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GLOBAL MINIMIZATION OF A CONCAVE FUNCTION SUBJECT TO MIXED LINEAR AND REVERSE CONVEX CONSTRAINTS(1)

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INTRODUCTION

In this paper we shall be concerned with the global minimization of a oncave function over a nonconvex set which is a kind of « excised » polyhedral onvex set, i. e., a polyhedral convex set from which an open convex subset as been removed. In more precise terms, we shall consider the following roblem:

$$\left.\begin{array}{c}\text{Minimize } f(x), \text{ subject to}\\ x \in D, \quad g(x) \geqslant 0\end{array}\right\},$$

'here $f: R^n \to R$ is a concave function, D is a polyhedral convex set in R^n given explicitly by a system of linear inequalities, and $g: R^n \to R$ is a onvex function. Setting

$$G = \{x \in R^n : g(x) < 0\},$$

e can also formulate this problem as

$$\begin{array}{c}\text{Minimize } f(x), \text{ subject to}\\ x \in D \setminus G.\end{array} \qquad \text{(P)}$$

ere G is an open convex set (since g is continuous). The constraint $g(x) \geqq 0$. e., $x \notin G$) is called *reverse convex* ([9], [6]), so problem (P) differs from the 'dinary concave program under linear constraints only by the presence of an lditional reverse convex constraint.

Optimization problems with reverse convex constraints have been studied rst by Rosen [12] and subsequently by Avriel and Williams ([1], [2]), Meyer], Ueing [17], Bansal and Jacobsen [4], and Hillestad [5]. In [7] an algorithm as given for solving linear programs with an additional reverse convex nstraint, i. e., problems of the form (P) in the special case where f is linear

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and D is bounded (is a polytope). There, Hillestad and Jacobsen established the important property that the convex hull of the feasible region $D \setminus G$ is a polytope, if D itself is a polytope.

In the sequel, this property will be extended, by proving that, in the general case where D is a polyhedral convex set (may be unbounded), the closure of the convex hull of $D \setminus G$, written $\overline{co}(D \setminus G)$, is still a polyhedral convex set. From the concavity of the objective function f it will then follow that Problem (P) is equivalent to minimizing $f(x)$ over the polyhedral convex set $\overline{co}(D \setminus G)$, which theoretically, is a concave programming problem already studied by many authors [14] ; see [8] and [10] for a review). The main difficulty with this concave program is, however, that the constraint set, $\overline{co}(D \setminus G)$, is not given in the explicit form via a system of linear inequalities as in the usual case. Nevertheless, we shall show that this problem can be solved by using an extended version of the method elborated by V.T. Ban in [3] (see also [16]).

The paper is divided into several sections. After the Introduction, we begin, in Section 1, with establishing some basic properties of Problem (P). Next, in Section 2, we present a general method for partitioning a polyhedral convex set into generalized simplices. Since this method proceeds through successive bisections of an initial generalized simplex containing the original polyhedral convex set, it seems to be appropriate for use as a branching scheme in branch-and-bound procedures for solving optimization problems of the type under consideration. In Section 3, we examine the partitiono f $(D \setminus G)$ into « elementary pieces » such that each subproblem obtained by restricting the constraint set to an elementary piece is quite easy to solve. In Section 4, we show how this partition can be incorporated into a branch-and-bound algorithm to solve Problem (P). Finally, to illustrate how this algorithm works, a two-dimensional example is given in Section 5.

1. BASIC PROPERTIES

Before embarking on the solution of Problem (P), let us establish some basic properties of it.

PROPOSITION 1. *If (P) is solvable, it has an optimal solution $\bar{x}$ lying on some edge of the polyhedral convex set D.*

Proof. Let $\widehat{x}$ be an optimal solution of (P). Since $\widehat{x} \notin G$ and G is convex, there is by the separation theorem a hyperplane passing through $\widehat{x}$ and disjoint with G. Let S be th section of D by this hyperplane. Then S is a polyhedral convex set contained in $D \setminus G$. Since the concave function f is bounded below on S by the value $f(\widehat{x})$, its minimum over S is achieved at some extreme point $\bar{x}$ of S ([12], Corollary 32 3.4). It is easy to see that $\bar{x}$ lies on an edge of D. Indeed, let F be the smallest face of D containing $\bar{x}$. If dim $F > 1$, then F would have in common with S a line segment containing $\bar{x}$ in its relative interior, which would onflict with the fact that $\bar{x}$ is an extreme point of S. Therefore, $\dim F \leq 1$, and o $\bar{x}$ lies on at least one edge of D. Since $f(\bar{x}) \leq f(\hat{x})$, $\bar{x}$ is an optimal solution of (P). //

In view of this Proposition, to solve Problem (P) it suffices to seek the best among the optimal solutions of the one-dimensional subproblems:

$$\text{Minimize } f(x), \text{ s.t. } x \in E \setminus G, \qquad (*)$$

where E is any edge of D. Since $E \setminus G$ consists of one or two connected components, each of which is a line segment or a ray, an optimal solution of (*) can be found simply by comparing the values of f at the endpoints of these components. Thus, Problem (P) is decomposed into a finite number of easy one-dimensional subproblems. The difficulty is, however, that the number of edges of D is usually very large, making this crude approach very difficult to implement.

To provide a better insight into the problem, let us establish a deeper property which extends to the general case an earlier result of Hillestad and Jacobsen [7].

