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ON STATISTICS ANALOGOUS TO PEARSON'S CHI-SQUARE IN TESTING THE NULL HYPOTHESIS INCLUDING UNKNOWN PARAMETERS

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I. INTRODUCTION

E. Csáki and I. Vincze [2] investigated the test (1.4) which is analogous to Pearson's Chi-square, utilizing, apart from the sample frequencies lying in the partition intervals, the sample positions too. We consider here the similar test (1.11) under the null hypothesis $\mathscr{H}_0$ depending on s unknown parameters.

Let $X_1, \ldots, X_n$ be independent and have the common distribution function $F(x)$. Let $-\infty = x_0 < x_1 < x_2 < \ldots < x_r = \infty$ be a given partition of R^1 such that

$$p_i = P\{X_1 \in \Delta_i = (x_{i-1}, x_i)\} = F(x_i) - F(x_{i-1}) > 0,\ i = 1, \ldots, r,$$

and let $v_i = \text{card}\{j : X_j \in \Delta_i,\ j = 1, \ldots, n\}$, $i = 1, \ldots, r$.

It is well-known that

$$\chi^2 = \sum_{i=1}^{r} \frac{(v_i - np_i)^2}{np_i} \xrightarrow{\mathscr{L}} \chi^2(r-1) \tag{1.1}$$

as $n \to \infty$, where $\xrightarrow{\mathscr{L}}$ denotes the convergence in law.

Let

$$\begin{cases} e_i = E(X_1 | X_i \in \Delta_i), \\ \sigma_i^2 = V(X_1 | X_1 \in \Delta_i), \end{cases} \tag{1.2}$$

and

$$\overline{X}_i = \frac{1}{v_i} \sum_{X_j \in \Delta_i} X_j,\ i = 1, \ldots, r. \tag{1.3}$$

Csáki and Vincze [2] investigated the test

$$\overline{\chi}^2 = \sum_{i=1}^{r} \left(\frac{\overline{X}_i - e_i}{\sigma_i}\right)^2 \cdot v_i \tag{1.4}$$

and showed that under F

$$\overline{\chi}^2 \xrightarrow{\mathcal{L}} \chi^2(r) \quad \text{as} \quad n \to \infty, \tag{1.5}$$

provided that $p_i > 0$ and $\sigma_i^2 > 0$, $i = 1, \dots, r$ (cf. Th. 2.1 [2]) and

$$\frac{\overline{\chi}^2 - r}{\sqrt{2r}} \xrightarrow{\mathcal{L}} N(0,1) \tag{1,6}$$

as $n \to \infty$ and $r = O(n^\alpha)$, $0 < \alpha < 1$, (cf. Th. 3.1[2]).

Now suppose that the common d. f. of X_is is

$$F(x, \theta) = F(x, \theta_1, \dots, \theta_s), \tag{1.7}$$

where $\theta = (\theta_1, \dots, \theta_s)^T$, $s < r$, is a vector of unknown parameters taking values in an open set $\Theta \subset R^s$. Then

$$p_i = p_i(\theta) = P_\theta\{X_1 \in \Delta_i\} = F(x_i, \theta) - F(x_{i-1}, \theta), \quad i = 1, \dots, r.$$

It is well-known that (see [1], [3]) under suitable assumptions the equations

$$\sum_{i=1}^{r} \frac{\nu_i}{p_i(\theta)} \cdot \frac{\partial p_i}{\partial \theta_j} = 0, \; j = 1, \dots, s \tag{1.8}$$

have a uniquely consistent solution $\widetilde{\theta} = (\widetilde{\theta}_1, \dots, \widetilde{\theta}_s)^T$ and

$$\widetilde{\chi}^2 = \sum_{i=1}^{r} \frac{(\nu_i - np_i(\widetilde{\theta}))^2}{np_i(\widetilde{\theta})} \xrightarrow{\mathcal{L}} \chi^2(r-s-1), \tag{1.9}$$

as $n \to \infty$.

The aim of this paper is to study the asymptotic behaviour of the test similar to (1.4) with F given by (1.7).

