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A STOCHASTIC TAYLOR FORMULA FOR FUNCTIONALS OF TWO-PARAMETER SEMIMARTINGALES

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1. INTRODUCTION

The aim of this note is to prove the stochastic Taylor formula for functionals of two-parameter semimartingales. This result extends those of W. Wagner and E. Platen for the one-parameter case.

Throughout the paper, for all unexplained notations and terminology we refer to [1] and [5].

Let $R_+^2 = [0, +\infty) \times (0, +\infty)$ with the natural ordering. For two points $s = (s_1, s_2)$, $t = (t_1, t_2)$, we shall denote by $s \wedge t$ the condition $s_1 \leqslant t_1$ and $t_2 \leqslant s_2$, and by $s \vee t$ the point $(\max(s_1, t_1), \max(s_2, t_2))$. The indicator function of the set $\{(s, t) \in R_+^2 \times R_+^2 : s \wedge t\}$ is denoted by $I(s \wedge t)$. For $z \in R_+^2$, let $R_z = \{s \in R_+^2 : 0 \leqslant s \leqslant z\}$ and let $R_z \otimes R_z = \{(s, t) \in R_z \times R_z : s \wedge t\}$. The Lebesgue measure on R_+^2 is denoted by μ.

Let $W = \{W(z) : z \in R_+^2\}$ be the two-parameter Wiener process defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_z : z \in R_+^2\}$ where $\mathcal{F}_z$ is generated by the functions $\{W_s\ s \in R_z\}$ and the null sets of $\mathcal{F}$.

For $1 \leqslant p \leqslant +\infty$, denote by L_1^p (resp. $\overline{L}_1^p$) the collection of measurable functions $q(s, \omega)$ on $(R_{z_0} \times \Omega, B(R_{z_0}) \otimes \mathcal{F})$ (where B denotes Borel subsets) such that $q(s)$ is $\mathcal{F}_s$-measurable for each s and $\int_{R_{z_0}} |q(s. \omega)|^p ds < +\infty$ a.s. (resp. $\int_{R_{z_0}} E|q(s)|^p ds < \infty$) iff $p < +\infty$ and $\sup_s |q(s)| < +\infty$ a.s. (resp. $\operatorname{ess\,sup}_w \sup_s |q(s, w)| < +\infty$) if $p = +\infty$. Define L_2^p (resp. $\overline{L}_2^p$) to be the collection of measurable functions $r(s,s',\omega)$ on $(R_{z_0}\times R_{z_0}\times\Omega, B(R_{z_0})\otimes B(R_{z_0})\otimes\mathcal{F})$ such that $r(s,s')$ is $\mathcal{F}_{s\vee s'}$-measurable for each $s, s'\in R_{z_0}$ and $\int_{R_{z_0}\otimes R_{z_0}} |r(s,s')|^p\,ds\,ds' < +\infty$ a.s. (resp. $\int_{R_{z_0}\otimes R_{z_0}} E|r(s,s')|^p\,ds\,ds' < +\infty$) if $p<+\infty$ and $\sup_{s,s'} |r(s,s')| < +\infty$ a.s. (resp. $\operatorname{esssup}_{w} \sup_{s,s'} |r(s,s')| < +\infty$) if $p=+\infty$.

For $2\leqslant p\leqslant+\infty$ let $\mathcal{S}^p$ (resp. $\overline{\mathcal{S}}^p$) be the linear space of processes of the form

$$X_z=\int_{R_z} q(s)\,dW(s)+\int_{R_z\otimes R_z} r(s,s')\,dW(s)\,dW(s')+\int_{R_z\otimes R_z}\alpha(s,s')\,ds\,dW(s')$$

$$+\int_{R_z\otimes R_z}\beta(s,s')\,dW(s)\,ds'+\int_{R_z} b(s)\,ds, \tag{1.1}$$

or, briefly,

$$X=q\,.\,W+W\,.\,r\,.\,W+\mu\,.\,\alpha\,.\,W+W\,.\,\beta\,.\,\mu+b\,.\,\mu, \tag{1.1'}$$

where b, $q\in L_1^p$ and $r,\alpha,\beta\in L_2^p$ (resp. b, $q\in\overline{L}_1^p$ and $r,\alpha,\beta\in\overline{L}_2^p$). In the sequel, the processes in $\mathcal{S}^2$ will be called (two-parameter) semi-martingales (cf. [3, 5].)