PROPOSITION 2. *Assume that $n \geqslant 2$ and D contains no line. Then the set $\overline{co}(D \setminus G)$, i.e., the closure of the convex hull of $D \setminus G$, is a polyhedral convex set, whose extreme points are endpoints of sets of the form $\overline{co}(E \setminus G)$, where E is an edge of D, and whose extreme directions are extreme directions of sets of the form $\overline{co}\ (0^+F \setminus C)$, where 0^+F is the recession cone of a two-dimensional face of D, while C is made up of all $v \in 0^+G$ (the recession cone of G) such that for every $x \in D$ the ray $\{x + \lambda v : \lambda \geqslant 0\}$ meets G.*

In other words,

$$\overline{co}(D \setminus G) = coU + \text{cone}V,$$

where

$$U = \bigcup \{\text{ext}\ (\overline{co}\ (E \setminus G) : E \text{ edge of } D\},$$
$$V = \bigcup \{\text{dext}\ (\overline{co}\ (0^+F \setminus C) : F \text{ face of } D, \dim F = 2\}$$

(ext A denotes the set of extreme poinrs of A, dext A the set of extreme directions of A).

Note that the sets U, V are finite, because each $\overline{co}(E \setminus G)$ is a line segment or a ray, each cone $(0^+F \setminus C)$ is a two-dimensional cone.

Proof. Consider an arbitrary point $x \in D \setminus G$ and let S be the section of D by a hyperplane passing through x and disjoint with G. Then S is a polyhedral convex set and $x \in$ co (ext S) + cone (dext S). But, as in the proof of Proposition 1, it is easily seen that any extreme point u of S lies on some edge E of D, hence

$$u \in \text{co}\ (E \setminus G) \subset \text{co}\ U \qquad \text{if } E \setminus G \text{ is bounded,}$$

while

$$u \in \text{co}(E \setminus G) \subset \text{co}\ U + \text{cone}\ V \text{ if } E \setminus G \text{ is unbounded.}$$

ote that in the latter case the direction of E is an element of V, since E is an treme ray of any two-dimensional face of D containing it.) Furthermore, st as an extreme point of S must lie on an edge of D, so must any extreme y of S lie in a two-dimensional face F of D. Therefore, the direction of such extreme ray belongs to $0^+F\setminus C$. We thus have $x \in \text{co}\, U + \text{cone}\, V$, which iplies $\overline{\text{co}}(D\setminus G) \subset \text{co}\, U + \text{cone}\, V$. To prove the converse inclusion we observe at, obviously, $U \subset \overline{\text{co}}(D\setminus G)$, since $E\setminus G \subset D\setminus G$ for every edge E of D. On other hand, if $v \in 0^+F\setminus C$ for some two-dimensional face F of D, then ere is at least one $x \in D$ such that $\{x + \lambda v : \lambda \geqslant 0\} \subset D\setminus G$. This means at any $v \in 0^+F\setminus C$ must belong to the recession cone of $\overline{\text{co}}(D\setminus G)$. Since this cession cone is convex and closed (see e.g., [11]), it follows that it contains ery cone $\overline{\text{co}}(0^+F\setminus C)$. Hence it contains V and, consequently, $\overline{\text{co}}(D \setminus G) \supset U + \text{cone}\, V$, as was to be proved //

COROLLARY 1. *Problem (P) is solvable if and only if so is the problem*

$$\text{Minimize } f(x), \text{ s.t. } x \in \overline{\text{co}}\,(D\setminus G) \qquad \text{(R)}$$

d then any optimal basic solution of (R) is an optimal solution of (P).

Proof. If (P) is solvable, then the function f is bounded below over the set $\setminus G$, hence over the set $\overline{\text{co}}\,(D\setminus G)$ too, by the concavity of f. This implies the lvability of (R), because $\overline{\text{co}}\,(D\setminus G)$ is a polyhedral convex set by Proposition 2 e e.g., [11]. Corollary 32.3.4). Moreover, when (R) is solvable, the optimum achieved in at least one extreme point of $\overline{\text{co}}\,(D\setminus G)$ and since any such point longs to $D\setminus G$ by Proposition 2, it follows that any optimal basic solution of) is also an optimal solution of (P) //

The above results show that solving (P) amounts to solving (R), which is a ncave *programming problem under linear constraints.* It should be stressed, wever, that the linear constraints are given only in an *implicit* form, so that en when the objective funcfion f is linear one cannot use standard methods linear programming for solving it.

Fortunately, the extreme points and extreme directions of the constraint $\overline{\text{co}}\,(D\setminus G)$ are related in a simple way to the extreme points and extreme ections of the polyhedral convex set D, as shown by Proposition 2. In the quel we shall show how this property can be exploited to construct a finite thod for solving (R).

2. PARTITIONING A POLYHEDRAL CONVEX SET INTO GENERALIZED SIMPLICES

A subset S of R^n is called a *generalized simplex* if it is a polyhedral convex containing no lines and having $n + 1$ or fewer extreme points and direc- ns. Thus, a generalized simplex in R^n is a polyhedral convex set generated $k_1 \geq 1$ points and $k_2 \leq (n + 1) - k_1$ directions.

The method to be developed below for solving (R) uses a partition of the polyhepral convex set D into generalized simplices. Therefore, in this section, we shall first examine a procedure for constructing such a partition. The essential idea of the procedure is borrowed from a cone partitioning procedure presented in [3] (see also [16]).