Let

$$\begin{cases} p_i = p_i(\theta) = F(x_i, \theta) - F(x_{i-1}, \theta), \\ e_i = e_i(\theta) = E_\theta\{X_1 \mid X_1 \in \Delta_i\}, \\ \sigma_i^2 = \sigma_i^2(\theta) = V_\theta\{X_1 \mid X_1 \in \Delta_i\}, \; i = 1, \dots, r \end{cases} \tag{1.10}$$

Then the test under consideration is of the form

$$\overline{\chi}^2(\widehat{\theta}) = \sum_{i=1}^{r} \left(\frac{\overline{X}_i - e_i(\widehat{\theta})}{\sigma_i(\widehat{\theta})} \right)^2 \cdot \nu_i \tag{1.11}$$

where $\widehat{\theta}$ is an estimator of θ. It will be show that $\overline{x}^2(\widehat{\theta}) \xrightarrow{\mathcal{L}} \chi^2(r-s)$ as $n \to \infty$ and r is fixed (Th. 2.1), and $(\overline{\chi}^2(\widehat{\theta}) - r) / \sqrt{2r} \xrightarrow{\mathcal{L}} N(0, 1)$ as $n \to \infty$ and $r = O(n^\alpha)$, $0 < \alpha < 1$ (Th. 2.2).

For any function $f(\theta)$ let us adopt the following notations

$$\partial_i f = \frac{\partial f(\theta)}{\partial \theta_i}, \quad \partial_{ij} f = \frac{\partial^2 f(\theta)}{\partial \theta_i \, \partial \theta_j},$$

$$\partial_i^0 f = \partial_i f \Big|_{\theta=\theta_0}, \quad \partial_{ij}^0 f = \partial_{ij} f \Big|_{\theta=\theta_0}, \quad i, j = 1, \dots, s,$$

$$f^0 = f(\theta_0), \ \widehat{f} = f(\widehat{\theta}).$$

When r is fixed and n tends to infinite we can state

THEOREM 2.1: *Let Θ be an open set in R^s. Let $\theta_0 = (\theta_1^0, \dots, \theta_s^0)^T$ be a fixed point in Θ and let $\mathscr{H}_0$ denote the null hypothesis: $\theta = \theta_0$. Assume that for $i = 1, \dots, r$, $j, k = 1, \dots, s$:*

(i) $p_i^0 = p_i(\theta_0) > 0$,

(ii) $\sigma_i^2(\theta) \geq C^2$ *for some constant* $C > 0$, *and* $\sigma_i^2(\theta)$ *and* $\partial_j \sigma_i^2(\theta)$ *are continuous in* Θ,

(iii) $e_i(\theta)$, $\partial_j e_i(\theta)$, $\partial_{jk} e_i(\theta)$ *are continuous in* Θ,

(iv) *The rank of the matrix* $E = (E_{ij}) = (\partial_j^0 e_i)$ *equals* s: *rank* $(E) = s$.

Then

(a) *There exists a solution* $\widehat{\theta} = \widehat{\theta}_n$ *of*

$$\sum_{i=1}^{r} \frac{\overline{X}_i - e_i(\theta)}{\sigma_i^2(\theta)} \, v_i \, \partial_j e_i = 0, \quad j = 1, \dots, s \tag{2.1}$$

such that under $\mathscr{H}_0$

$$P_{\theta_0}\text{-}\lim_{n \to \infty} \widehat{\theta} = \theta_0, \quad \text{i.e.} \quad \widehat{\theta} \xrightarrow{P_{\theta_0}} \theta_0. \tag{2.2}$$

The solution $\widehat{\theta}$ *is unique in the sense that if* $\overline{\theta}$ *is any other solution of (2.1) and* $\overline{\theta} \xrightarrow{P_{\theta_0}} \theta_0$ *then*

$$P_{\theta_0}(\widehat{\theta} = \overline{\theta}) \to 1 \text{ as } n \to \infty.$$

(b) *Under* $\mathscr{H}_0$

$$\overline{\chi}^2(\widehat{\theta}) = \sum_{i=1}^{r} \left(\frac{\overline{X}_i - \widehat{e}_i}{\widehat{\sigma}_i} \right)^2 \cdot v_i \xrightarrow{\mathcal{L}} \chi^2(r-s) \tag{2.3}$$

as $n \to \infty$.

