IMPLICIT FUNCTION THEOREMS FOR SET-VALUED MAPS

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Let P be a topological space, X and Y be Banach spaces, and $F : P \times X \to Y$ be a set-valued map. Then the set-valued map $G : P \to X$

$$G(p) := \{ x \mid 0 \in F(p, x)\}$$

will be called the *implicit map* defined by the inclusion $0 \in F(p, x)$. The purpose of this paper is to study the behaviour of $G(.)$ near a given point (p_0, x_0) satisfying the condition $0 \in F(p_0, x_0)$. To this end, we shall use some results from P. H. Sach's theory of prederivatives of set-valued maps as developed in [9].

In the first part of the paper we are concerned with the lower semicontinuity property of implicit maps which is closely related to the stability of a system of inequalities (see for example [7]). From Theorem 2. 1 to be proved in this part we recover a result established by H. Methlouthi [6]. This theorem is also near to the implicit function theorem of S.M. Robinson [7], but does not imply the latter.

In the second part of the paper we are interested in the pseudo—Lipschitz property of set-valued maps introduced in [1]. This property was established in [1] and [2] for the inverse of set-valued maps taking values in a finite-dimensional space. Our result (Theorem 3. 1) differs from [1, 2] in that it is derived for Banach spaces by using prederivatives [9] instead of the Clarke tangent cones to the graphs of set-valued maps. Moreover, a sufficient condition for the pseudo-Lipschitz property of implicit maps will be obtained. As a by-product, we shall also have a result on the stability of perturbed nonsmooth inequalities which can be interpreted as the stability of the feasible set of a nonsmooth mathematical programming problem.

1. PRELIMINARIES

1. Given a set-valued map $F : X \to Y$ from a topological space X into a Banach space Y, we denote its graph and support function by gr F and $c_F(y^*, .)$, respectively. Recall that gr $F := \{(x, y) \in X \times Y \mid y \in F(x)\}$,

$C_F\{(y^*, x) := \sup < y^*, y > \mid y \in F(x)\}$, where y^* belongs to the dual space Y^* of Y and $< y^*, y >$ denotes the canonical pairing between Y and Y^*.

In what follows, the symbols $d[a, M]$, $\overline{B}(a, \delta)$ and $N(a)$ will denote the distance from a point a to a set M, the closed ball with radius δ centered at a and the collection of all open neighbourhoods of a, respectively.

DEFINITION 1. 1, [3]. We say that F is *lower semicontinuous* (l. s. c.) at $x_0 \in X$ if for every $y_0 \in F(x_0)$ and every $\varepsilon > 0$ there exists $U \in N(x_0)$ such that $\forall x \in U \ \exists y \in F(x)$ satisfying $\| y - y_0 \| < \varepsilon$. F is *upper semicontinuous* (u.s.c.) at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $U \in N(x_0)$ such that $F(x) \subset F(x_0) + \varepsilon \overline{B}_Y(0, 1)$ for all $x \in U$. F is u.s.c. (resp. l.s.c.) on a set $A \subset X$ if it is u.s.c. (resp. l.s.c.) at every point of A. When X is a Banach space we say that F is *positively homogeneous* if $F(\lambda x) = \lambda F(x)$ for any $\lambda \geqslant 0$ and $x \in X$.

2. Now suppose that both X and Y are Banach spaces and denote by $\mathcal{L}$ the collection of all set-valued maps $t: X \to Y$ that are positively homogeneous and u.s.c. on X [9].

DEFINITION 1.2, [9]. The *Banach constant* of a $t \in \mathcal{L}$ is the number

$$p(t) := -\sup_{y^* \in S^*} \ \inf_{x \in \overline{B}_X(0,1)} c_t(y^*, x), \tag{1.1}$$

where S^* is the unit sphere in Y^* and $c_t(y^*, x) = \sup \{< y^*, y > \mid y \in t(x)\}$.

Remark 1.1. $p(t)$ is a finite real number which coincides with the ordinary Banach constant $C(t)$ of t if t is a bounded linear map. If t is generated by a convex compact set Δ of bounded linear operators from X into Y (so that $t(h) = \{Ah \mid A \in \Delta\}$), then $p(t) = \inf \{C(A) \mid A \in \Delta\}$.