The equation (1 . 1) can be rewritten as

$$X_z=\int_{R_z} X_{W_1}(z,s')\,dW_{s'}+\int_{R_z} X_{\mu_1}(z,s')\,ds', \tag{1.2}$$

$$X_z=\int_{R_z} X_{W_2}(z,s)\,dW_s+\int_{R_z} X_{\mu_2}(z,s)\,ds, \tag{1.3}$$

where

$$X_{W_1}(z,s')=q(s')+\int_{R_z} I(s\wedge s')\,r(s,s')\,dW_s+\int_{R_z} I(s\wedge s')\,\alpha(s,s')\,ds, \tag{1.4}$$

$$X_{\mu_1}(z,s')=b(s')+\int_{R_z} I(s\wedge s')\,\beta(s,s')\,dW_s, \tag{1.5}$$

$$X_{W_2}(z,s)=q(s)+\int_{R_z} I(s\wedge s')\,r(s,s')\,dW_{s'}+\int_{R_z} I(s\wedge s')\,\alpha(s,s')\,ds', \tag{1.6}$$

$$X_{\mu_2}(z,s)=b(s)+\int_{R_z} I(s\wedge s')\,\beta(s,s')\,dW_{s'}. \tag{1.7}$$

Let X be a semi-martingale with representation (1. 1') and X be the semi-martingale

$$\tilde{X}=\tilde{q}\,.\,W+W\,.\,\tilde{r}\,.\,W+W\,.\,\tilde{\beta}\,.\,\mu+\tilde{b}\,.\,\mu\,.$$

Following B. Hajek (cf. [5]) we put

$$[X,\tilde{X}]=(q\tilde{q})\,.\,\mu+\mu\,.\,(r\tilde{r})\,.\,\mu, \tag{1.8}$$

$$\langle X, \tilde{X}\rangle_1(z) = \int_{R_Z} X_{W_1}(z, s')\tilde{X}_{W_1}(z, s')\, ds', \tag{1.9}$$

$$\langle X, \tilde{X}\rangle_2(z) = \int_{R_Z} X_{W_2}(z, s)\, \tilde{X}_{W_2}(z, s) ds, \tag{1.10}$$

$$\begin{aligned} X_*\tilde{X} = {} & W.(X_{W_2}(svs', s)\,\tilde{X}_{W_1}(svs', s')).\,W \\ & + \mu.(X_{\mu_2}(svs', s)\,\tilde{X}_{W_1}(svs', s)).\,W \\ & + W.(X_{W_2}(svs', s)\,\tilde{X}_{\mu_1}(svs', s')).\mu \\ & + \mu.(X_{\mu_2}(svs', s)\,\tilde{X}_{\mu_1}(svs', s')).\,\mu, \end{aligned} \tag{1.11}$$

$$\begin{aligned} \psi . X = {} & (q\psi).W + W.(r(s, s')\,\psi(svs')).W \\ & + \mu.(\alpha(s, s')\,\psi(svs')).W \\ & + W.(\beta(s, s')\,\psi(svs')).\mu \\ & + (b\psi).\mu, \end{aligned} \tag{1.12}$$

where $\psi \in L_1^p$.

