ACTA MATHEMATICA VIETNAMICA
Vol. 12 No 2 (1987)

# ON DEGREE BOUNDS FOR THE DEFINING EQUATIONS OF ARITHMETICALLY COHEN-MACAULAY AND BUCHSBAUM VARIETIES

N.V. TRUNG* and G. VALLA**

## 1. INTRODUCTION

Let $k$ be an algebraically closed field. Let $V \subset P_k^m$ be a reduced irreducible non-degenerate variety of codimension $n < m$ and degree $s > n$. Let $p_V$ denote the defining prime ideal of $V$. Recently, Treger [$T$] showed that if $V$ is arithmetically Cohen-Macaulay, $p_V$ may be generated by forms of degree $\leqslant \left]\frac{s}{n}\right[$, where $\left]\frac{s}{n}\right[$ denotes the least integer $\geqslant \frac{s}{n}$. Some authors [$G$], [$M$—$V$] have discussed Treger's result and the aim of this paper is to solve the problems raised in their approach.

In [$G$] Geramita has shown that those cases in which $p_V$ actually needs a generator of degree $\left]\frac{s}{n}\right[$ are rather special. For instance, he has proved that if $n \geqslant 5$ (res. $n \geqslant 2$), Treger's bound could be only sharp for varieties lying on quadric (res. cubic) hypersurfaces. Similar results have been also obtained by Maroscia and Vogel [M—V] for varieties of codimension 2. The first part of this paper will explain this phenomenon by giving a finer degree bound for the generators of $p_V$ which involves the number of independent quadric (res. cubic) hypersurfaces not containing $V$.

Let $H$ denote the Hilbert function of $V$, i. e. $H(t)$ is the number of independent hypersurfaces of degree $t$ not containing $V$. Put

$$a_t = \sum_{i=0}^{t} (-1)^i \binom{m-n}{i} H(t-i)$$

for all $t \geqslant 0$. Write

$$s = b_t (a_t - 1) + c_t + 1.$$

$0 \leqslant c_t < a_t - 1$, and define

$$\varepsilon_t = \inf \{i \geqslant 0;\, c_t - n + 1 < a_i\},$$
$$\tau_t = tb_t + \varepsilon_t + 1$$

Then the main result of the first part may be formulated as follows.

THEOREM 1.1. *Let $V$ be an arithmetically Cohen-Macaulay variety. Then $p_V$ may be generated by forms of degree $\leqslant \tau_t$ for all $t > 0$.*

We will see that $\tau_1 = \left]\frac{s}{n}\right[$ and $\tau_t \leqslant \left]\frac{s}{n}\right[$ for $t > 1$. In particular,

$$\left]\frac{s}{n}\right[ - \tau_t \geqslant \left[\frac{b_t\,(a_t - tn - 1)}{n}\right],$$

where for a rational number $q$, $[q]$ denotes the largest integer $\leqslant q$. Therefore, $\tau_t < \left]\frac{s}{n}\right[$ if $s$ and $a_t$ are large enough in comparison with $n$, e. g. in the case considered by Geramita, Maroscia and Vogel. This explains why Treger's bound is rather special.

The second part of this paper will deal with the following problem raised by Maroscia and Vogel in [M—V]. Let $V$ be an arithmetically Buchsbaum variety, i.e. the local ring $A$ of the affine cone over $V$ at the vertex is a Buchsbaum ring, and let $i(A)$ be the difference between colength and multiplicity $l(A/q) - e_A(q)$ of a parameter ideal $q$ of $A$ which is independent of the choice of $q$ [S — V]. Is then Treger's result still valid if $\left]\frac{s}{n}\right[$ is replaced by $\left]\frac{s}{n}\right[ + i(A)$? Note that $i(A) = 0$ if $V$ is arithmetically Cohen-Macaulay. Following a technique of [Tg3], we are able to answer this question affirmatively by the following result.

THEOREM 1.2. *Let $V$ be an arithmetically Buchsbaum variety. Then $p_V$ may be generated by forms of degree $\leqslant \left]\frac{s}{n}\right[ + \min\,\{2, i(A)\}$.*

The proofs of the above theorems will be found in Section 2 and Section 4, respectively. Section 3 is devoted to the relationship between $\left]\frac{s}{n}\right[$ and $\tau_t$.