Proof. Put

$$X_{0n} = \left[\frac{\bar{X}_1 - e_1^0}{\sigma_1^0}\sqrt{\nu_1}, \ldots, \frac{\bar{X}_r - e_r^0}{\sigma_r^0}\sqrt{\nu_r}\right]^T, \tag{2.4}$$

$$X_n = \left[\frac{\bar{X}_1 - \hat{e}_1}{\hat{\sigma}_1}\sqrt{\nu_1}, \ldots, \frac{\bar{X}_r - \hat{e}_r}{\hat{\sigma}_r}\sqrt{\nu_r}\right]^T \tag{2.5}$$

$$\left\{\begin{aligned} B_n &= (B_{ij}^n) = \left[\frac{\sqrt{\nu_i/n}}{\sigma_i^0}\,\partial_j^0 e_i\right], \; i = 1,\ldots, r, \; j = 1,\ldots, s \\ B &= (B_{ij}) = \left[\frac{\sqrt{p_i^0}}{\sigma_i^0}\,\partial_j^0 e_i\right], \; i = 1,\ldots, r, \; j = 1,\ldots, s. \end{aligned}\right. \tag{2.6}$$

By the Chebyshev inequality:

$$P_{\theta_0}\left(\left|\nu_i - np_i^0\right| \leqslant \frac{np_i^0}{2}, \; 1 \leqslant i \leqslant r\right) > 1 - \frac{a}{n},$$

where $a = \sum_{i=1}^{r} (4q_i^0 / p_i^0)$, $q_i^0 = 1 - p_i^0$.

Therefore, from (i) it follows that under $\mathcal{H}_0$

$$\nu_i / n \geqslant p_i^0 / 2, \; i = 1,\ldots, r \tag{2.7}$$

with probability larger than $Q = 1 - \frac{a}{n}$ for each $n \geqslant 1$ (abbr.: with $Q \mid \mathcal{H}_0$ for $n \geqslant 1$).

Put

$$D_n = \operatorname{diag}\left(\frac{\sqrt{\nu_1/n}}{\sigma_1^0}, \ldots, \frac{\sqrt{\nu_r/n}}{\sigma_r^0}\right),$$

$$D = \operatorname{diag}\left(\frac{\sqrt{p_1^0}}{\sigma_1^0}, \ldots, \frac{\sqrt{p_r^0}}{\sigma_r^0}\right).$$

Then

$$\left\{\begin{aligned} & B_n = D_n E, \; B = DE, \; \operatorname{rank}(B) = \operatorname{rank}(E) = s, \\ & \operatorname{rank}(B^T B) = s, \; |B^T B| \neq 0, \\ & D_n \xrightarrow[n]{P_{\theta_0}} D, \; B_n \xrightarrow[n]{P_{\theta_0}} B, \end{aligned}\right. \tag{2.8}$$

and

$$\operatorname{rank}(B_n) = s, \operatorname{rank}(B_n^T B_n) = s, \ | B_n^T B_n | \neq 0 \tag{2.9}$$

with $Q \mid \mathscr{H}_0$ for $n \geqslant 1$.

Let us rewrite (2.1) as

$$\sum_{k=1}^{s} (\theta_k - \theta_k^0) \sum_{i=1}^{r} \frac{v_i/n}{\sigma_i^{02}} \partial_j^0 e_i \partial_k^0 e_i =$$

$$= n^{-1/2} \sum_{i=1}^{r} \frac{\sqrt{v_i/n}}{\sigma_i^0} \partial_j^0 e_i \frac{\overline{X}_i - e_i^0}{\sigma_i^0} \sqrt{v_i} + w_j(\theta), \ 1 \leqslant j \leqslant s, \tag{2.10}$$

where

$$w_j(\theta) = \sum_{i=1}^{r} (\overline{X}_i - e_i^0) \frac{v_i}{n} \left(\frac{1}{\sigma_i^2} \partial_j e_i - \frac{1}{\sigma_i^{02}} \partial_j^0 e_i \right) +$$

$$- \sum_{i=1}^{r} \frac{v_i}{n} (e_i - e_i^0) \left[\frac{1}{\sigma_i^2} \partial_j e_i - \frac{1}{\sigma_i^{02}} \partial_j^0 e_i \right] +$$

$$- \sum_{i=1}^{r} \frac{v_i/n}{\sigma_i^{02}} \partial_j^0 e_i \left[e_i - e_i^0 - \sum_{k=1}^{s} (\theta_k - \theta_k^0) \partial_k^0 e_i \right],$$

or in matrix form

$$B_n^T B_n (\theta - \theta_0) = n^{-1/2} B_n^T X_{on} + w(\theta) \quad , \tag{2.11}$$

where $w(\theta) = \left(w_1(\theta), \ldots, w_s(\theta)\right)^T$.