PROPOSITION 1.1 (cf. [5]). *Let $2 \leqslant r, s, t < +\infty$ be such that $1/r + 1/s = 1/t$ and suppose $X \in \mathcal{S}^r$, $\tilde{X} \in \mathcal{S}^s$ and $\psi \in L_1^s$. Then $[X, \tilde{X}]$, $\langle X, \tilde{X}\rangle_1$, $\langle X, \tilde{X}\rangle_2$, $X_*\tilde{X}$ and $\psi.X$ are well-defined semi-martingales in $\mathcal{S}^t$. If $s = +\infty$ (so $2 \leqslant r = t < +\infty$) then X is still a well-defined semi-martingale in $\mathcal{S}^t$.*

Given $X \in \mathcal{S}^{4p}$ $(p \geqslant 2)$ we set

$$\begin{aligned} S^1(X) &= X, \\ S^2(X) &= X_* X, \\ S^3(X) &= (\langle M, X\rangle_1 + \langle X, X\rangle_2 - [X, X])/2, \\ S^4(X) &= (X_*\langle X, X\rangle_1 + \langle X, X\rangle_{2\,*} X + 2[X, X_* X])/2 \\ S^5(X) &= (\langle X, X\rangle_{2\,*}\langle X, X\rangle_1)/4. \end{aligned} \tag{1.13}$$

It follows from Proposition 1.1 that $S^i(X) \in \mathcal{S}^p$ for each $i = 1, \dots, 5$.

Let $B = \{1, 2, \dots, 5\}$ and $A = (\bigcup^{\infty} B^k) \cup \{\varnothing\}$, B^k being the k-fold Cartesian product of the set B. Further, let φ be the function from the set B into N such that $\varphi(1) = 1$. $\varphi(2) = \varphi(3) = 2$, $\varphi(4) = 3$, $\varphi(5) = 4$. Given $\alpha \in A$ set

$$\|\alpha\| = \begin{cases} 0 & \text{if } \alpha = \varnothing, \\ \sum_{i=1}^{k} \varphi(\alpha_i) & \text{if } \alpha = (\alpha_1, \dots, \alpha_k),\ k \geqslant 1, \end{cases}$$

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = \varnothing \\ k & \text{if } \alpha \in B^k,\ k \geqslant 1 \end{cases}$$

and if $\alpha = (\alpha_1, \dots, \alpha_k)$ $k \geqslant 1$,

$$\alpha- = \begin{cases} \varnothing & \text{if } k = 1, \\ (\alpha_1, \dots, \alpha_{k-1}) & \text{if } k \geqslant 2, \end{cases}$$

$$-\alpha = \begin{cases} \varnothing & \text{if } k = 1, \\ (\alpha_2, \dots, \alpha_k) & \text{if } k \geqslant 2, \end{cases}$$

suppose $h \in L_1^{\infty}$ and $X \in \mathcal{S}^{4p}$. Define

$$J_i(h, z) = (h \,.\, S^i(X))_z \text{ for } i = 1, \dots, 5.$$

If $\alpha \in A$ is a multi-index we define a multiple stochastic integral J_α recursively as follows

$$J_\alpha(h, z) = \begin{cases} h(z) & \text{if } \alpha = \varnothing, \\ J_{\alpha_1}(J_{-\alpha}(h, z)) & \text{if } \alpha = (\alpha_1, \dots, \alpha_k),\ k \geqslant 1. \end{cases} \tag{1.14}$$

From Proposition 1.1 it follows that for $X \in \mathcal{S}^{4p}$, $\alpha \in A \setminus \{\varnothing\}$ and a sample continuous and adapted stochastic function h

$$J_\alpha(h) \in \mathcal{S}^p.$$

Using multiple stochastic integrals for semi-martingales $S^1(X), \dots, S^5(X)$ and the Ito formula (cf. [2, 4, 5]) we shall derive the stochastic Taylor formula for functionals of a two-parameter semi-martingale X. This result extends those of W. Wagner and E. Platen for the one-parameter case (cf. [6]) and of the author for two-parameter martingales. Eurthermore, we also obtain the estimation of errors in the Taylor formula which determines the rate of the convergence in L^2.