## 2. PROOF OF THEOREM 1.1

Let $d = m - n$ be the dimension of the arithmetically Cohen-Macaulay variety $V$. It is well-known that one can find (generically) linear forms $L_1, \ldots,$

$L_d$ such that $(p_V, L_1, \dots, L_d)$ is a radical ideal which defines a set of $s$ points in $H$-uniform position in the subspace $P^n_k = \bigcap_{i=1}^{d} (L_i = 0)$ of $P^m_k$[G — H, (2.13)].

Let $X$ be a set of $s$ points in $P^n_k$. Let $H_X$ denote the Hilbert function of X. Then $X$ is called in $H$-uniform position (the «$H$» is for Harris) if $X$ is non-degenerate and $H_Y(t) = \inf \{ Y, H_X(t)\}$ for any subset $Y$ of $X$ and $t \geqslant 0$. In particular, this condition implies that X is in general position, i.e.no subset of $n + 1$ points of $X$ lies on a hyperplane of $P^n_k$.

Let $p_X$ denote the defining ideal of X. It is also well-known that if $X$ arises as a generic section of $V$, then a minimal basis for $p_V$ can be lifted to one $p_X$ and $H_X(t) = a_t$ fot all $t$. Therefore, to prove Theorem 1,1 we need only to prove the following

PROPOSITION 2.1. *Let $X$ be in H-uniform position with $H_X(i) = a_i$, $i = 1, \dots, t$. Then $p_X$ may be generated by forms of degree $\leqslant \tau_t$, $t < 0$.*

*Let $r_X$ denote the regularity index of $H_X$, i.e.*

$$r_X = \inf \{t,\ H_X(t) = s\}.$$

The proof of Proposition 2.1 is based on the following degree bounds for the generators of $p_X$ in terms of $r_X$.

LEMMA 2.2. [G—M]. *For any X, $p_X$ may be generated by forms of degree* $\leqslant r_X + 1$.

LEMMA 2.3 [$M$ — $V$]. *Let X be in general position. If $r_Y < r_X$ for every subset $Y$ of $s - n + 1$ points of X, $p_X$ may be generated by forms of degree $\leqslant r_X$.*

If X is in $H$—uniform position, any upper bound for $r_X$ in terms of $H_X(t)$ may be also used to estimate $r_Y$ for every subset $Y$ of $s - n + 1$ points of $X$ because $Y$ is also in $H$—uniform position and $H_Y(t)$ is known. Now the problem is to find an appropriate bound for $r_X$. First, we observe that the Hilbert function of $X$ obeys certain rule.

LEMMA 2.4. *Let $X$ be in H — uniform position; Let $t_1 \dots, t_r$ be any system of not necessarily distinct positive integers such that $t = t_1 + \dots + t_r < r_X$. Then*

$$H_X(t) \geqslant \sum_{i=1}^{r} (H_X(t_i) - 1) + 1.$$

*Proof.* This was already proved for $r = 2$ [H, Corollary 3. 5]. For $r > 2$, use induction.

For all $t > 0$, define

$$\rho_t = \inf \{i > 0 ;\ c_t < a_i\}.$$

COROLLARY 2.5. *Let $X$ be as in Proposition 2. 1. Then*

$$r_X \leqslant tb_t + \rho_t.$$

*Proof*. If $r_X > tb_t + \rho_t$, by Lemma 2.4.

$$H_X(tb_t + \rho_t) \geqslant b_t(a_t - 1) + (a_{\rho t} - 1) + 1 \geqslant s,$$

a contradiction because the Hilbert function of a Cohen—Macaulay ring is non-decreasing.

*Proof of Proposition 2.1.* If $\varepsilon_t \geqslant \rho_t$, the statement follows from Lemma 2.2 and Corollary 2.5. If $\varepsilon_t < \rho_t$, we have $\varepsilon_t + 1 = \rho_t$ because by Lemma 2.4, $a_{\varepsilon_t} + 1 \geqslant a_{\varepsilon_t} + a_1 - 1 = a_{\varepsilon_t} + n > c$. Hence by Lemma 2.2, we may assume that $r_X = tb_t + \rho_t$. According to Corollary 2.5, $r_Y \leqslant tb_t + \varepsilon_t < r_X$ for every subset $Y$ of $s-n+1$ points of $X$. Hence the statement follows from Lemma 2.3.