DEFINITION 1.3. [9]. A map $t \in \mathcal{L}$ is called a *prederivative* of a set-valued map $F: X \to Y$ at a point $z_0 = (x_0, y_0) \in \text{gr}\, F$ if for every $\varepsilon > 0$ there exists $U \in N(x_0)$ such that: $\forall x \in U \ \exists y \in F(x)$ satisfying

$$< y^*, y - y_0 > - c_t(y^*, x - x_0) \leqslant \varepsilon \| x - x_0 \| \tag{1.2}$$

for all y^* from the unit ball B^* of Y^*.

Remark 1.2. Assume that F is single-valued, then a linear continuous map t is a prederivative of F at $(x_0, F(x_0))$ iff it is the Frechet derivative of F at x_0.

Remark 1.3. A linear continuous map A is a prederivative of a set-valued map F at (x_0, y_0) iff A is a lower-derivative of F at (x_0, y_0) in the sense of H. Methlouthi [6]. In this case we simply call A a linear prederivative of F at (x_0, y_0).

DEFINITION 1.4. [9]. A positively homogeneous upper semicontinuous function $g: X \to R$ is an *upper ε-approximation* of a function $f: X \to R$ at a point $x_0 \in X$ if there is $U \in N(x_0)$ such that

$$f(x) - f(x_0) \leqslant g(x - x_0) + \varepsilon \| x - x_0 \|$$

for all $x \in U$.

The following two lemmas will play a fundamental role in our proofs.

LEMMA. 1.1, [9]. *Assume that $F: X \to Y$ is a set-valued map, and $t \in \mathcal{L}$ is a prederivative of F at $z_0 = (x_0, y_0) \in gr\ F$. Suppose that the norm-functional in Y is Frechet differentiable at every point different from the origin. Let there be given a point $y' \notin F(x_0)$ such that y_0 is the closest point to y' in $F(x_0)$. Denote by $\tilde{y}^*$ the Frechet derivative at $y' - y_0$ of the norm-functional in Y. Then, for each $\varepsilon > 0$, the function $c_t(\tilde{y}^*...)$ is an upper ε-approximation of $f(.) = d[y', F(.)]$ at x_0.*

LEMMA 1.2. [5]. *Let V be a complete metric space, $f: V \to R_+ \cup \{+\infty\}$ be a lower semicontinuous real valued function. For every point $x_0 \in V$ satisfying $f(x_0) \leqslant \varepsilon$ and every $\lambda > 0$ there exists a point $\bar{x} \in V$ such that $f(\bar{x}) \leqslant f(x_0)$, $d(\bar{x}, x_0) \leqslant \lambda$ and $f(\bar{x}) \leqslant f(x) + \frac{\varepsilon}{\lambda} d(x, \bar{x})$ for every $x \in V$.*

2. LOWER SEMICONTINUITY OF IMPLICIT SET-VALUED MAPS

Let P be a topological space, X and Y be two Banach spaces and $F: P \times X \to Y$ be a set-valued map. Assume that the norm-functional in Y is Frechet differentiable at every point different from the origin. Throughout this section we suppose that $0 \in F(p_0, x_0)$ and, for any $(p, x) \in P \times X$, there is a point $y \in F(p, x)$ satisfying $\| y \| = d[0, F(p, x)]$.

THEOREM 2.1. *Suppose the following:*

(a) For every $x \in X$ the map $F(.,x)$ is l. s. c. on P and for every $p \in P$ the map $F(p,.)$ is u. s. c. on X;

(b) There exist neighbourhoods A and B of p_0 and x_0, respectively, such that for each $w = (p, x, y) \in gr\ F$, where $(p, x) \in A \times B$, the map $F(p, .)$ has a prederivative t_w at (x, y) and $p(t_w) \geqslant \gamma$, for some $\gamma > 0$.

Then, there exist neighbourhoods U and V of p_0 and x_0, respectively, such that

1. $\tilde{G}(p) := \{x \in V \mid 0 \in F(p, x)\}$ (2.1)

is nonempty for all $p \in U$;

2. The map $\widetilde{G} : U \to V$ is l. s. c. on U. (2.2)

Besides, if F is l. s. c. at (p_0, x_0), then there are neighbourhoods $\widetilde{U} \in N(p_0)$, $\widetilde{V} \in N(x_0)$ such that

3. $d[x, G(p)] \leqslant \frac{1}{\gamma} d[0, F(p, x)],$ (2.3)

where $(p, x) \in \widetilde{U} \times \widetilde{V}$ *and* $G(p) = \{x \in X \mid 0 \in F(p, x)\}$.