2. STOCHASTIC TAYLOR EORMULA

THEOREM 2. 1. *Let $p \geqslant 2$ and suppose that $X \in \mathcal{S}^{4p}$ and $F \in C^{4n}(R)$ $(n \geqslant 1)$ are given. Then $F(X) \in \mathcal{S}^p$ and*

$$F(X_z) = \sum_{0 \leqslant |\alpha| < n} J_\alpha(F_\alpha(X_0), Z) + \sum_{|\alpha| = n} J_\alpha(F_\alpha(X), Z) \tag{2.1}$$

where $F_\alpha(x) = D^{\|\alpha\|} F(x)$.

Proof. Observe that for each $\alpha \in \bigcup_{k=l}^{n} B^k$, $S^i(X)$ and $F_\alpha(X)$ are sample continuous.

This fact and Proposition 1. 1 show that the right-hand side of (2.1) is in $\mathcal{S}_p$. We prove the formula (2. 1) by induction with respect to n. For $n=1$, it follows from the Wong-Zakai-Ito differentiation formula (cf. [5], Theorem 2. 3) that

$$
\begin{aligned}
F(X) &= F(X_0) + F_1(X) . X + F_2(X) . (X * X) + \\
&+ \frac{1}{2} F_2(X) . (\langle X, X\rangle_1 + \langle X, X\rangle_2 - [X, X]) \\
&+ \frac{1}{2} F_3(X) . (X * \langle X, X\rangle_1 + \langle X, X\rangle_2 * X + 2 [X, X * X]) \\
&+ \frac{1}{4} F_4(X) . (\langle X, X\rangle_2 * \langle X, X\rangle_1) \\
&= F(X_0) + \sum_{j=1}^{5} F_j(X) . S^j(X) \\
&= F(X_0) + \sum_{|\alpha|=1} J_\alpha(F_\alpha(X)).
\end{aligned} \tag{2. 2}
$$

Thus, (2. 1) holds for $n = 1$.

Suppose now that (2. 1) is true for $n-1$. Then we have

$$F(X_z) = F(X_0) + \sum_{0<|\alpha|<n-1} J_\alpha(F_\alpha(X_0), Z) + \sum_{|\alpha|=n-1} J_\alpha(F_\alpha(X), Z) \tag{2. 3}$$

Applying (2. 2) for each $F_\alpha(X)$ with $\alpha = n-1$ and using the relation $D^{\varphi(k)} F_\alpha = F_{k * \alpha}$ for $k=1,\ldots, 5$ we get

$$
\begin{aligned}
F_\alpha(X_z) &= F_\alpha(X_0) + \sum_{k\in B} J_k(D^{\varphi(k)} F_\alpha(X), Z) \\
&= F_\alpha(X_0) + \sum_{k\in B} J_k(F_{k*\alpha}(X), Z).
\end{aligned} \tag{2. 4}
$$

Replacing $F_\alpha(X)$ in (2. 3) by the right hand side of (2. 4) we obtain

$$
\begin{aligned}
F(X_z) &= \sum_{|\alpha|<n-1} J_\alpha(D^{\|\alpha\|} F(X_0), Z) + \sum_{|\alpha|=n-1} J_\alpha(D^{\|\alpha\|} F(X_0), Z) \\
&+ \sum_{|\alpha|=n-1} \sum_{k\in B} J_\alpha(J_k(D^{\|\alpha*k\|} F(X), .), Z) \\
&= \sum_{|\alpha|\leqslant n-1} J_\alpha(D^{\|\alpha\|} F(X_0), Z) + \sum_{|\beta|=n} J_\beta(D^{\|\beta\|} F(X), Z).
\end{aligned}
$$

This fact proves that (2.1) holds for n, which completes the proof.