Suppose the polyhedral convex set D is given as the solution set of the system:

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \geq 0 \qquad (i = 1,\ldots, m) \tag{1}$$

$$x_j \geq 0 \qquad (j = 1,\ldots, n). \tag{2}$$

Introducing an additional variable x_{n+1}, we can write this system in the form

$$h_i(y) \geq 0 \qquad (i = 1,\ldots, m) \tag{3}$$

$$y = (x, t) \in R_+^{n+1} \tag{4}$$

$$t = 1 \tag{5}$$

where

$$h_i(y) = \sum_{j=1}^{n} a_{ij} x_j - b_i t. \tag{6}$$

Define

$$M = \{y \in R^{n+1} : h_i(y) \geq 0 \ (i = 1,\ldots, m)\}$$

$$Q = \{y = (x, t) \in R^{n+1} : t = 1\}$$

(M is a cone, Q a hyperplane). Then, denoting by $\widetilde{D}$ the set of all points $(x, 1) \in R^{n+1}$ such that $x \in D$, we have

$$\widetilde{D} = M \cap R_+^{n+1} \cap Q. \tag{7}$$

Consider now a system U_0 of $n + 1$ vectors of R_+^{n+1} such that

$$M \cap R_+^{n+1} \subset \text{cone}\, U_0 \subset R_+^{n+1}.$$

The idea of the procedure is to divide cone U_0 (the cone generated by U_0) into subcones, each generated by $n + 1$ or fewer vectors of co U_0 and lying either *entirely outside* M or *entirely inside* M. Then, clearly the collection of all subcones lying inside M forms a partition of $M \cap (\text{cone } U_0) = M \cap R_+^{n+1}$ and hence, the collection of their intersections with Q will give a partition of D into generalized simplices (see (7)).

Let U be a system of $n + 1$ vectors of co U_0 (so that cone U is a subcone of cone U_0):

$$U = \{u^1, \ldots, u^{n+1}\}.$$

e $u^k \in M$ if and only if $h_{ik}=h_i(u^k)\geq 0$ for all $i=1,\dots,m$, we have cone M if and only if the matrix

$$(h_{ik}) \tag{8}$$

no negative entry. On the other hand, if at least one row of this matrix ists solely of negative entries then (cone U) $\cap M=\phi$. Based on this obser- on, we shall define some simple operations to be performed on U, so that operation either cuts off the part of cone U outside M, or splits cone U two subcones, and after a finite number of these operations the part of U inside M will be split into subcones with the desired properties.

Given a system $U=\{u^1,\dots,u^{n+1}\}$ whose associated matrix (h_{ik}) has at least negative entry, we shall call the first row of this matrix that contains nega- entries the *test row* of the matrix (or of U). Set s be the index of the test of a given system $U=\{u^1,\dots,u^{n+1}\}$.

I. If the test row has no positive entry, i. e., if $h_{sk}\leq 0$ $(k=1,\dots,n+1)$, ote by $U(s*)$ the system that obtains from U by replacing each vector u^k such $h_{sk}<0$ with the zero vector.

II. If the test row has just one positive entry, say $h_{sp}>0$, denote by $U(s,p)$ system that obtains from U by replacing each vector u^k such that $h_{sk}<0$

$$v^k=h_{sp}\,u^k-h_{sk}\,u^p \tag{9}$$

e that $h_s(v^k)=h_{sp}h_s(u^k)-h_{sk}h_s(u^p)=h_{sp}h_{sk}-h_{sk}h_{sp}=0$, therefore ies in the intersection of the hyperplane $H_s=\{y: h_s(y)=0\}$ with the cone erated by u^p and u^k).

III. If the test row has both positive and negative entries, and p, q are two ces such that $h_{sp}>0$, $h_{sq}<0$, denote by $U(s,p,q)$ (or $U(s,q,p)$, respectively) system that obtains from U by replacing u^q (u^p respectively) with the vector

$$v=h_{sp}\,u^q-h_{sq}u^p \tag{10}$$

ce $h_s(v)=0$, this vector lies in the intersection of the hyeerplane H_s with cone generated by u^p and u^q).

We shall call the operation of passing from U to $U(s*)$ (or $U(s,p)$, or $U(s,q,p)$, $J(s,q,p)$) an *elementary operation*.

PROPOSITION 3. *We have*

$$(\text{cone } U (s*)) \cap M = (\text{cone } U) \cap M \tag{11}$$

$$(\text{cone } U (s, p)) \cap M = (\text{cone } U) \cap M \tag{12}$$

$$\text{cone } U = \text{cone } U (s, p, q) \cup \text{cone } U (s, q, p). \tag{13}$$

An elementary operation increases either the number of zeros in the test row or the index of the test row.

Proof. In the case (I) discussed above, every vector of U lies in the halfspace $H_s^- = \{y: h_s(y) \leq 0\}$, and so

$$(\text{cone } U) \cap M = (\text{cone } U) \cap H_s \cap M = (\text{cone } U (s *)) \cap M$$

(H_s being the hyperplane $h_s(y) = 0$). Hence, (11). In the case (II), let $N = \{k : h_{sk} < 0\}$. Since by (9) every v^k $(k \in N)$ is a positive combination of u^k and u^p, it is clear that cone $U(s, p) \subset$ cone U. Therefore, to prove (12) we need only show that

$$(\text{cone } U) \cap M \subset (\text{cone } U (s, p)) \cap M.$$

Let $y \in (\text{cone } U) \cap M$, so that $y = \Sigma\, \theta_k u^k$ with $\theta_k \geq 0$, and $h_s(y) \Sigma\, \theta_k h_{sk} \geq 0$.