Proof. Select a number $\delta > 0$ such that $\overline{B}(x_0, \delta) \subset B$. Since $0 \in F(p_0, x_0)$ and $F(., x_0)$ is l. s. c. on P there is $A_1 \in N(p_0)$ such that : $\forall p \in A_1 \exists y_p \in F(p, x_0)$ satisfying $\| y_p \| < \gamma\delta$.

For each $p \in U := A \cap A_1$, consider the restriction of the function $v_p(x) = d[0, F(p, x)]$ to the ball $\overline{B}(x_0, \delta)$. From the u. s. c. property of $F(p,.)$ it follows that $v_p(.)$ is a lower semicontinuous real valued function. Besides, $v_p(x) \geqslant 0$ and $v_p(x_0) = d[0, F(p, x_0)] \leqslant \| y_p \| < \gamma\delta$. Hence, $v_p(x_0) < \gamma'\delta$ for some $\gamma' \in (0, \gamma)$. We set $\varepsilon = \gamma - \gamma'$. By Lemma 2.2 there is a point $\overline{x} \in \overline{B}(x_0, \delta)$ such that

$$v_p(\overline{x}) \leqslant v_p(x_0), \quad \| \overline{x} - x_0 \| < \delta, \tag{2.4}$$

$$v_p(\overline{x}) \leqslant v_p(x) + \gamma' \| x - \overline{x} \|, \ \forall x \in \overline{B}(x_0, \delta). \tag{2.5}$$

We now show that $v_p(\overline{x}) = 0$, i.e. $0 \in F(p, \overline{x})$. Indeed, otherwise we could find a point $\overline{y} \in F(p, \overline{x})$ such that $0 \neq \| \overline{y} \| = d[0, F(p, \overline{x})]$. By assumption (b) there is a prederivative t of $F(p,.)$ at $(\overline{x}, \overline{y})$ such that

$$p(t) \geqslant \gamma. \tag{2.6}$$

According to Lemma 2.1, if we denote by $\widetilde{y}^*$ the Frechet derivative of the norm-functional in Y at $\overline{y}$ then $c_t(y^*,.)$ is an upper $\frac{\varepsilon}{2}$ — approximation of $v_p(x)$ at $\overline{x}$. Hence, using (2.4), for sufficiently small $\mu > 0$ we have $\overline{B}(\overline{x}, \mu) \subset \overline{B}(x_0, \delta)$ and

$$v_p(x) \leqslant v_p(\overline{x}) + c_t(\widetilde{y}^*, x - \overline{x}) + \frac{\varepsilon}{2} \| x - \overline{x} \| \tag{2.7}$$

for all $x \in \overline{B}(\overline{x}, \mu)$.

Combining (2.5) with (2.7) yields

$$c_t(y^*, x - \overline{x}) + \left(\gamma' + \frac{\varepsilon}{2}\right) \| x - \overline{x} \| \geqslant 0 \ (\forall x \in \overline{B}(\overline{x}, \mu)). \tag{2.8}$$

Deviding both sides of (2.8) by $\| x - \overline{x} \|$ and taking account of the fact that t is positively homogeneous, we get $c_t(\tilde{y}^*, x) + \left(\gamma' + \frac{\varepsilon}{2}\right) \geqslant 0$, for every x from the unit sphere S of X.

Then

$$\inf_{x \in S} c_t(\tilde{y}^*, x) + \left(\gamma' + \frac{\varepsilon}{2}\right) \geqslant 0. \tag{2.9}$$

Noting that $\tilde{y}^* \in S^*$ and that the inclusion $x \in \overline{B}_X(0, 1)$ in (1.1) can be replaced by $x \in S$, we obtain from (2.6)

$$0 \geqslant \gamma + \inf_{x \in S} c_t(\tilde{y}^*, x). \tag{2.10}$$

Upon adding (2.9) and (2.10) we then have $\gamma' + \frac{\varepsilon}{2} \geqslant \gamma$, which is a contradiction.