Then, by (2.9), the equations (2.1) are equivalent to

$$\theta = \theta_0 + n^{-1/2} (B_n^T B_n)^{-1} B_n^T X_{on} + (B_n^T B_n)^{-1} w(\theta) \tag{2.12}$$

with $Q \mid \mathscr{H}_0$ for $n \geqslant 1$.

Now construct a sequence $\{\theta_m\}$ such that

$$\theta_m = \theta_0 + n^{-1/2} (B_n^T B_n)^{-1} B_n^T X_{on} + (B_n^T B_n)^{-1} w(\theta_{m-1}), \ m \geqslant 1, \tag{2.13}$$

Noting that $w(\theta_0) = 0$, it follows from (2.13) that

$$\begin{cases} \theta_1 - \theta_0 = n^{-1/2} (B_n^T B_n)^{-1} B_n^T X_{on} \\ \theta_{m+1} - \theta_m = B_n^T B_n)^{-1} [w(\theta_m) - w(\theta_{m-1})], \ m \geqslant 1. \end{cases} \tag{2.14}$$

Let $\{\lambda = \lambda(n, \alpha)\}$ be a sequence of constants satisfying

$$\lambda \to \infty \qquad \text{and } \lambda / n^{\frac{\alpha}{4}} \to 0 \tag{2.15}$$

for some α, $0 < \alpha < 1$, as $n \to \infty$.

By the Chebyshev inequality one obtains from (2.7)

$$P_{\theta_o}\left(|\overline{X}_i - e_i^o| < \frac{\lambda}{\sqrt{n}}, \frac{v_i}{n} \geqslant \frac{p_i^o}{2}, v_i \geqslant 1, i = 1, \dots, r\right) \tag{2.16}$$

$$\geqslant 1 - \lambda^{-2} \sum_{i=1}^{r} \frac{\sigma_i^{o2}}{v_i/n} - \frac{a}{n}$$

$$\geqslant 1 - 2\lambda^{-2} \sum_{i=1}^{r} \frac{\sigma_i^{o2}}{p_i^o} - \frac{a}{n} = 1 - b\lambda^{-2} - \frac{a}{n}$$

$$\geqslant 1 d\lambda^{-2} := Q_\lambda$$

for $n \geqslant N_1 \geqslant 1$, where $b = 2\sum_{i=1}^{r} \frac{\sigma_i^{o2}}{p_i^o}$, and $d > b$ is given, while $N_1 \geqslant 1$ can be chosen such that

$$b\lambda^{-2} + \frac{a}{n} \leqslant d\lambda^{-2}, \text{ for } n \geqslant N_1, \text{ e. g. } N_1 = \max\left\{1, \left[\frac{\lambda^2 a}{d-b} + 1\right]\right\}.$$

Therefore, by (ii), for $n \geqslant N_1$

$$P_{\theta_o}\left(\left|\frac{\overline{X}_i - e_i^o}{\sigma_i^o} \sqrt{v_i}\right| < \frac{\lambda}{C}, i = 1, \dots, r\right) \geqslant Q_\lambda . \tag{2.17}$$

It follows from (i), (2.14), (2.4), (2.6), (2.8), (2.9), (2.16) and (2.17) that

$$\| \theta_1 - \theta_o \| \leqslant K_1 \lambda/\sqrt{n} \text{ with } Q_\lambda \mid \mathcal{H}_o \text{ for } n \geqslant N_1, \tag{2.18}$$

where K_1 is a positive constant independent of n.