Substituting $\dfrac{h_{sk} u^p + v^k}{h_{sp}}$ for u^k $(k \in N)$ we can write

$$y = \sum_{k \notin N \cup \{p\}} \theta_k u^k + \left[\theta_p + \sum_{k \in N} \theta_k \frac{h_{sk}}{h_{sp}}\right] u^p + \sum_{k \in N} \left[\frac{\theta_k}{h_{sp}}\right] v^k$$

where, as just seen, $\theta_p + \sum_{k \in N} \theta_k \dfrac{h_{sk}}{h_{sp}} = \dfrac{h_s(y)}{h_{sp}} \geq 0$. This implies $y \in$ cone $U(s,p)$,

hence (cone $U) \cap M \subset$ cone $U(s, p) \cap M$, proving (12). Finally, in the case (III), since v defined by (10) is a positive combination of u^p and u^q, it is obvious that

$$\text{cone } U (s, p, q) \cup \text{cone } U (s, q, p) \subset \text{cone } U.$$

To prove the converse inclusion, let $y \in$ cone U, so that $y = \Sigma\, \theta_k u^k$ with $\theta_k \geqslant 0$.

If $\eta = \theta_p h_{sp} + \theta_q h_{sq} \geq 0$, then, substituting $u^q = \dfrac{h_{sq} u^p + v}{h_{sp}}$, we can write

$$y = \sum_{\substack{k \neq p \\ k \neq q}} \theta_k u^k + \left[\frac{\theta_q}{h_{sp}}\right] v + \left[\frac{\eta}{h_{sp}}\right] u^p ,$$

which implies $y \in$ cone $U(s, p, q)$. Similarly, if $\eta \leq 0$ then $y \in$ cone $U(s, q, p)$. Therefore, cone $U \subset$ cone $U(s, p, q) \cup$ cone $U(s, q, p)$ as was to be proved. Thus, (11), (12), (13) hold.

Turning to the second part of the Proposition, observe that in each ele-
ıtary operation some elements y of U with $h_s(y) \neq 0$ are replaced by some ors v satisfying $h_s(v) = 0$, such that each v is a nonnegative combination of ıents of U. The latter fact implies that $h_i(v) \geqslant 0$ for all $i < s$, and so the x of the test row never decreases by an elementary operation. Therefore, er the test row gets some more zeros, or its index increases. //

We shall say that a point $u \in R_+^{n+1}$ is D-feasible if $h_i(u) \geqslant 0$ $(i = 1,..., m)$; stem $U = \{u^1, ..., u^{n+1}\}$ is D-feasible if every u^k is D-feasible, in other ds if the associated matrix (h_{ik}) has no negative entry. For any R_+^{n+1} let $\pi(u) = \frac{(u_1, ..., u_n)}{u_{n+1}}$ if $u_{n+1} > 0$; $\pi(u) = (u_1, ..., u_n)$ if $_{+1} = 0$. Clearly for every $U = \{u^1, ..., u^{n+1}\} \subset R_+^{n+1}$, the set $S = S(U)$ erated by $\pi(U)$ is a generalized simplex in R^n, with extreme points $u^k) : u_{n+1}^k > 0\}$ and extreme directions $\{\pi(u^k) : u_{n+1}^k = 0\}$. We have D if and only if U is D-feasible.

PROPOSITION 4. *Let* $U^0 = \{u^{0,1}, .., u^{0,n+1}\} \subset R_+^{n+1}$ *be such that cone* $\subset M \cap R_+^{n+1}$. *The collection of all D-feasible systems U that can be ved from* U_0 *by a sequence of elementary operations is finite and the esponding generalized simplices constitute a partition the polyhedral convex* D.

Proof. Let us associate to the given system U_0 a tree $\Gamma(U_0)$ whose root $_0$ and whose nodes are the systems U that can be derived from U_0 by a ıence of elementary operations, two nodes U, V being adjacent if and only ›ne is derived from the other by a single elementary operation (is a ccessor» of the other). Then, clearly a node of $\Gamma(U_0)$ is terminal (i. e., no successor) if and only if it is D-feasible. From Proposition follows that at most mn elementary operations are needed to transform U_0 a D-feasible system. Therefore, every path in $\Gamma(U_0)$ from the root to a ıinal node has a length of at most mn. Noting that each node has at most successors, we see that the number of terminal nodes is at most equal he number of 0-1 sequences of length mn. This proves the finiteness of the ection of D-feasible systems derived from U_0. On the other hand, it follows n (11), (12), (13) that (cone U_0) $\cap M = \cup$ (cone U) $\cap M$ where the union is extended over all D-feasible systems derived from U_0. On the basis of (7) and the assumption cone $U_0 \supset M \cap R_+^{n+1}$ we then conclude that the collection of all generalized simplices $S(U)$, where U is any D-feasible system derived from U_0, yields a partition of D. //

Thus, to obtain a partition of a given polyhedral convex set D into generalized simplices, it suffices to take any $U_0 = \{u^{0,1}, \ldots, u^{0,n+1}\} \subset R_+^{n+1}$ such that cone $U_0 \supset M \cap R_+^{n+1}$ and to generate the tree $\Gamma(U_0)$ — which can be done in many different ways.