We have thus proved that for each $p \in U$ there is a point $\overline{x}$ such that $\| \overline{x} - x_o \| < \delta$ and $0 \in F(p, \overline{x})$. Now it is clear that (2.1) holds if we set $V = \{x \in X \mid x - x_o \| < \delta\}$.

To prove (2.2) we take arbitrary $p \in U$, $x \in \tilde{G}(p)$. We have to show that, for every $\tau > 0$, there exists $U' \in N(p)$ such that; $\forall p' \in U' \; \exists x' \in \tilde{G}(p')$ satisfying $\| x' - x \| \leqslant \tau$. Pick $\delta' \in (0, \tau)$ such that $\overline{B}(x, \delta') \subset \overline{B}(x_o, \delta)$. Arguing as above one can find $U' \in N(p)$ such that: $\forall p' \in U' \; \exists \; x' \in \overline{B}(x_o, \delta')$ satisfying $0 \in F(p', x')$. From this it follows that $x' \in \tilde{G}(p')$ and $\| x' - x \| < \tau$.

Turning to the proof of (2.3) we note that by the lower semicontinuity of F at (p_o, x_o) there exist neighbourhoods $\tilde{U} \in N(p_o)$, $\tilde{V} \in N(x_o)$ such that $\tilde{U} \subset U$,

$$\tilde{V} \subset \overline{B}\left(x_o, \frac{\delta}{2}\right) \text{ and } d[0, F(p, x)] < \frac{\gamma\delta}{2} \quad (\forall (p, x) \in \tilde{U} \times \tilde{V}). \tag{2.11}$$

Given any $(p, x) \in \tilde{U} \times \tilde{V}$ we set $\alpha = d[0, F(p, x)]$. All we have to show is that

$$d[x, G(p)] \leqslant \frac{\alpha}{\gamma}. \tag{2.12}$$

Since $\alpha < \frac{\gamma\delta}{2}$, by (2.11) we can pick $\lambda \in \left(\frac{2\alpha}{\delta}, \gamma\right)$. Defining $v_p(z) = d[0, F(p, z)]$ for all $z \in X$, one has $v_p(x) = \alpha < \frac{\alpha\gamma}{\lambda}$. Repeating the proof of (2.1) with x and $\frac{\alpha}{\lambda}$ playing the role of x_o and δ, we find $\overline{x} \in X$ such tha

$$\| x - \overline{x} \| < \frac{\alpha}{\lambda} \text{ and } 0 \in F(p, \overline{x}).$$

This means that $\overline{x} \in \tilde{G}(p)$, since

$$\| x_0 - \overline{x} \| \leqslant \| x_0 - x \| + \| x - \overline{x} \| < \frac{\delta}{2} + \frac{\alpha}{\lambda} \leqslant \delta.$$

Therefore, $d\,[x, \widetilde{G}(p)] \leqslant \| x - \overline{x} \| \leqslant \frac{\alpha}{\lambda}$. By letting $\lambda \to \gamma$ we obtain (2.12). This completes the proof of the theorem.

The following proposition is a « local version » of Theorem 2.1.

THEOREM 2.2. *The conclusions of Theorem 2.1 still hold if we replace condition (b) by the following one:*
(c) The map F is l.s.c at (p_o, x_o) and there exists $\gamma > 0$ such that, for every $w = (p, x, y) \in gr\, F$ in some neighbourhood of $(p_o, x_o, 0)$, one can find a prederivative t_w of $F(p,.)$ at (x, y) such that $p(t_w) \geqslant \gamma$.

We shall further need

DEFINITION 2. 1. For $t_1, t_2 \in \mathcal{L}$ the *excess* of t_1 over t_2 is the number

$$e(t_1, t_2) = \sup_{x \neq o} \sup_{a \in t_1(x)} \frac{d\,[a, t_2(x)]}{\| x \|}.$$

We say that a prederivative (resp. a linear prederivative) t of $F(p_0,.)$ at (x_0, p_0) has *property (C)* if for every $\varepsilon > 0$ there is a neighbourhood W of (p_0, x_0, y_0) such that for every $(p, x, y) \in W \cap \text{gr}\, F$ there exists a prederivative (resp. a linear prederivative) t' of $F(p,.)$ at (x, y) satisfying $e(t', t) < \varepsilon$.

It can easily be proved that $p(t_2) \leqslant p(t_1) + e(t_1, t_2)$.

