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APPROXIMATE CONTROLLABILITY OF NONLINEAR DISCRETE SYSTEMS IN BANACH SPACES

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In this paper, using essentially the investigation method of stability theory [1—3], we present new sufficient conditions for the global approximate controllability of nonlinear discrete-time systems with rather general constrained controls in infinite-dimensional spaces. We shall also point out a class of nonlinear discrete-time systems which are not globally approximately controllable.

The results of this note extend the corresponding results in [4, 5].

Consider the following nonlinear discrete-time control system

$$x_{k+1} = Ax_k + f_k(x_k, u_k),\ k = 0, 1, 2,\dots \tag{1}$$

where $x_k \in X$, $u_k \in \Omega \subset U$; X, U are infinite-dimensional Banach spaces; Ω is a given nonempty subset of U; $A : X \to X$, $f_k : X \times U \to X$, $k = 0, 1,\dots$ are linear and nonlinear operators.

Throughout this paper, the resolvent of A, the spectrum of A and the set of eigenvalues of A are denoted by $R(A)$, $\sigma(A)$ and $\sigma_T(A)$, respectively. The open ball of radius ε centered at x is denoted by $B_\varepsilon(x)$.

DEFINITION 1. *The system (1) is said to be globally ε-controllable if, for some $\varepsilon > 0$ and for every $x \in X$, there exist a positive integer N and controls $u_k \in \Omega$, $k = 0, 1,\dots, N-1$ such that the corresponding solution x_k, $k = 0, 1,\dots, N$, of (1) satisfies $x_0 = x$, $x_N \in B_\varepsilon(0)$.*

DEFINITION 2. *A set Ω is called radially convex if for every $u \in \Omega$, $\lambda \in (0,1]$, $\lambda u \in \Omega$.*

Clearly any convex set containing 0 is radially convex, but not every radially convex set is convex.

LEMMA 1 [3]. *Assume that for every* $c > 0$, $m > 1$, $a_k \geqslant 0$, $y_k \geqslant 0$, $0 \leqslant y_0 \leqslant c$

$$y_n \leqslant c + \sum_{k=0}^{n-1} a_k y_k^m,$$

and

$$(m-1)c^{m-1} \sum_{k=0}^{n-1} a_k < 1.$$

Then

$$y_n \leqslant c\,[1-(m-1)\, c^{m-1} \sum_{k=0}^{n-1} a_k]^{-\frac{1}{m-1}}.$$

LEMMA 2. *Let* $A : X \to X$ *be a linear bounded operator. Assume that* $\sigma(A) \subseteqq \{z : |z| \leqslant q < 1\}$ *for some* $q > 0$. *Then there exist numbers* $\alpha > 0$, $M > 0$ *such that*

$$\| A^n \| \leqslant M \exp(-\alpha n),\ n = 1, 2, \ldots \tag{2}$$

Proof. Since A is a linear bounded operator in a Banach space, it follows that the operator A has the following spectral expansion (see e. g. [2])

$$A^n = -\frac{1}{2\pi i} \int_\Gamma \lambda^n R_\lambda(A) d\lambda,$$

where

$$\Gamma = \{ z : |z| = q \}.$$

Therefore, by taking

$$\alpha = -\ln q,$$
$$M = \max_{\lambda \in \Gamma} \| R_\lambda(A) \|,$$

we obtain (2).

THEOREM 1. *Let* A *be a linear bounded operator,* $\{a_k\}$ *be a sequence of nonnegative numbers convergent to zero,* Ω *be a radially convex subset of* U. *Assume that*

$$\sigma(A) \subseteqq \{ z : |z| \leqslant q < 1\},$$

for some $q > 0$ *and, moreover, that*

i) $\sum_{k=0}^{\infty} e^{\beta k} \| f_k(x_k, 0) \| < +\infty$, $f_k(0, 0) = 0$ (3)

ii) $\| f_k(x, u) \| \leqslant a_k \| x \|^m + b \| u \|^c$, $\forall x \in X, u \in \Omega$, (4)

where $\beta > 0$, $b > 0$, $c > 0$, $m > 1$ *and* $a_k \to 0$ as $k \to +\infty$. *Then the system* (1) *is globally* ε*-controllable.*

Proof. Let x be an arbitrary element of X. For every control $u = (u_0, u_1, \ldots, u_{n-1})$ the solution of (1) with $x_0 = x$ is given by

$$x_n = A^n x + \sum_{k=0}^{n-1} A^{n-k-1} f_k(x_k, u_k).$$

Define

$$\alpha = -\ln q,$$
$$M = \max_{\lambda \in \Gamma} \| R_\lambda \|,$$
$$\gamma = \min (\alpha, \beta),$$

where $\beta > 0$ is defined by (3).