The above Taylor formula represents a decomposition of the increment $F(X_z) - F(X_0)$ into a finite sum of multiple stochastic integrals with constant integrands depending only on X_0 and a remainder which is a finite sum of other multiple stochastic integrals with integrands depending on $X_{z'}$, for $z' < z$. Now for $M \subset A$ we set:

$$\widehat{M} = \{ \alpha \in A \setminus M : \alpha- \in M \}.$$

In many applications, the following generalization of the Taylor formula (2.1) for two parameter semi-martingales is useful.

THEOREM 2.1. *Let $p \geqslant 2$. Suppose that $X \in \mathcal{S}^{4p}$ and F are given. Suppose in addition that for $\varnothing \neq M \subset A$ the following conditions are satisfied*

(a) $\sup_{\alpha \in M} |\alpha| < +\infty.$

(b) $\alpha- \in M$ *for all* $\alpha \in M$.

(c) $F_\alpha = D^{\|\alpha\|} F \in C^0(R)$ *for* $\alpha \in M \cup \widehat{M}$.

Then $F(X_z) = F(X_0) + \sum_{\alpha \in M} J_\alpha(F_\alpha(X_0), Z) + \sum_{\alpha \in \widehat{M}} J_\alpha(F_\alpha(X), Z).$

3. ESTIMATION OF ERRORS

THEOREM 3.1. *Suppose that X is a semi-martingale such that $S^l(X) \in \overline{\mathcal{S}}^\infty$ $(j = 1, \ldots, 5)$. Furthermore, suppose that $F \in C^{4n}(R)$ and for each $\alpha \in B^n$*

$$\sup_{s \in R_{11}} E|F_\alpha(X_s)|^2 \leqslant B_2 (|\alpha|!) B_1^{|\alpha|} \tag{3.1}$$

where B_1, B_2 are some constants. Then for any $Z \in R_{11}$

$$E|F(X_z) - F(X_0) - \sum_{0 < |\alpha| < n} J_\alpha(F_\alpha(X_0), z)|^2 \leqslant B_2 [B_3 \mu(R_z)]^n / n! \tag{3.2}$$

where B_3 is a constant depending only on B_1.

The proof of the theorem is based on the following lemmas.

LEMMA 3.2. *Suppose $Y \in \overline{\mathcal{S}}^\infty$ and*

$$Y = q \cdot W + W \cdot r \cdot W + \mu \cdot \alpha \cdot W + b \cdot W.$$

Then for every function $h \in \overline{L}_1^2$ we have

$$E|h \cdot Y(z)|^2 \leqslant C_0 \left(\int_{R_z} E|h(s)|^2 ds + \int_{R_z \times R_z} E|h(s \vee s')|^2 ds\, ds'\right) \tag{3.3}$$

where C_0 is a constant depending only on Y.

Proof. By the properties of stochastic integrals and mixed stochastic integrals (cf. X. Guyon and R. Prum [4]) and Hölder's inequality, for each $z \in R_{11}$ we have

$$I_1 := E\left|\int_{R_z} h(s)\, q(s)\, dW(s)\right|^2 = \int_{R_z} E|h(s)\, q(s)|^2 ds$$

$$\leqslant \|q\|_1^2 \int_{R_z} E|h(s)|^2 ds, \tag{3.4}$$

$$I_2 := E\left|\int_{R_z \otimes R_z} h(s \vee s')\, r(s, s')\, dW(s)\, dW(s')\right|^2$$

$$= \int_{R_z \otimes R_z} E\,|h(s \vee s')\, r(s, s')|^2\, ds\, ds'$$

$$\leqslant \|r\|_2^2 \int_{R_z \otimes R_z} E\,|h(s \vee s')|^2\, ds\, ds', \tag{3.5}$$

$$I_3 := E\left|\int_{R_z \otimes R_z} h(s \vee s')\, \alpha(s, s')\, ds\, dW(s')\right|^2$$