Now using the Taylor expansion for each $w_j(\theta') - w_j(\theta'')$, $\theta', \theta'' \in U(\theta_o)$ a closed ball of centre θ_o, one obtains from (ii), (iii), (2.16)

$$|w_j(\theta') - w_j(\theta'')| \leqslant \|\theta' - \theta''\| K_2 \left(\|\theta' - \theta_o\| + \|\theta'' - \theta_o\| + \lambda/\sqrt{n}\right)$$

with $Q_\lambda \mid \mathcal{H}_o$ for $n \geqslant N_1$, where K_2 is independent of n. Therefore

$$\| w(\theta') - w(\theta'')\| \leqslant \sqrt{s} K_2 \|\theta' - \theta''\| \left(\|\theta' - \theta_o\| + \|\theta'' - \theta_o\| + \lambda/\sqrt{n}\right), \tag{2.19}$$

with $Q_\lambda \mid \mathcal{H}_o$ for $n \geqslant N_1$.

From (2.14) one has, assuming that $\theta_m, \theta_{m-1} \in U(\theta_o)$,

$$\|\theta_{m+1} - \theta_m\| \leqslant K_3 \|\theta_m - \theta_{m-1}\| \left(\|\theta_m - \theta_o\| + \|\theta_{m-1} - \theta_o\| + \lambda/\sqrt{n}\right), \tag{2.20}$$

$m \geqslant 1$, with $Q_\lambda \mid \mathcal{H}_o$ for $n \geqslant N_1$, where K_3 is independent of n.

However from (2.18), (2.19) and (2.20) one can choose $N_2 \geqslant N_1$ such that for $n \geqslant N_2$ the sequence $\{\theta_m\}$ belongs to $U(\theta_o)$ with $Q_\lambda \mid \mathcal{H}_o$. Then (2.20) is satisfied with $Q_\lambda \mid \mathcal{H}_o$ for $n \geqslant N_2$.

By induction one can verify that

$$\| \theta_{m+1} - \theta_M \| \leqslant K_1[(4K_1 + 1)K_3]^m [\lambda/\sqrt{n}]^{m+1}, m = 0, 1, 2, \ldots \tag{2.11}$$

and

$$\| \theta_m - \theta_o \| \leqslant 2K_1 \lambda/\sqrt{n}, m = 1, 2, \ldots \tag{2.22}$$

with $Q_\lambda \mid \mathcal{H}_o$ for $n \geqslant N_3$, where $N_3 \geqslant N_2$ is chosen such that $(4K_1 + 1) K_3 \lambda/\sqrt{N_3} < 1/2$.

Hence

$$\theta_m = \theta_o + \sum_{i=0}^{m} (\theta_{i+1} - \theta_i) \to \hat{\theta} := \theta_o + \sum_{i+1}^{\infty} (\theta_{i+1} - \theta_i) \tag{2.23}$$

as $m \to \infty$, and $\| \hat{\theta} - \theta_o \| \leqslant 2K_1 \lambda/\sqrt{n}$ with $Q_\lambda \mid \mathcal{H}_o$, $n \geqslant N_3$.

From (2.12), (2.13), (2.23) one can conclude that $\hat{\theta}$ is a solution of (2.1) and $\hat{\theta} \in U(\theta_o)$ with $Q_\lambda \mid \mathcal{H}_o$, $n \geqslant N_3$.

Since Q_λ, as defined in (2.16), converges to 1 as $n \to \infty$, (2.23) implies (2.2).

Suppose there is another solution $\tilde{\theta}$ of (2. 1) such that $\tilde{\theta} \overset{P_{\theta_0}}{\to} \theta_0$. Then $\tilde{\theta} \in U(\theta_0)$ with $1 - \varepsilon \mid \mathcal{H}_0$ for $n \geqslant N_\varepsilon \geqslant 1$, where N_ε can be chosen for a given $\varepsilon > 0$. It follows from (2. 12) and (2. 19) that

$$\| \hat{\theta} - \tilde{\theta} \| = \| (B_n^T B_n)^{-1} [w(\hat{\theta}) - w(\tilde{\theta})] \| \leqslant K \| \hat{\theta} - \tilde{\theta} \| (\| \hat{\theta} - \theta_0 \| +$$

$$+ \| \tilde{\theta} - \theta_0 \| + \lambda/\sqrt{n}),$$ with $Q_\lambda - \varepsilon \mid \mathcal{H}_0$ for $n \geqslant N^* = \max(N_\varepsilon, N_3)$, where $K = \sqrt{S} K_2$. Since $\hat{\theta}$ and $\tilde{\theta}$ both tend to θ_0 in probability under $\mathcal{H}_0$ and $\lambda/\sqrt{n}$ tends to 0 as $n \to \infty$, the above inequality is valid iff $\| \hat{\theta} - \tilde{\theta} \| = 0$ for sufficiently large n. Thus $P_{\theta_0} (\hat{\theta} - \tilde{\theta}) \geqslant Q_\lambda - \varepsilon$ for $n > N^*$.