In the remainder of this section we suppose that F has compact values in Y.

REMARK 2.1. It is a simple matter to show that condition (b) of Theorem 2.1 can be replaced by any one of the following conditions:

(b_1) F is u.s.c. at (p_0, x_0) and for every $y \in F(p_0, x_0)$ there is a prederivative t of $F(p_0,.)$ at (x_0, y) having property (C) and satisfying the inequality $p(t) > 0$.

(b_2) F is u.s.c. at (p_0, x_0) and for every $y \in F(p_0, x_0)$ there is a surjective linear map which is a prederivative of $F(p_0,.)$ at (x_0, y) having property (C).

(b_3) F is u.s.c. at (p_0, x_0) and for every $y \in F(p_0, x_0)$ there is a linear prederivative t of $F(p_0,.)$ at (x_0, y) having property (C). In addition, the map t_{w_0}, where $w_0 = (p_0, x_0, 0)$, is invertible and $\| t_{w_0}^{-1} \| . \| t_w - t_{w_0} \| < 1$ for each $w = (p_0, x_0, y)$, $y \in F(p_0, x_0)$.

Obviously, $(b_3) \Rightarrow (b_2) \Rightarrow (b_1)$. The implicit function theorem given in [6] is proved under assumptions (a) and (b_3). It should be noted that the argument used in the proof of Methlouthi [6] requires the upper semicontinuity of F at (p_0, x_0), but this condition was not formulated in [6].

Remark 2.2. Condition (c) may be replaced by the following one:
(c') The map F is l.s.c. at (p_0, x_0) and there is a prederivative t of $F(p_0,.)$ at $(x_0, 0)$ having property (C) and satisfying the inequality $p(t) > 0$.

Before going further we recall the notion of pseudo-Lipschitz maps.

DEFINITION 3.1, [1]. A set-valued map $G : X \to Y$ is said to be pseudo-Lipschitz around $(x_0, y_0) \in \operatorname{gr} G$ with modulus $k > 0$ if there exist a neighbourhood U of x_0 and neighbourhoods V, V_1 of y_0 such that $G(x) \cap V$ is nonempty for all $x \in U$ and

$$G(x) \cap V \subset G(x') \cap V_1 + k \| x - x' \| \bar{B}(0, 1) \tag{3.1}$$

for all $x, x' \in U$.

PROPOSITION 3.1. *If G is pseudo-Lipschitz around $(x_0, y_0) \in gr\ G$ then there are neighbourhoods U and V of x_0 and y_0, respectively, such that $\tilde{G}(.) := G(.) \cap V$ is l.s.c. on U.*

Proof. Since G is pseudo-Lipschitz, there are $U \in N(x_0)$, $V \in N(y_0)$, $k > 0$ such that $G(x)$ is nonempty for all $x \in U$ and (3.1) holds with $V_1 = Y$. Let $\tilde{G}$ be defined as above. Take a point $(\bar{x}, \bar{y}) \in \operatorname{gr} \tilde{G}$. Given $\varepsilon > 0$, we select $\delta > 0$ such that $k\delta < \varepsilon$ and $\bar{B}(\bar{y}, k\delta) \subset V$. Now, for every $x \in \bar{B}(\bar{x}, \delta) \cap U$ we have from (3.1) that $\bar{y} \in G(x) + k \| x - \bar{x} \| \bar{B}(0, 1)$. Thus, there is $y \in G(x)$ such that

$$\bar{y} \in y + k \| x - \bar{x} \| \bar{B}(0, 1) \subset y + k\delta\, \bar{B}(0, 1).$$

From this it implies that $y \in G(x) \cap V$, $\| \bar{y} - y \| < \varepsilon$. Hence $\tilde{G}$ is l.s.c. on U.

The main result of this section deals with the pseudo-Lipschitz property of implicit maps. Since this property is stronger than lower semicontinuity, to obtain it we have to impose an additional requirement on the map $F(p, x)$ under consideration. Namely, we shall suppose this map to be locally Lipschitz with respect to the first variable. Let us state the result.

Given Banach spaces P, X, Y and a set-valued map $F : P \times X \to Y$ such that for every $(p, x) \in P \times X$ there is $y_p \in F(p, x)$ satisfying $\| y_p \| = d[0, F(p, x)]$. As in the previous section we assume $0 \in F(p_0, x_0)$ and the norm-functional in Y has Frechet derivative at every point different from the origin.