## 3. THE RELATIONSHIP BETWEEN THE BOUNDS

First we note that

$$\left]\frac{s}{n}\right[ = \left[\frac{s-1}{n}\right] - 1.$$

Using this representation of $\left]\frac{s}{n}\right[$, it is easy to check that $\tau_1 = \left]\frac{s}{n}\right[$. To compare $\left]\frac{s}{n}\right[$ with the other $\tau_t$'s we note that by Lemma 2.4, $a_t \geqslant t(a_1 - 1) + 1 = tn+1$. Then $\tau_t \leqslant \left]\frac{s}{n}\right[$ by the following

LEMMA 3.1. $\left]\frac{s}{n}\right[ \geqslant \tau_t + \left[\frac{b_t(a_t - tn - 1)}{n}\right]$ *for all* $t > 0$.

*Proof*. We have

$$\begin{aligned} s &= b_t(a_t - 1) + c_t + 1 \\ &\geqslant b_t(a_t - 1) + a_{\varepsilon_t} - 1 + n \\ &\geqslant b_t tn + b_t(a_t - tn - 1) + \varepsilon_t n + 1. \end{aligned}$$

Hence

$$\left[\frac{s-1}{n}\right] \geqslant b_t t + \varepsilon_t + \left[\frac{b_t(a_t - tn - 1)}{n}\right]$$

which implies the statement.

$\left[\dfrac{b_t(a_t - tn - 1)}{n}\right]$ can be very large. To see this we consider the case $V$ not lying on any hypersurface of degree $t$ of $P_k^m$. For $t = 2$ (res . $t = 3$). Maroscia and Vogel [$M - V$] showed that if $V$ is an arithmetically Cohen — Macaulay variety of codimension 2 and degree large enough, $P_V$ may be generated by forms of degree $\leqslant \left[\dfrac{s-1}{2}\right]$ (res. $\left[\dfrac{s-1}{2}\right] - 2$). It was not apparent why they got these bounds. Using Theorem 1.1 and Lemma 3.1 we can not only give an explanation for this result but also show that these bounds still hold for varieties of higher codimension and arbitrary degree.

PROPOSITION 3.2. *Let $V$ be an arithmetically Cohen-Macaulay variety not lying on any hypersurface of degree $t$. Then $p_V$ may be generated by forms of degree* $\leqslant \left]\dfrac{s}{n}\right[ - \dfrac{t(t-1)}{2}$ *if $n > 2$ or if $n = 2$ and $s \geqslant t(t+3)+1$.*

*Proof*. We have $a_t = \dfrac{(n+1)\dots(n+t)}{t!}$. Write $a_t - tn - 1 = nF(n)$. Then $F(n)$ is a polynomial of the form $\alpha_t n^t + \dots + \alpha_1 n - \alpha_0$ with $\alpha_i \geqslant 0$ for $i = 1, \dots, t$. Since $F(n)$ is a strictly increasing function with

$$E(2) = \frac{t(t-1)}{4}$$

$$F(3) = \frac{t(t-1)(t+7)}{12} > \frac{t(t-1)}{2},$$

we can conclude that

$$\frac{b_t(a_t - tn - 1)}{n} = b_t F(n) \geqslant \frac{t(t-1)}{2}$$

if $n > 2$ or if $n = 2$ and $b_t \geqslant 2$, i. e. $s \geqslant t(t+3)+1$. By Theorem 1.1 and Lemma 3. 1, this implies the statement.

**Remark.** In general, by comparing $b_t F(n)$ with a given polynomial $g(t)$, one can find a number $n_o$ and a function $s_o(t)$ such that $p_V$ may be generated by forms of degree $\leqslant \left]\dfrac{s}{n}\right[ - g(t)$ if $n \geqslant n_o$ or if $s \geqslant s_o(t)$.

Using Proposition 3.2 we can also improve the result of Garamita mentioned in Section 1.