Furthermore, set

$M_1 = \max \{1, M\}$,

$c_1 = \sum_{k=0}^{\infty} M_1 e^{\gamma(k+1)} \| f_k(x_k, 0) \|$,

$\delta = \gamma (m - 1)$,

$h = (1 - e^{-\delta})/2(m-1) M_1 e^{\gamma}(M_1 x_0 + c_1)^{m-1}$.

We have

$$\| x_n \| e^{\gamma n} \leqslant M_1 \| x \| + \sum_{k=0}^{n-1} M_1 e^{\gamma(k+1)} \| f_k(x_k, u_k) \| . \tag{5}$$

Let $p \in (0, h)$ be an arbitrary given number and N be a positive integer such that $a_k < p$ for all $k \in N$. From (5) it follows that

$$=x_n \| e^{\gamma n} \leqslant M_1 \| x \| + \sum_{k=0}^{N-1} M_1 e^{\gamma(k+1)} \| f_k (x_k, u_k) \|$$
$$+ \sum_{k=N}^{n-1} M_1 e^{\gamma(k+1)} \| f_k(x_k, u_k) \| .$$

In view of (4) we have

$$\| x_n \| e^{\gamma n} \leqslant M_1 \| x \| + \sum_{k=0}^{N-1} M_1 e^{\gamma(k+1)} \| f_k (x_k, u_k) \|$$
$$+ \sum_{k=N}^{n-1} M_1 e^{\gamma(k+1)} b \| u_k \| c + \sum_{k=N}^{n-1} M_1 e^{\gamma - \delta k} p(\| x_k \| e^{\gamma k})^m .$$

Pick $\eta > \gamma$ so large that for $k \geqslant N$,

$$e^{-\eta k} \| x \| / M e^{\gamma} p < 1.$$

Define the following positive numbers

$$c_2 = \left(\frac{1 - e^{-\delta}}{2(m-1) M_1 e^{\gamma} p} \right)^{\frac{1}{m-1}} - c_1,$$
$$\sigma = \eta - \gamma,$$
$$q = \left(\frac{c_2}{\| x \|} - M_1 \right) \left(1 - e^{-\sigma} \right).$$

Let $\hat{u} \in \Omega$ be an arbitrary control such that

$$0 < \hat{u} < q^{\frac{1}{c}} .$$

Set

$$u_k = \begin{cases} 0 & , \text{ for } k = 0, 1, \ldots, N-1, \\ \left(\dfrac{e^{-\eta k} x}{M e^{\gamma} p}\right)^{\frac{1}{c}} \widehat{u}, & \text{for } k \geqslant N. \end{cases}$$

It is easily seen that $c_2 > M_1 \| x \|$ and

$$(m-1) c_3^{m-1} M_1 e^{\gamma} p (1 - e^{-\delta})^{-1} < 1/2, \tag{6}$$

where

$$c_3 = (M_1 + \| \widehat{u} \|^{c} (1 - e^{-\delta})^{-1}) \| x \| + c_1.$$

Therefore

$$\| x_n \| e^{\gamma n} \leqslant M_1 x + c_1 + \frac{\| \widehat{u} \|^{c} \| x \|}{1 - e^{-\delta}} + \sum_{k=0}^{n-1} M e^{\gamma - \delta k} p (\| x_k \| e^{\gamma k})^m$$

Setting

$$z_n = \| x_n \| e^{\gamma n},$$

$$q_k = M_1 e^{\gamma - \delta k} p,$$

yields

$$z_n \leqslant c_3 + \sum_{k=0}^{n-1} q_k z_k^m .$$

Now, using Lemma 1 we have

$$z_n \leqslant c_3 (1 - (m-1) c_3^{m-1} M_1 p e^{\gamma} (1 - e^{-\delta})^{-1})^{-\frac{1}{m-1}} .$$