THEOREM 3.1. Suppose that

(a) For every $p \in P$, $F(p,.)$ is u.s.c. on X;

(b) There are $\gamma > 0$ and a neighbourhood $A \times B$ of (p_0, x_0) such that for every $w = (p, x, y) \in (A \times B \times Y) \cap gr\ F$ there is a prederivative t_w of $F(p,.)$ at (x, y) with $p(t_w) \geq \gamma$;

(c) There is $l > 0$ such that

$$F(p, x) \subseteq F(p', x) + l \| p - p' \| .\bar{B}(0,1) \tag{3.2}$$

for all $p, p' \in A$ and $x \in B$,

Then, there are neighbourhoods U and V of p_0 and x_0, respectively, such that $\tilde{G}(p) := \{x \in V \mid 0 \in F(p, x)\}$ is nonempty for all $p \in U$ and $\tilde{G} : U \cdot\to V$ is pseudo-Lipschitz around (p_0, x_0) with modulus $k = \frac{2l}{\gamma}$.

Proof. Choose $\delta > 0$, $U \in N(p_0)$ such that $\bar{B}(x_0, \delta l) \subset B$ and diam $U := \sup\{\|p - p'\| \mid p, p' \in U\} < \frac{\gamma\delta}{4}$. Setting $V = \{x \mid \|x - x_0\| < l\delta\}$ we shall show that U and V are the desired neighbourhoods. Observe that if the inequality

$$d[x, \tilde{G}(p')] \leqslant k \|p - p'\| \tag{3.3}$$

holds for arbitrary p, $p' \in U$ and $x \in \tilde{G}(p) \cap V_1$, where $V_1 = \left\{x \mid \|x - x_0\| < \frac{l\delta}{2}\right\}$, then the theorem follows. Indeed, from (3.3) we have $G(p) \cap V_1 \subset G(p') \cap V + k\|p - p'\|\bar{B}(0,1)$, for all $p, p' \in U$. Besides, the proof of Theorem 2.1 shows that $G(p) \cap V_1$ is nonempty for all p in a neighbourhood $U' \subset U$ of p_0.

To prove (3.3), we consider the function

$$v(z) = d[0, F(p', z)] + \varepsilon \|z - x\|.$$

Since diam $U < \frac{\gamma\delta}{4}$ we can take ε such that

$$\frac{2\|p - p'\|}{\delta} < \varepsilon < \frac{\gamma}{2}.$$

Clearly, the function $v(.)$ is lower semicontinuous because $F(p', .)$ is u.s.c. Let $\alpha = v(x) = d[0, F(p', x)]$. By (3.2) we get

$$\alpha = d[0, F(p', x)] = d[0, F(p', x)] - d[0, F(p, x)] \leqslant l\|p - p'\| < \frac{l\varepsilon\delta}{2}.$$

Lemma 1.2 shows that there is $\bar{x} \in \bar{B}(x_0, \delta l)$ such that

$$v(\bar{x}) \leqslant v(x),$$

$$\|\bar{x} - x\| < \frac{l\delta}{2},$$

$$v(\bar{x}) \leqslant v(z) + \varepsilon \|z - \bar{x}\|, \quad \forall z \in \bar{B}(x_0, \delta l),$$

which implies

$$f(\bar{x}) + \varepsilon\|x - \bar{x}\| \leqslant f(x), \tag{3.4}$$

$$\|\bar{x} - x\| < \frac{l\delta}{2}, \tag{3.5}$$

$$f(\bar{x}) \leqslant f(z) + 2\|z - \bar{x}\|, \quad \forall z \in \bar{B}(x_0, \delta l), \tag{3.6}$$

where $f(.) = d[0, F(p', .)]$.

