\|x-\bar{x}\| < \|x_1''\| \leqslant C\{-\sum_{i\neq i_0}\langle f_i'(\bar{x}), x_1''\rangle\}. \tag{4. 17}$$

It follows from (4.11), (4.17) that there is a number $\rho_3 > 0$ ($\rho_3 \leqslant \rho_1$) such that for every $x \in M_1 \cap B(\overline{x}, \rho_3)$,

$$f(x) = \mathcal{L}(x, \lambda_1, \ldots, \lambda_m, y^*) - \sum_{i=1}^{m} \lambda_i f_i(x) = \mathcal{L}(\overline{x}, .) + < \mathcal{L}'_x(\overline{x}, .), x - \overline{x} >$$

$$- \sum_{i=1}^{m} \lambda_i f_i(\overline{x}) - \sum_{i=1}^{m} \lambda_i < f'_i(\overline{x}), x - \overline{x} > + 0_3(\| x - \overline{x} \|)$$

$$= f_o(\overline{x}) - \sum_{i=1}^{m} \lambda_i < f'_i(\overline{x}), x - \overline{x} > + 0_3(\| x - \overline{x} \|)$$

$$\geqslant f_o(\overline{x}) - \sum_{i \neq i_o} \lambda_i < f'_i(\overline{x}), x_1 > - \sum_{i=1}^{m} \lambda_i < f'_i(\overline{x}), x_2 > + 0_3(\| x - \overline{x} \|)$$

$$\geqslant f_o(\overline{x}) + \frac{A\varepsilon}{C} \min_{i \neq i_o} \lambda_i \| x - \overline{x} \| - \varepsilon \max_i (\lambda_i \| f'_i(\overline{x}) \|) \| x - \overline{x} \| - \varepsilon \| x - \overline{x} \|$$

$$= f_o(\overline{x}) + \varepsilon \| x - \overline{x} \| \left(\frac{A}{C} \min_{i \neq i_o} \lambda_i - \max_i (\lambda_i \| f'_i(\overline{x}) \|) - 1 \right) \geqslant f_o(\overline{x}).$$

The proof is thus complete.

By the same argument as that used in the proof of Theorem 4.3, we obtain

THEOREM 4.4. *Assume that $F, f_o, f_1, \ldots, f_m$ are Fréchet differentiable at $\overline{x}$ and $F'(\overline{x}) X = Y$. Suppose, furthermore, that there exists Lagrange multipliers $y^* \in Y^*$, $\lambda_i \geqslant 0$, $\lambda_j > 0$ ($i = 1, \ldots, m_1$; $j = m_1 + 1, \ldots, m$) and a number $\beta > 0$ such that*

a) $\mathcal{L}'_x(\overline{x}, \lambda_1, \ldots, \lambda_m, y^*) = 0$,

b) $< f'_o(\overline{x}), x > \geqslant \beta \| x \|$ *for all* $x \in L$, *where*

$$L = \{ x \in X \mid < f'_i(\overline{x}), x > \leqslant 0, < f'_j(\overline{x}), x > = 0, F'(\overline{x}) x = 0, \\ i = 1, \ldots, m_1 ; j = m_1 + 1, \ldots, m \}, \ (1 \leqslant m_1 \leqslant m).$$

Then, $\overline{x}$ is a local minimum of Problem (II).

REFERENCES

[1] J. Zowe and S. Kurcyusz, *Regularity and stability for the mathematical programming problem in Banach spaces*, Appl. Math. Optim. 5(1979), 49-62.

[2] A.D. Ioffe and V.M. Tikhomirov, *The theory of extremum problems*, Moscow 1974 (in Russian).

[3] V.M. Alekseev, V.M. Tikhomirov, and S.V. Fomin, *Optimal control*, Moscow 1979 (in Russian).

[4] E.S. Levitin, A.A. Milyutin, and N.P. Osmolovskii, *On necessary and sufficient conditions for local minimum in the presence of constraints*, DAN SSSR, Vol. 210 (1973), N^{0}5, 1022-1025 (in Russian).

[5] D.V. Lưu, *Necessary and sufficient conditions for some general optimization problems*, Acta Math. Vietnamica, 7(1982), 37-57.

[6] H. Maurer and J. Zowe, *First and second-order necessary and sufficient optimality conditions for infinite dimensional programming problems*, Math. Programming 16 (1979), 98-110,

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