Remark 1. Every extreme point or direction of D is an extreme point (or direction, respectively) of some generalized simplex of the partition. This follows from the fact that an extreme ray of the polyhedral convex cone M is an extreme ray of any subcone of M containing this ray.

To test whether a given element u^k of a D-feasible system U corresponds to an extreme point or direction of D is a simple matter. Indeed, suppose that the matrix $A = (a_{ij})$ of (1) is of rank m. Then it can easily be verified that u^k gives an extreme point or direction of D if and only if there exists a set $I \subset \{i : h_{ik} = 0\}$ such that $u_j^k = 0$ for all $j \in \{1, \ldots, n+1\} \setminus I$ and the square matrix $(a_{ij}, i \in I, j \in I)$ is nonsingular.

Therefore, a partition of D into generalized simplices provides all the extreme points and directions of D. What makes this partition useful for our purpose is that it proceeds by successive bisections and generates in each intermediate step a collection of generalized simplices whose union covers all of D. As will be clear shortly, this allows to incorporate a branch-and-bound procedure in the partition, so as to reduce the number of generalized simplices to be explored.

Remark 2. If the system of constraints defining the set D is of form

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, \ldots, m)$$

$$x_j \geqslant 0 \qquad (j = 1, \ldots, n)$$

one should convert it into a system of form (1) (2) by using the following transformations. Assuming the coefficients matrix (a_{ij}) to be of rank m, select a set of m independent columns of this matrix: a_j $(j \in J)$, $|J| = m$. Then, introducing an additional variable x_{n+1} and expressing the «basic» variables

$j \in J$, in terms of the « nonbasic » ones, x_k, $k \in K = \{1,..., n+1\} \setminus J$, we
:

$$x_j = \sum_{k \in K} z_{jk} x_k \qquad j \in J,\ x_{n+1} = 1.$$

e the above system is equivalent to the following one

$$\sum_{k \in K} z_{jk} x_k \geqslant 0 \qquad (j \in J) \tag{3'}$$

$$x_k \geqslant 0 \qquad (k \in K) \tag{4'}$$

$$x_{n+1} = 1 \tag{5'}$$

h is of the form (3), (4), (5) required, with $y = (x_k,\ k \in K)$, $h_j(y) = x_j = \sum_{k \in K} z_{jk} x_k$, $(j \in J)$.

According to the previous scheme we start from a system U_0 of $|K| = + 1 - m$ vectors spanning a subcone of R_+^{n+1} and using elementary ations on U_0 we transform this subcone into a collection of subcones of ^{1} whose union equals just $R_+^{n+1} \cap M$, where M is the cone defined he m inequalities (3'). An alternative approach is to start from a m U_0 of $n+1-m$ vectors spanning a subcone of M and to transform subcone (also by elementary operations performed on U_0) into a collection ubcones of M whose union covers just $M \cap R_+^{n+1}$. In the latter approach, s regarded as spanned by z^k, $k \in K$, where z^k is such that $z_j^k := z_{jk}$ for J, $z_k^k = 1$, $z_j^k = 0$ for $j \in K, j \neq k$, that is, M is the set of all $\mathbf{y} = \sum_{\in K} x_k z^k$, with $x_k \geqslant 0$ $(k \in K)$. Therefore, we start, for example, from the em $U_0 = \{z^k, k \in K\}$ (spanning M). A vector y is now D-feasible if and if $y \geqslant 0$ and all the elementary operations are defined, of course. with ect to this notion of D-feasibility.

3. PARTITIONING D∖G INTO ELEMENTARY PIECES

A subset C of $D \setminus G$ is called an *elementary piece* of $D \setminus G$ if $C = S/G$, where S generalized simplex with all its extreme points in $D \setminus G$ and all its extreme nearly entirely in $D \setminus G$ (i. e., having an unbounded intersection with D/G). our purpose the basic property of such a set is that, as shown in Section 1, the infimum of a concave function over it, whenever finite, must be attained in at least one extreme point of the associated generalized simplex S. Therefore, if we know a finite partition of $D \setminus G$ into elementary pieces, then Problem (P) will thereby be decomposed into a finite number of trivial subproblems.

To construct a partition of D/G into elementary pieces we proceed as follows.

Suppose that D has been partitioned into generalized simplices and consider any one of them, say S. Let S correspond to a D-feasible system $U = \{u^1, \ldots, u^{n+1}\}$, i.e., a system with the associated matrix $(h_{ik}) \geq 0$. We shall perform on U some elementary operations defined as follows.

For every $u = (x, t) \in R_+^{n+1}$ let

$$h_{m+1}(u) = \begin{cases} tg(t^{-1}x) & \text{if } t > 0 \\ \lim\limits_{\lambda \to 0} \lambda g(\lambda^{-1}x) & \text{if } t = 0 \end{cases} \tag{14}$$

(so $h_{m+1}(\cdot)$ is a positively homogeneous convex function on R_+^{n+1}, see, e. g., [12], section 8).

I. If $h_{m+1}(u^k) \leqslant 0$ for all $k = 1, \ldots, n+1$, then we denote by $U(m+1, *)$ the system obtained from U by replacing each vector u^k such that $h_{m+1}(u^k) < 0$ with the zero vector.