$$= E\left[\int_{R_z \otimes R_z} I(s \vee s')\, h(s \vee s')\, \alpha(s, s')\, ds)^2\, ds'\right]$$

$$\leqslant \mu(R_z) \int_{R_z \otimes R_z} E\,|h(s \vee s')\, \alpha(s, s')|^2\, ds\, ds'$$

$$\leqslant \|\alpha\|_2^2 \cdot \int_{R_z \otimes R_z} E\,|h(s \vee s')|^2\, ds\, ds', \tag{3.6}$$

$$I_4 := E\left|\int_{R_z \otimes R_z} h(s \vee s')\, \beta(s, s')\, dW(s)\, ds'\right|^2$$

$$\leqslant \|\beta\|_2^2 \cdot \int_{R_z \otimes R_z} E\,|h(s \vee s')|^2\, ds\, ds', \tag{3.7}$$

$$I_5 := E\left|\int_{R_z} h(s)\, b(s)\, ds\right|^2 \leqslant \mu(R_z) \int_{R_z} E\,|h(s)\, b(s)|^2\, ds$$

$$\leqslant \|b\|_1^2 \int_{R_z} E\,|h(s)|^2\, ds \tag{3.8}$$

where $\|\cdot\|_1$ (resp. $\|\cdot\|_2$) is the norm in $\bar{L}_1^\infty$ (resp. $\bar{L}_2^\infty$). Applying the Buniakovsky inequality we obtain

$$E\,|h.Y(z)|^2 \leqslant 5 \sum_{k=1}^{S} I_k$$

$$\leqslant 5(\|q\|_1^2 + \|b\|_1^2) \int_{R_z} E\,|h(s)|^2\, ds$$

$$+\ 5(\|r\|_2^2 + \|\alpha\|_2^2 + \|\beta\|_2^2) \int_{R_z \times R_z} E\,|h(s \vee s')|^2\, ds ds'$$

$$\leqslant c_0 \left(\int_{R_z} E\,|h(s)|^2\, ds + \int_{R_z \times R_z} E\,|h(s \vee s')|^2\, ds ds'\right)$$

where $C_0 = 5\, max\,(\|q\|_1^2 + \|b\|_1^2, \|r\|_2^2 + \|\alpha\|_2^2 + \|\beta\|_2^2)$

The lemma is proved.

LEMMA 3.3. *Suppose that* X *is a semi-martingale such that* $S^j(X) \in \mathfrak{S}^\infty$ $(j = 1,\dots, 5)$. *Then for any* $h \in \overline{L}^2_1$, $z \in R_{11}$ *and* $\alpha \in A \setminus \{\phi\}$

$$E \mid J_\alpha(h, z) \mid^2 \leqslant K_h [c\, \mu(R_z)]^{|\alpha|} / (\mid \alpha \mid !)^2 \tag{3.9}$$

where $K_h = \sup_{s \in R_{11}} E \mid h(s) \mid^2$ *and* C *is a constant depending only on* X.

Proof (by induction).

(i) $|\alpha| = 1$. Suppose $\alpha = k \in B$. By the assertion of Lemma 3.1 and the assumption $S^k(X) \in \mathfrak{S}^\infty$ there exist constants C_k such that

$$E \mid J_k(h, z) \mid^2 = E \mid h \,.\, S^k(X)_z \mid^2$$

$$\leqslant C_k \left(\int_{R_z} E \mid h(s) \mid^2 ds + \int_{R_z \times R_z} E \mid h(s \vee s') \mid^2 ds ds'\right)$$

$$\leqslant C_k K_h [\mu(R_z) + (\mu(R_z))^2 / 4]$$

$$\leqslant 2 C_k K_h \mu(R_z) \leqslant C K_h \mu(R_z)$$

where $C = 2 \max \{ C_1, \dots, C_5 \}$.