Since $\| \bar{x} - x_0 \| \leqslant \| \bar{x} - x \| + \| x - x_0 \| < \frac{l\delta}{2} + \frac{l\delta}{2} = l\,\delta$ we conclude that $0 \in F(p', \bar{x})$. Indeed, suppose the contrary. Then there is $\bar{y} \in F(p', \bar{x})$ such that $0 \neq \| \bar{y} \| = d\,[0, F(p', \bar{x})\,]$. For $\bar{w} = (p', \bar{x}, \bar{y})$ there is a prederivative t of $F(p',.)$ at $(\bar{x}, \bar{y})$ with $p(t) \geqslant \gamma$. Let $\tilde{y}^*$ denote the Frechet derivative of the norm—functional in Y at $\bar{y}$. According to Lemma 1. 1 the function $c_t(\tilde{y}^*, .)$ is an upper $\left(\frac{\gamma}{2} - \varepsilon\right)$— approximation of f at $\bar{x}$. Since $\| \bar{x} - x_0 \| < l\,\delta$ we find $\mu > 0$ such that $\bar{B}(\bar{x}, \mu) \subset \bar{B}(x_0, \delta\, l)$ and

$$f(z) \leqslant f(\bar{x}) + c_t(\tilde{y}^*, z - \bar{x}) + \left(\frac{\gamma}{2} - \varepsilon\right) \| z - \bar{x} \|$$

for all $z \in \bar{B}(\bar{x}, \mu)$. Combining this with (3.6) we have

$$c_t(\tilde{y}^*, z - \bar{x}) + \left(\frac{\gamma}{2} + \varepsilon\right) \| z - \bar{x} \| \geqslant 0$$

for all $z \in \bar{B}(\bar{x}, \mu)$, which implies

$$\inf_{z \in S} c_t(\tilde{y}^*, z) + \left(\frac{\gamma}{2} + \varepsilon\right) \geqslant 0,$$

where S denotes the unit sphere in Y. We have thus arrived at a contradiction because $p(t) \geqslant \gamma$ and $\varepsilon < \frac{\gamma}{2}$. Hence $\bar{x} \in \tilde{G}(p')$. Using (3. 4) we obtain

$$\| \bar{x} - x \| \leqslant \frac{1}{\varepsilon} \cdot f(x) = \frac{1}{\varepsilon}\{ d\,[0, F(p', x)\,] - d\,[0, F(p, x)\,]\} \leqslant \frac{l}{\varepsilon} \| p - p' \|$$

Letting $\varepsilon \to \frac{\gamma}{2}$ we have (3.3), as desired.

Remark 3.1. Theorem 3.1 still holds if instead of (b) we suppose the following conditions : The map F is l.s.c. at (p_0, x_0) and for each $w = (p, x, y) \in$ gr F near $(p_0, x_0, 0)$ one can find a prederivative t_w of $F(p,.)$ at (x, y) such that $p(t_w) \geqslant \gamma$ for some $\gamma > 0$.

Suppose $F : X \to Y$ is a set—valued map such that for every $(x, y) \in X \times Y$ there is a point $\bar{y} \in F(x)$ satisfying $\| \bar{y} - y \| = d\,[y, F(x)]$.

THEOREM 3. 2. (Inverse function theorem). *Assume that F is continuous on X and $y_0 \in F(x_0)$. Let the following condition be satisfied: There exists $\gamma > 0$ such that for every $w = (x, y) \in gr\, F$ near (x_0, y_0) one can find a prederivative t_w of F at (x, y) satisfying $p(t_w) \geqslant \gamma$. Then, there are neighbourhoods $U, \tilde{U}$ of x_0 and V of y_0 such that*

1. $\tilde{G}(y) := \{ x \in \tilde{U} \mid y \in F(x) \}$ is nonempty for all $y \in V$,

2. *The map* $\widetilde{G}$ *is l. s. c. on* V *and* $d\,[x, G\,(y)] \leqslant \frac{1}{\gamma}\, d\,[y, \ F\,(x)]$ *for every* $(x, y) \in U \times V$,

3. *The map* $G\,(y) = \{\, x \in X \mid y \in F\,(x)\}$ *is pseudo—Lipschitz around* (y_o, x_o) *with modulus* $k = \frac{2}{\gamma}$.

The proof is immediate by applying Theorem 3. 1 and Remark 3. 1 to the map

$$F\,(p, x) := F\,(x) - p, \qquad (p \in Y).$$

Remark 3. 2. The results obtained in Section 2 and 3 still hold if instead of $\mathcal{L}$ we take the class of locally Lipschitz set-valued maps. The Banach constant is then defined as in [8].

The following simple example shows that Theorem 3. 2 is useful even in the case the corresponding results of [1] and [2] fail.