II. If $h_{m+1}(u^p) > 0$ for just one p, we denote by $U(m+1, p)$ the system obtained from U by replacing each vector u^k such that $h_{m+1}(u^k) < 0$ with the vector v^k that satisfies

$$v^k \in [u^k, u^p], \quad h_{m+1}(v^k) = 0.$$

III. If $h_{m+1}(u^k) > 0$ for more than one k, and $h_{m+1}(u^k) < 0$ for at least one k, let p, q be two indices such that $h_{m+1}(u^p) < 0$, $h_{m+1}(u^q) < 0$. Then we denote by $U(m+1, p, q)$ (or $U(m+1, q, p)$, respectively) the system obtained from U by replacing u^q (or u^p, respectively) with the vector v^k such that

$$v^k \in [u^p, u^q], \; h_{m+1}(v^k) = 0.$$

We shall assume that *either* of the following conditions holds:

A) No direction of recession x of D exists such that

$$\lim_{\lambda \searrow 0} \lambda g(\lambda^{-1}x) \geq 0,$$

B) No direction of recession x of D exists such that

$$\lim_{\lambda \searrow 0} \lambda g(\lambda^{-1}x) \leq 0.$$

This is satisfied, in particular, if D has no common direction of recession G).

Jnder this assumption, it is easily seen that whenever

$$h_{m+1}(u) > 0, \qquad h_{m+1}(v) < 0 \qquad \text{for } u, v \in M,$$

at least one of the numbers u_{n+1}, v_{n+1} is positive (i. e., u, or v corres-is to an extreme point of D). Therefore, the vector v^k defined in Cases (II), above can always be computed.

ROPOSITION 5. *Each $U(m+1, *)$, $U(m+1, p)$, $U(m+1, p, q)$, $U(m+1, q, p)$ is D—feasible, and the corresponding generalized simplices $S(m+1, *)$, -1, p), $S(m+1, p, q)$, $S(m+1, q, p)$ satisfy:*

$$S(m+1, *) \supset S \setminus G, \tag{15}$$

$$S(m+1, p) \supset S \setminus G, \tag{16}$$

$$S = S(m+1, p, q) \cap S(m+1, q, p). \tag{17}$$

iermore, the passage from U to any one of the above sets increases the num-f elements u such that $h_{m+1}(u) = 0$ at least by one.

roof. The proof is similar to that of Proposition 3, by noting that each e mentioned operations replaces one or several vectors of U by positive inations of vectors of U, and that whenever a point a and a direction u uch that $g(a) \leqq 0$, $\lim\limits_{\lambda \to 0} \lambda g(\lambda^{-1}u) < 0$ then the ray emanating from a in irection u must lie entirely in $\overline{G}$ (see, e.g., [11], Section 8). //

lows that at most n elementary operations of the above described type eeded to transform a D-feasible system $U = \{u'^1, \dots, u'^{n+1}\}$ into another sible system $U' = \{u'^1, \dots, u'^{n+1}\}$ such that $h_{m+1}(u'^k) \geqq 0$ $(k = 1, \dots, n+1)$. the corresponding generalized simplex provides an elementary piece of by virtue of the following

ROPOSITION 6. *If a D-feasible system $U = \{u^1, \dots, u^{n+1}\}$ satisfies $h_{m+1}(u^k) \geqq 0$ 1, ..., n+1), then for the corresponding generalized simplex $S = S(U)$ the $\setminus G$ is an elementary piece of $D \setminus G$.*

roof. If $u = (x, t) \in U$ with $t > 0$, then $g(t^{-1}x) = t^{-1}h_{m+1}(u) \geqq 0$, hence the sponding extreme point is $t^{-1}x \in S \setminus G$. On the other hand, if $v = (y, 0) \in$ n, from assumption (A) or (B), $h_{m+1}(v) = \lim\limits_{\lambda \to 0} \lambda g(\lambda^{-1}y) > 0$ hence, for xtreme point x of S, $g(x + \theta y) \to +\infty$ as $\theta \to +\infty$. This implies $x + \theta y \in$ G for all $\theta > 0$ sufficiently large. Therefore the extreme ray emanating x in the direction y nearly entirely lies in $D \setminus G$. //

Thus, given any generalized simplex $S \subset D$ we can subdivide $S \setminus G$ into elementary pieces by using a finite number of appropriate bisections. If we do this for every generalized simplex in a partition of D, we obtain a partition of $D \setminus G$ into elementary pieces. This process can of course be thought of as a prolongation of the process of partitioning D into generalized simplices. So we get a unified process, starting from a system $U_0 \subset R_+^{n+1}$ such that cone $U_0 \supset M \cap R_+^{n+1}$ and terminating with a partition of $D \setminus G$ into elementary pieces. In this unified process, each system $U = \{u^1, ..., u^{n+1}\}$ is associated with an $(m+1) \times (n+1)$-matrix

$$(h_{ik})$$

where

$$h_{ik} = h_i(u^k), \qquad i = 1, ..., m+1, k = 1, ..., n+1,$$

$h_i(.)$ being defined by (6) for $i = 1, ... m$, and $h_{m+1}(.)$ being defined by (14). As in Section 2, the test row is the first row s having at least one negative entry. The elementary operations are defined as in Section 2, if $s = 1, ..., m$, and as in the present section if $s = m + 1$. (Note that the definitions of elementary operations given in this section reduce to the previous ones when the function g is affine.)