COROLLARY 3.3. *Let $V$ be an arithmetically Cohen-Macaulay variety. Then $p_V$ may be generated by forms of degree* $> \left]\dfrac{s}{n}\right[$ *if one of the following conditions is satisfied:*

(*i*) $n \geqslant 3$ *and* $V$ *lie on no quadric hypersurface,*
(*ii*) $n \geqslant 2$ *and* $V$ *lie on no more than one cubic hypersurface.*

*Proof.* We may assume that $V$ does not lie on any quadric hypersurface. By Proposition 3.2, $p_V$ may be generated by forms of degree $< \left]\frac{s}{n}\right[$ if $n \geqslant 3$ or if $n = 2$ and $s \geqslant 11$. It remains to consider the case $n = 2$ an $s < 11$. Replace $V$ by a set $X$ of points in $H$-uniform position in $P_k^2$ arising as a generic section of $V$. Since $X$ lies on no more than one cubic, $a_3 \geqslant 9$. Hence, using Lemma 2.4, one can show that $a_4 = s$ ($s$ could be 9 or 10 only). By Lemma 2.2 and Lemma 2.3, $p_X$ and therefore $p_V$ may be generated by forms of degree $< 5 = \left]\frac{s}{n}\right]$.

Finally, we show that the bound $\tau_t$ may be replaced by a simpler bound which depends only on $s$, $n$, and $a_t$. For all $t > 0$, define

$$\tau'_t = tb_t + \left]\frac{c_t + 1}{n}\right[.$$

Note that $\tau'_1 = \left]\frac{s}{n}\right[ = \tau_1$.

LEMMA 3. 4. $\tau'_t \geqslant \tau_t$.

*Proof.* Using Lemma 3.4 we can show that $c_t + 1 \geqslant a_{\varepsilon_i - 1} + n \geqslant \varepsilon_t n+1$.

Hence $\left]\frac{c_t + 1}{n}\right[ \geqslant \varepsilon_t + 1$,

which implies the statement.

### 4. PROOF OF THEOREM 1.2.

Let $X \subset P_k^n$ be a set of $s$ points in $H$-uniform position arising as a generic section of the variety $V$ with a linear subspace of $P_k^m$ defined by $d$ linear forms $L_1, \ldots, L_d$, $d = m - n$ [G — H, (2.13)]. As we have seen in the preceding sections, one knows much about the defining ideal $p_X$ of $X$. We shall use this knowledge to prove Theorem 1.2. For that we may replace $p_X$ by the radical ideal $p$ of $(p_V, L_1, \ldots, L_d)$, the defining ideal of $X$ as a set of points in $P_k^m$.

LEMMA 4.1. *P is the unmixed part of* $(p_V, L_1, \ldots, L_d)$.

*Proof*. We will work over the field $k(u)$, where $u = \{u_{ij};\ i = 1, \ldots, d$ and $j = 0, \ldots, m\}$ is a set of indeterminates over $k$. Set $H_i = u_{i0}X_0 + \ldots + u_{im}X_m$, $i = 1, \ldots, d$. Then $(p_V, H_1, \ldots, H_d) = P \cap Q$, where $P$ is a prime ideal and $Q$ an $(X_0, \ldots, X_m)$ — primary ideal of $k(u)[X]$ [Tg1]. Thus, $(X_0, \ldots, X_m)^t P \subseteq (p_V, H_1, \ldots, H_d)$ for some positive integer $t$. If one specializes $H_1, \ldots, H_d$ to $L_1, \ldots, L_d$, then $p$ is the specialization of $P$ [Tg2]. Therefore, $(X_0, \ldots, X_m)^t\, p \subseteq (p_V, L_1, \ldots, L_d)$. Hence $p$ is the unmixed part of $(p_V, L_1, \ldots, L_d)$

COROLLARY 4.2. *Let $V$ be an arithmetically Buchsbaum variety. Then*

$$(X_0, \ldots, X_m)p \subseteq (p_V, L_1, \ldots, L_d).$$

*Proof.* This is a standard property of the graded Buchsbaum ring $k[X]/p_V$ [S—V].

The proof of Theorem 1.2 consists of two parts, and in both parts we shall need Corollary 4.2.

First, the case $i(A) > 1$ of Theorem 1.2 is a consequence of the following stronger result.