(ii) $|\alpha| \geqslant 2$. Suppose $|\alpha| = n$. From (3.3) and the induction assumption it follows that

$$E \mid J_\alpha(h, z) \mid^2 = E \mid J_{\alpha_1}(J_{-\alpha}(h, .), z) \mid^2$$

$$\leqslant C_{\alpha_1} \left[\int_{R_z} E \mid J_{-\alpha}(h, s) \mid^2 ds + \int_{R_z \otimes} E_{R_z} \mid J_{-\alpha}(h, s \vee s') \mid^2 ds\, ds' \right]$$

$$\leqslant K_h C_{\alpha_1} \frac{C^{n-1}}{[(n-1)!]^2} \left[\int_{R_z} \mu(R_s)^{n-1} ds + \int_{R_z \otimes} \mu_{R_z}(R_{s \vee s'})^{n-1} ds\, ds' \right]$$

$$= K_h C_{\alpha_1} \frac{C^{n-1}}{[(n-1)!]^2} \left[\frac{\mu(R_z)^n}{n^2} + \frac{\mu(R_z)^{n+1}}{(n+1)^2} \right]$$

$$\leqslant K_h [C \,.\, \mu(R_z)]^n / (n!)^2$$

$$\leqslant K_h [C\, \mu(R_z)]^{|\alpha|} / (|\alpha| !)^2 .$$

Proof of Theorem 3.1. Put

$$R_n^2(z) = E \mid F(X_z) - F(X_0) - \sum_{0 < |\alpha| < n} J_\alpha(F_\alpha(X_0), z) \mid^2$$

By the Minkowski inequality, Theorem 2.1, Lemma 3.2 and the inequality (3.1) we get

$$\mathcal{R}_n(z) \leqslant \sum_{|\alpha|=n} [E|J_\alpha(F_\alpha(X), z)|^2]^{1/2}$$

$$\leqslant \sum_{|\alpha|=n} \left[\frac{B_2 \cdot (|\alpha|!)\, (C B_1 \mu(R_z))^{|\alpha|}}{(|\alpha|!)^2} \right]^{1/2}$$

$$= \sum_{|\alpha|=n} B_2^{1/2} \frac{[C B_1 \mu(R_z)]^{n/2}}{(n!)^{1/2}}$$

$$= \frac{B_2^{1/2} [25\, C\, B_1 \mu(R_z)]^{n/2}}{(n!)^{1/2}} .$$

Therefore

$$R_n^2(z) \leqslant B_2 \frac{[B_3 \mu(R_z)]^n}{n!} .$$

where $B_3 = 25\, C\, B_1$.

COROLLARY 3.4. *Suppose that $S^i(X) \in \mathfrak{J}^\infty$, $F \in C^\infty(R)$ and condition (3.1) is satisfied for each $\alpha \in \Lambda \setminus \{\varnothing\}$. Then*

$$\lim_{n \to \infty} \sup_{z \in R_{11}} E\, |F(X_z) - F(X_0) - \sum_{0 < |\alpha| \leqslant n} J_\alpha(F_\alpha(X_0), z)|^2 = 0 .$$

Example. Consider the semi-martingale $X = W$. By (1.13) we get

$$S^1(W) = W, \quad S^2(W) = W . 1 . W, \quad S^3(W) = 1 . \mu,$$

$$S^4(W) = W . \frac{1}{2} . \mu + \mu . \frac{1}{2} . W,$$

$$S^5(W) = \mu . \frac{1}{4} . \mu.$$

It is clear that $S^j(W) \in \mathfrak{J}^\infty$ for every $j = 1, \ldots, 5$, then

$$\sum_{k=0}^{n} a_k W_z^k = a_0 + \sum_{0 < \|\alpha\| \leqslant} (\|\alpha\|!)\, a_{\|\alpha\|} J_\alpha(1, z).$$

In particular,

$$W^2 = 2W . W + W . 2 . W + 1 . \mu.$$

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