This proves the uniqueness of the solution.

From (2 . 14), (2. 21), and (2. 23) one can write

$$\| \sqrt{n} (\hat{\theta} - \theta_0) - (B_n^T B_n)^{-1} B_n^T X_{on} \| = \sqrt{n} \| \sum_{m=1}^{\infty} (\theta_{m+1} - \theta_m) \| \leqslant K^* \lambda^2 / \sqrt{n}$$

with $Q_\lambda \mid \mathcal{H}_0$ for $n \geqslant N_3$, where K^* is some constant independent of n. Then

$$\sqrt{n} (\hat{\theta} - \theta_0) \overset{P_{\theta_0}}{\approx} (B_n^T B_n)^{-1} B_n^T X_{on}, \tag{2. 24}$$

where $A \overset{P_{\theta_0}}{\approx} B$ stands for $P_{\theta_0}\text{-}\lim_{n \to \infty} (A - B) = 0$.

Let us now examine the asymptotic behaviour of X_n. We can write

$$\frac{\overline{X}_i - \widehat{e}_i}{\widehat{\sigma}_i}\sqrt{v_i} = \frac{\sigma_i^o}{\widehat{\sigma}_i}\left[\frac{\overline{X}_i - e_i^o}{\sigma_i^o}\sqrt{v_i} - \sqrt{n}\,\frac{\sqrt{v_i/n}}{\sigma_i^o}(\widehat{e}_i - e_i^o)\right],\ i = 1,\dots,r.$$

By (iii), using the Taylor expansion, one can write

$$\widehat{e}_i - e_i^o = \sum_{j=1}^{s} \partial_j^o e_i\,(\widehat{\theta}_j - \theta_j^o) + 0(\|\widehat{\theta} - \theta_0\|^2),\ i = 1,\dots,r.$$

Therefore

$$\frac{\overline{X}_i - \widehat{e}_i}{\widehat{\sigma}_i}\sqrt{v_i} =$$

$$= \frac{\sigma_i^o}{\widehat{\sigma}_i}\left[\frac{\overline{X}_i - e_i^o}{\sigma_i^o}\sqrt{v_i} - \sqrt{n}\,b_i(\widehat{\theta} - \theta_o) + \sqrt{n}.0(\|\widehat{\theta} - \theta_o\|^2)\right],\ i = 1,\dots,r,$$

where b_i is the i-th row of B_n.

From (ii) and (2.2) it follows that

$$\widehat{\sigma}_i = \sigma_i(\widehat{\theta}) \xrightarrow{P_{\theta_0}} \sigma_i(\theta_0) = \sigma_i^o,\ i = 1,\dots,r.$$

Further, by (2.23), one has

$$\sqrt{n}.0(\|\widehat{\theta} - \theta_o\|^2) \leqslant 0(\lambda^2/\sqrt{n}) \text{ with } Q_\lambda \mid \mathcal{H}_o,\ n \geqslant N_\varepsilon.$$

Hence

$$\frac{\overline{X}_i - \widehat{e}_i}{\widehat{\sigma}_i}\sqrt{v_i} \overset{P_{\theta_0}}{\approx} \frac{\overline{X}_i - e_i^o}{\sigma_i^o}\sqrt{v_i} - \sqrt{n}\,b_i(\widehat{\theta} - \theta_o),\ i = 1,\dots,r,$$

or, in matric form, by taking account of (2. 24) and (2. 8),

$$X_n \overset{P_{\theta_0}}{\approx} X_{on} - \sqrt{n}\,B_n(\widehat{\theta} - \theta_o)$$

$$\overset{P_{\theta_0}}{\approx} X_{on} - B_n(B_n^T B_n)^{-1} B_n^T X_{on} = [I_r - B_n(B_n^T B_n)^{-1} B_n^T]\,X_{on}$$

$$\overset{P_{\theta_0}}{\approx} [I_r - B(B^T B)^{-1} B^T]\,X_{on}, \tag{2.25}$$

where I_r is the unit matrix of order r.