Taking (6) into account, we get

$$z_n \leqslant 2^{m-1} c_3 .$$

Then

$$\| x_n \| \leqslant 2^{m-1} c_3 e^{-\gamma n} .$$

On the other hand for every $\varepsilon > 0$ there exists a number $N_1 > N$ such that

$$e^{-\gamma n} < \varepsilon / 2^{m-1} c_3 .$$

Consequently,

$$\| x_n \| < \varepsilon \qquad \text{for all } n \geqslant N_1$$

The proof of Theorem 1 is complete.

The following theorem on the local ε-controllability of the system (1) can be proved by an analogous argument.

THEOREM 2. *Let $A : X \to X$ be a linear bounded operator and*

$$\delta(A) \subset \{ z : \| z \| \leqslant q < 1 \}.$$

Let $\{ a_k \}$ be a monotone bounded sequence of nonnegative numbers satisfying

$$\| f_k(x, 0) \| \leqslant a_k \| x \|^m, \; m > 1, \, x \in X.$$

Then there is a neighbourhood of zero $V \subset X$ such that the system (1) si globally ε-controllable in V.

On the other hand we can state:

THEOREM 4. *Let $A : X \to X$ be a linear bounded operator. Assume that*

$$\delta_T(A) \cap \{z : |z| > 1\} \neq \varnothing, \tag{7}$$

and

$$\sup_{u \in \Omega} \| f_k(x,u) \| \leqslant a_k \| x \|^{\alpha},$$

where

$$\alpha > 0,\ a_k \geqslant 0 : \sum_{k=o}^{\infty} a_k < +\infty.$$

Then the system (1) is not globally ε-controllable.

Proof. We shall prove that if the conditions of the Theorem are satisfied, thenthere exists a point $x \in X$ such that the solution x_k, $k = 0, 1, \ldots$ of the system

$$x_{k+1} = Ax_k + f_k(x_k)$$

with $x_o = x$, converges to infinity.

Indeed, let

$$\lambda \in \delta_T(A) \cap \{ z : |z| > 1 \},$$

and x be an eigenvector of A corresponding to λ. We shall prove that the solution x_n, $n = 0, 1, \ldots$ of the above system with $x_o = 0$ converges to infinity. Assume the contrary, i. e., there is a number $M > 0$ such that for any positive number N

$$\| x_n \| < M, \text{ for some } n > N.$$

Since

$$x_n = \lambda^n x + \sum_{k=0}^{n-1} f_k(x_k),$$

we have

$$\| x_n \| \geqslant |\lambda|^n \| x \| - \sum_{k=0}^{n-1} \| f_k(x_k) \|$$

$$\geqslant |\lambda|^n \| x \| - M^{\alpha} \sum_{k=N}^{n-1} a_k - \sum_{k=0}^{N-1} a_k \| x_k \|^{\alpha}.$$

Setting

$$h = \sum_{k=0}^{N-1} a_k \| x_k \|^{\alpha},$$

yields

$$M > \| x_n \| \geqslant |\lambda|^n \| x \| - h - M^{\alpha} \sum_{k=N}^{n-1} a_k.$$

Consequently,

$$M\ (1 + M^{\alpha-1} \sum_{k=N}^{n-1} a_k) > |\lambda|^n \|x\| - h. \tag{8}$$

Since $|\lambda| > 1$, $\sum_{k=0}^{\infty} a_k < +\infty$ this leads to a contradiction when $n \to \infty$. This proves the theorem.

For example, if

$$f_k(x, u) = e^{-\alpha k} \sin^2 x \cos u,$$

where $\alpha > 0$, $u \in \left[0, \frac{\pi}{2}\right)$, the system (1) is globally ε-controllable if

$$\sigma_T(A) \subseteq \{ z : |z| \leqslant q < 1\},$$

but is not globally ε-controllable if

$$\sigma_T(A) \cap \{ z : |z| > 1\} \neq \phi.$$

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