4. ALGORITHM

By partitioning the set $D \setminus G$ into elementary pieces, as described above, we reduce Problem (P) to a finite number of trivial subproblems, namely:

$$\min \{f(x) : x \in S_\nu\} \tag{18}$$

where each S_ν is an elementary piece of the partition. However, since the partitioning process is by successive bisections, it can be incorporated into a branch-and-bound scheme in order to avoid solving a large number of subproblems (18).

Specifically, let U_0 be a system of $n + 1$ vectors of R_+^{n+1} such that cone $U_0 \supset M \cap R_+^{n+1}$. For example, one can take $U_0 = \{e^1, ..., e^{n+1}\}$, with e^i being the i^{th} unit vector of R^{n+1}. In the case where a nondegenerate extreme point x^0 of D is known, one can, of course, take $u^{0,n+1} = x^0$, while $u^{0,1}, ..., u^{0,n}$ are the n extreme points of D adjacent to x^0.

Denote by $\tilde{\Gamma}(U_0)$ a tree rooted at U_0 such that, whenever U is a node of $\tilde{\Gamma}(U_0)$, then either $S = S(U)$ gives an elementary piece of $D \setminus G$ (which occurs when the matrix (h_{ik}) associated with U has no negative entry) and in that vent U is a terminal node of the tree, or U has one or two successors, which re derived from U by a single elementary operation. As seen above, such tree is finite, and the set of all its terminal nodes provides just a artition of $D \setminus G$ into elementary pieces. Since an optimal solution of Problem P) lies in some terminal node, the shortest way to this optimal solution is bviously the path from the root of the tree to this terminal node. Therefore, o avoid generating the whole tree $\widetilde{T}(U_0)$, which would require a great deal of omputation, one should attempt to follow the mentioned path as closely as ossible, on the basis of the information available at each stage of the process. 'or this purpose, observe that for any intermediate node $U = \{u^1, \ldots, u^{n+1}\}$ of he tree, where $u^k = (x^k, t^k) \in R^n_+ \times R_+$, the number

$$\mu(U) = \min \{f(x^k) : t^k > 0\} \tag{19}$$

ives the minimum of $f(x)$ over the generalized simplex $S = S(U)$ generated by he set $x^1, \ldots, x^{n+1}$, provided f is unbounded below on every extreme ray of S. Ience $\mu(U)$ provides a *lower bound* of the minimum of f over $S \setminus G$, i.e., the art of $D \setminus G$ contained in S. So, if $\bar{x}$ is the best among all feasible solutions nown up to this moment («*the incumbent*»), then the inequality $\mu(U) \geq f(\bar{x})$ vould mean that the node U and all the branch emanating from U can be liscarded from further examination, and, consequently, this branch need not e generated. In other words, at each stage, only those nodes U are worth onsidering which satisfy $\mu(U) < f(\bar{x})$.

We are thus led to the following

Algorithm (assuming $\inf \{f(x) : x \in R^n_+\} > -\infty$).

Initialization. Set $M = \{U_0\}$. Compute $\mu(U_0)$ according to (19). Set $\bar{x}$ = the best available feasible solution (for example, for any $u \in U_0$ such that $u_{n+1} > 0$, $h_i(u) \geq 0 (i = 1, \ldots, m+1)$, $x = \pi(u)$ is a feasible solution); set $\bar{x} = \phi$ if no feasible solution is available.

Step 1. Delete all $U \in M$ satisfying $\mu(U) \geq f(\bar{x})$. Let R be the list of remaining elements of M. If $R = \phi$, stop: $\bar{x}$ is an optimal solution. Otherwise, select an $U \in R$ with smallest $\mu(U)$ and go to Step 2.

Step 2. Compute the matrix (h_{ik}) associated with U. If $h_{ik} \geq 0$ for all i, k, set $R \setminus \{U\} \leftarrow M$ and return to Step 1. Otherwise, go to Step 3.

Step 3. Let s be the test row of U (the first row of the matrix (h_{ik}) that has a negative entry).

(a) If $h_{sk} \leq 0$ for all k, replace U by $U(s*)$ in the list R. Let M be the new list. Compute $\mu(\cdot)$ for the new member of the list. Update $\bar{x}$ and return to Step 1.

(b) If $h_{sp} > 0$ for just one p, replace U by $U(s,p)$ in the list R. Let M be the new list. Compute $\mu(*)$ for the new member of the list. Update $\bar{x}$ and return to Step 1.

(c) If neither (a) nor (b) occur, select p, q such that $h_{sp} > 0$, $h_{sq} < 0$ and replace U by $U(s,p,q)$, $U(s,q,p)$ in the list R. Let M be the new list. Compute $\mu(*)$ for each new member of the list. Update $\bar{x}$ and return to Step 1.

Remark 3. If the condition $\inf\{f(x) : x \in R^n\} > -\infty$ i. not fulfilled, then in computing $\mu(U)$ we may encounter a $u=(x,t) \in U$ with $t>0$, and a $v=(y,0) \in U$ such that $\lim_{\lambda \searrow 0} \lambda f(\lambda^{-1}y) < 0$ (i.e., f is unbounded below on the ray emanating from $x=\pi(u)$ in the direction $y=\pi(v)$.) In that event we have to set $\mu(U)=-\infty$. Furthermore, in Step 2, it may happen that $h_{ik} \geq 0$ for all i, k, and $\mu(U)=-\infty$: then the Algorithm stops with $\inf\{f(x): x \in D\setminus G\} = -\infty$ since this would mean that $S = S(U)$ gives and elementary piece of $D\setminus G$ an f is unbounded below on some ray of this elementary piece.