It follows from the proof of Th. 2. 1 in [2] that

$$X_{on} \xrightarrow{\mathcal{L}} Y = (Y_1,\dots,Y_r)^T, \tag{2.26}$$

where $Y_1,\dots,Y_r$ are mutually independent and have the common distribution $N(0, 1)$.

Since $A = I_r - B(B^T B)^{-1} B^T$ is idempotent and hence $\text{rank}(A) = \text{trace}(A) = r - \text{trace}[B(B^T B)^{-1} B^T] = r - \text{trace}[B^T B(B^T B)^{-1}] = r - \text{trace}(I_s) = r - s$, one can conclude from (2.5), (2.25) and (2.26):

$$\overline{\chi^2}(\widehat{\theta}) = X_n^T X_n \overset{\mathcal{L}}{\approx} X_{on}^T A X_{on} \overset{\mathcal{L}}{\rightarrow} Y^T A Y \,\mathcal{L}\, \chi^2(r-s),$$

where $\overset{\mathcal{L}}{\approx}$ means «having the same limit law». We have thus established (2. 3). Q. E. D.

THEOREM 2.2 *Let $r = r_n = O(n^\alpha)$, where $0 < \alpha < 1$. Let the assumptions (i) — (iv) of Theorem 2. 1 hold for every sufficiently large n and for $r = r_n$.*

Assume further

(v) $c_1 / r \leqslant p_i \leqslant c_2 / r$, $i = 1, \ldots, r$,

where c_1 and c_2 are constants, $0 < c_1 \leqslant 1 \leqslant c_2 < \infty$,

(vi) $\frac{1}{r} \sum_{i=1}^{r} \left[\mu_i^{(k)} / (\sigma_i^o)^k\right] < c_3$, *for $k = 4$ and 6, where c_3 is constant and*

$$\mu_i^{(k)} = E_{\theta_0}\left\{\left|X_1 - e_i^o\right|^k \,\middle|\, X_1 \in \Delta_i\right\}, \quad i = 1, \ldots, r.$$

Then under $\mathcal{H}_0$

$$\frac{\overline{\chi}^2(\widehat{\theta}) - r}{\sqrt{2r}} \overset{\mathcal{L}}{\longrightarrow} N(0,1) \text{ as } n \to \infty. \qquad (2.27)$$

Proof. From (2.25) one has

$$\frac{\overline{\chi^2}(\widehat{\theta}) - r}{\sqrt{2r}} = (1/\sqrt{2r})(X_n^T X_n - r)$$

$$\overset{\mathcal{L}}{\approx} (1/\sqrt{2r})\left(X_{on}^T [I_r - B(B^T B)^{-1} B^T] X_{on} - r\right) = T_1 - T_2,$$

where

$$T_1 = (1/\sqrt{2r})(X_{on}^T X_{on} - r),$$

$$T_2 = (1/\sqrt{2r}) X_{on}^T B (B^T B)^{-1} B^T X_{on} = (1/\sqrt{2r}) \| B (B^T B)^{-1} B^T X_{on} \|^2.$$

By Th. 3. 1 in [2] one obtains under $\mathcal{H}_0$

$$T_1 \overset{\mathcal{L}}{\rightarrow} N(0, 1) \text{ as } n \to \infty. \qquad (2. 28).$$

Since (i), (ii), (iii), and (2. 17) one can write

$$T_2 = 0(\lambda^2/\sqrt{r}) = 0(\lambda^2/n^{\frac{\alpha}{2}}) \text{ with } Q_\lambda \mid \mathcal{H}_0, n \geqslant N_1.$$

Hence, noting that $Q_\lambda \to l$ as $n \to \infty$, one can conclude by (2.15)

$$T_2 \xrightarrow[\theta]{P_\theta} 0. \tag{2.29}$$

Clearly (2.27) now follows from (2.28) and (2.29). Q.E.D.

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