PROPOSITIONS 4.3. *Let $V$ be an arithmetically Buchsbaum variety. Then $p_V$ may be generated by forms of degree* $\leqslant \left]\dfrac{s-1}{n}\right[ + 2$.

For the proof of Proposition 4. 3 we shall need the notation of the reduction exponent of a graded ring [Tg3]. Let $a$ be an $(d+1)$ — dimensional homogeneous ideal of $k[X]$. The reduction exponent $r(S)$ of the graded ring $S = k[X]/a$ is defined to be the least integer $t$ for which there exist $d+1$ linear forms $L_1, \ldots, L_{d+1}$ such that all forms of degree $t+1$ of $k[X]$ belong to the ideal $(a, L_1, \ldots, L_{d+1})$.

LEMMA 4.4. [Tg3]. *Let $S$ be a graded Buchsbaum ring. Then $a$ may be generated by forms of degree $\leqslant r(S) + 1$.*

*Proof of Proposition 4.3.* Set $S = k[X]/p_V$. By Lemma 4.4, it suffices to show that $r(S) \leqslant \left]\dfrac{s-1}{n}\right[ + 1$. Set $T = k[X]/(p_V, L_1, \ldots, L_d)$ and $\overline{T} = k[X]/p$. It is easily seen that $r(S) \leqslant r(T)$. On the other hand, since all forms of degree $r(\overline{T})$ belong to the ideal $(p, L_{d+1})$ for some linear form $L_{d+1}$, all forms of degree $r(\overline{T}) + 1$ belong to $(p_V, L_1, \ldots, L_{d+1})$ by applying Corollary 4. 2. Therefore, $r(T) \leqslant r(\overline{T}) + 1$. Since $\overline{T}$ is a Cohen—Macaulay ring, $r(\overline{T})$ is just the regularity index $r_X$ of $H_X$ [Sch]. Hence $r(\overline{T}) \leqslant b_1 + \rho_1 = \left]\frac{s-1}{n}\right[$ by Corollary 2. 5.

Summing up we get $r(S) \leqslant \left]\frac{s-1}{n}\right[ + 1$, as required.

It remains to prove the case $i(A) = 1$ of Theorem 1.2. Note that there is the following formula [S—V, Theorem 2]

$$i(A) = \sum_{i=0}^{d} \binom{d}{i} \, l\,(H_m^i(A)),$$

where $H_m^i(A)$ denotes the ith local cohomology module of $A$ with respect to the maximal ideal $m$. Thus, if $i(A) = 1$, $H_m^i(A) = 0$ for $i = 0, \ldots, d-1$, i.e. depth $(A) = d$. Hence the case $i(A) = 1$ of Theorem 1.2 is a consequence of the following result.

PROPOSITION 4.5. *Let $V$ be an arithmetically Buchsbaum variety wish depth $(A) = d$. Then $p_V$ may be generated by forms of degre $\leqslant \left]\frac{s}{n}\right[ + 1$.*

*Proof*. Let $F$ be a form of a minimal basis of $p_V$. Suppose that $t = \deg(F) \geqslant \left]\frac{s}{n}\right[ + 2$. Since $p_V \subset p$ and $p$ may be generated by forms of degree $\leqslant \left]\frac{s}{n}\right[$, $F \in (X_0, \ldots, X_m)^2 p$. Thus, using Corollary 4.2 we may write $F = G + G_1 L_1 + \ldots + G_d L_d$ for some form $G$ belonging to the ideal generated by the forms of degree $< t$ of $p_V$. Since depth$(A) = d$, $L_1, \ldots, L_d$ form a regular sequence of the ring $k[X]/p_V$. Therefore, $G_d \in (p_V, L_1, \ldots, L_{d-1}) : L_d = (p_V, L_1, \ldots, L_{d-1})$. Hence we can omit the term $G_d E_d$ in the above presentation of $F$. Proceeding like that, we can successively omit the terms $G_1 L_1, \ldots, G_d L_d$. That means $F$ belonging to the ideal generated by the forms of degree $< t$ of $p_V$, a contradiction (cf. the proof of [Tg3, Proposition 4.1]).