Remark 4. To facilitate the computation of the matrix (h_{ik}) in Step 2, it is convenient to work, not with U itself, but with the matrix $\tilde{U}=(\tilde{u}^1,\dots,\tilde{u}^{n+1})$; and the vector h_{m+1}, where

$$\tilde{u}^k = \begin{bmatrix} u^k_l \\ \vdots \\ u^k_{n+1} \\ h_{sk} \\ \vdots \\ h_{mk} \end{bmatrix}, \qquad h_{m+1,\cdot} = (h_{m+1}(u^1), \dots, h_{m+1}(u^{n+1})),$$

s being the test row index. Indeed, every elementary operation consists of one or several replacements of a vector $u^k \in U$ by a vector $v^k = \lambda u' + (1-\lambda u)''$, where $u', u'' \in U$ and $0 \leq \lambda \leq 1$. But in view of the linearity of h_i $(i=1,\dots,m)$, this implies the replacement of $\tilde{u}^k \in \tilde{U}$ by $\tilde{v}^k = \lambda\tilde{u}' + (1-\lambda)\tilde{u}''$. Therefore, if we work with $\tilde{U}$ everywhere, then Step 2 is entered with the knowledge of (h_{ik}) $(i = s, \dots, m;\ k=1,\dots, n+1)$ and we need to update only the vector $h_{m+1,\cdot}$

Remark 5. In step 3 (a), if for some $r \geq s$ one has $h_{rk} \leq 0$ for all $k=1,\dots,n+1$, one can replace U by $U(r*)$ in the list R.

Fig. 1. presents a two-dimensional example, in which both D and G are polyhedral convex sets and the constraint set $D \setminus G$ is disconnected, the objective function is linear. The constraints are numbered 1, 2, 3, 4, 5, 6, the last constraint being nonlinear (g is a piece-wise-linear convex function).

The algorithm starts from the extreme point (1) of the set D, together with the directions (2), (3), of the two edges of D emanating from (1). So the initial system U_0 corresponds to

$$\pi(U_0) = \{(1), (2), (3)\}.$$

Here the extreme point (1) is an optimal solution of the problem:

$$\text{Minimize} \quad f(x), \text{ s.t. } \quad x \in D.$$

The tree generated by the algorithm is depicted below the figure (a bracketed number denotes a point or a direction of a generalized

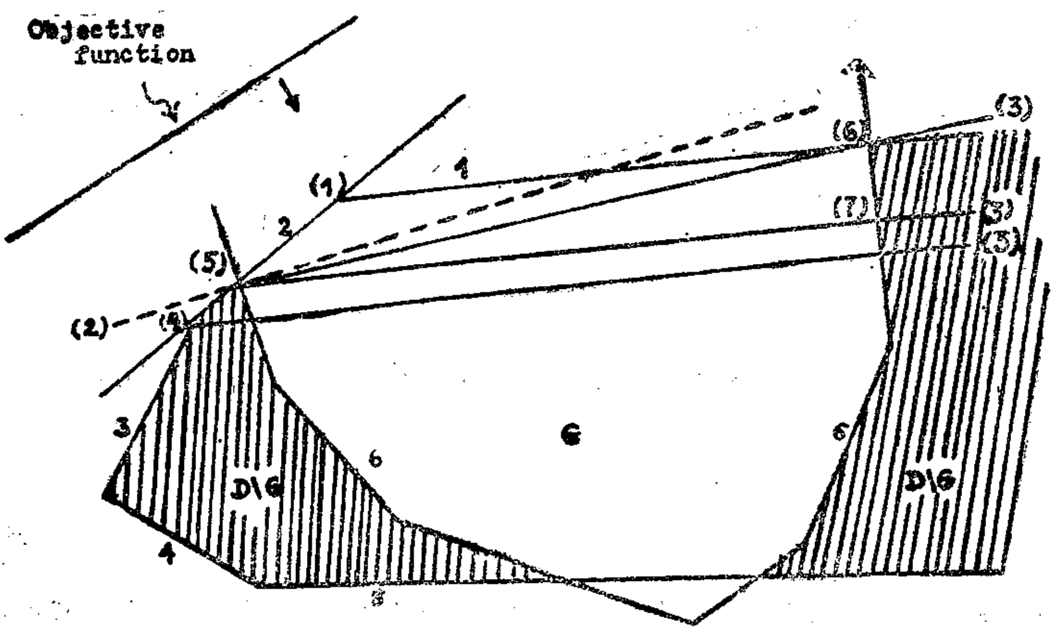

Figure 1.

```
      (1) (2) (3)
       /       \
(4)(2)(3)   (1)(4)(3)
             /      \
   (5)(4)(3)     (1)(5)(3)
                  /      \
         (6)(5)(3)    (1)(5)(6)
          /      \
(6)(7)(3)    (6)(5)(7)
```

(5) (4) (3) is double-underlined.

First incumbent: (4)

Second incumbent: (5)

(Optimal solution)

simplex generated during the algorithm; for example, (4) is a point, (3) is a direction). The terminal nodes of the tree correspond to systems U that are deleted according to the criterion mentioned in Step 1, namely $\mu(U) \geq f(\overline{x})$. Note that among these terminal nodes, only (6) (7) (3) and (5) (4) (3) (which contains the optimal solution (5)) are elementary pieces of $D \setminus G$.

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