**Remark.** We could not find an arithmetically Buchsbaum variety $V$ which needs a defining equation of degree $> \left]\frac{s}{n}\right[ + 1$.

Finally, we point out that there is no similar version of Theorem 1.1 for arithmetically Buchsbaum varieties. The reason is that the Hilbert function of a set $X \subset P_k^n$ arising as a generic section of $V$ can not be computed in terms of the one of $V$. However, some consequences of Theorem 1.1 may be extended to arithmetically Buchsbaum varieties.

PROPOSITON 4.6. *Let $V$ be an arithmetically Buchsbaum variety which lies on no more than $n$ hypersurfaces of degree $t+1$. Then $p_V$ may be generated by forms of degree* $\leqslant \left] \frac{s}{n} \left[ - \frac{t(t-1)}{2} + \min\{2, i(A)\}\right.\right.$ *if* $n > 2$ *or if* $n = 2$ *and* $s > t(t+3)+1$.

*Proof.* The proof is similar to the above proof of Theorem 1.2. It is based on Proposition 3.2. The key is the fact that $X$ lies on no hypersurface of degree $t$ of $P^n_k$. To prove that we may assume that $L_1 = X_{n+1}, \ldots, L_d = X_m$. Suppose that $p$ contains a form $F$ of degree $t$. Then $X_0 F, \ldots, X_n F \in (p_V, X_{n+1}, \ldots, X_m)$ by Corollary 4.2. Set $X_i F = F_i$ modulo $(X_{n+1}, \ldots, X_m)$ for some $F_i \in p_V$, $i = 0, \ldots, n$. It is easily seen that $F_0, \ldots, F_n$ are linearly independent over $k$, a contradiction.

*Acknowledgement.* The first author would like to thank Consiglio Nazionale delle Ricerche and the University of Genova for partially support and hospitality during the preparation of this paper.

*Added in proof.* Using the idea developed in the first part of this paper, the authors are able to describe arithmetically Cohen-Macaulay varieties which need a defining equation of degree $\left] \frac{s}{n} \right[$. It turns out that these varieties are closely related to the Castelnuovo varieties recently studied by J. Harris. The result of the second part of this paper has been improved by P. Maroscia and W. Vogel.

## REFERENCES

[G] A. V. Geramita, *The equations defining arithmetically Cohen-Macaulay varieties of dimension 1*, The Curves Seminar at Queen's Vol. III, Queen's Papers in Pure and Applied Mathematics, Queen's University at Kingston 1984.

[G – M] A. V. Geramita and P. Maroscia, *The ideal of forms vanishing at a finite set of points in $P^n$*, J. Algebra.

[G – H] P Griffiths and J. Harris, *Residues and zero-cycles on algebraic varieties*, Ann. Math. 108 (1978), 461 – 505.

[H] J. Harris, *Curves in projective space*, Les Presses de l'Université de Montreal 1982.

[M – V] P. Maroscia and W. Vogel, *On the defining equations of points in general position in $P^n$*, Math. Ann. 269 (1984), 183–189.

[Sch] P. Schenzel, *Über die freien Auflösungen extremaler* Cohen-Macaulay-Ringe, J. Algebra 64 (1980), 93–101.

[S – V] J. Stückrad and W. Vogel, *Toward a theory of Buchsbaum singularities*, Amer. J. Math. 100 1978), 727–746.

[T] R. Treger, *On equations defining arithmetically Cohen-Macaulay schemes II*, Duke Math. J. 48 (1981), 35–47.

[Tg1] N. V. Trung, *Über allgemeine Hyperflächenschnitte einer algebraischen Varietät*, Mh. Math. 89 (1980), 323–340.

[Tg2] N. V. Trung, *Spezialisierungen allgemeiner Hyperflächenschnitte und Anwendungen*, Seminar D. Eisenbud/ B. Singh/ W. Vogel, Vol. I, Teubner-Verlag, Leipzig 1980.

[Tg3] N. V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc., (to appear).

*Received December 5, 1986*

* INSTITUTE OF MATHEMATICS
P. O. BOX 631, BO HO, HANOI, VIETNAM
** DIPARTIMENTO DI MATEMATICA
UNIVERSITA DI GENOVA
VIA L. B. ALBERTI, 4, GENOVA, ITALY