Example. Let $F\,(x) = \begin{cases} \alpha x & \text{if} \quad x > 0 \\ \beta x & \text{if} \quad x < 0 \end{cases}$

where $x \in R$ and $\beta > \alpha > 0$. Using Theorem 3. 2 and the fact that

$$t(h) := \{\lambda\, h \mid \lambda \in [\alpha, \beta]\} \quad (\forall\, h \in R)$$

is a prederivative of E at (0,0), we can verify that the inverse set-valued map is pseudo-Lipschitz around (0, 0) with modulus $\frac{2}{\alpha}$.

4. APPLICATION : STABILITY OF INEQUALITIES

Given a closed convex cone K in a Banach space Y we shall write $y \leqslant 0$ if $-y \in K$. Let Ω be an open subset of $P \times X$, where P is a topological space and X is a Banach space. Assume that

$$f(p_0, x_0) \leqslant 0,$$

where $f : \Omega \to Y$ is a single-valued map. We say that (p_0, x_0) is a stable point of the inequality

$$f(p, x) \leqslant 0 \tag{4.1}$$

if, to any $\varepsilon > 0$ we can associate a neighbourhood U of p_0 such that

$$\forall\, p \in U \quad \exists\, x_p \in X \text{ satisfying } f(p, x_p) \leqslant 0,$$
$$(p, x_p) \in \Omega \text{ and } \| x_p - x_0 \| < \varepsilon.$$

Here the variable p plays the role of parameter. We shall suppose that $K_1 := \{y \in K \mid \| y \| \leqslant 1\}$ is a compact subset of Y. Theorem 2.1 applied to the map $F(p, x) := f(p, x) + K_1$ $((p, x) \in \Omega)$ yields the following

COROLLARY 4.1. *If f is contin ous and there is a neighbourhood $A \times B$ of (p_0, x_0) such that for every $(p, x) \in A \times B$ we can find a prederivative t of $f(p,.)$ at $(x, f(p, x))$ satisfying $p(t) \geqslant \gamma$ for some $\gamma > 0$, then (p_0, x_0) is a stable point of (4. 1).*

Remark 4.1. Under the assumption that for every (p, x) the map $f(p, .)$ is Frechet differentiable in x, its derivative $\nabla_x f(p, x)$ is continuous at (p_0, x_0) and $\nabla_x f(p_0, x_0)$ is an invertible map, H. Methlouhi [7] proved a similar result. In this case we have only to assume $\nabla_x f(p_0, x_0)$ to be surjective.

We now suppose that $X = R^n$, $Y = R^m$ and $f(p,.)$ is locally Lipschitz on X. Denote by $\Delta_x f(p, x_0)$ the generalized Jacobian [4] of $f(p,.)$ in x at x_0. If we set

$$t(h) = \{Ah \mid A \in \Delta_x f(p, x_0)\}, \ (h \in X)$$

then t is a prederivative of $f(p,.)$ at $(x_0, f(p, x_0))$. Using this fact and Corollary 4.1 we obtain:

COROLLARY 4.2. *Suppose that f is a continuous map. If there exists $\gamma > 0$ such that for every (p, x) in a neighbourhood of (p_0, x_0) we have inf $\{C(A) \mid A \in \Delta_x f(p, x)\} \geqslant \gamma$ then (p_0, x_0) is a stable point of (4.1).*

Remark 4.2. We can use the results of the previous Sections to derive sufficient conditions for lower semicontinuity and Lipschitz property of the solution set of (4.1).

Example. Let $P = R, X = Y = R^2$, $K = \{(y_1, y_2) \in R^2 \mid y_1 \geqslant 0, \ y_2 \geqslant 0\}$, $f(p, x_1, x_2) = (p^2 \mid x_1 \mid + x_2, (2 + \mid p \mid) x_1 + p \mid x_2 \mid)$. Then (4.1) reduces to the following system of nonsmooth inequalities

$$\begin{cases} p^2 \mid x_1 \mid + x_2 \leqslant 0 \\ (2 + \mid p \mid) x_1 + p \mid x_2 \mid \leqslant 0. \end{cases}$$

We have

$$\Delta_x f(p, x) = \left\{ \begin{bmatrix} p^2 \operatorname{sign} x_1 & 1 \\ 2 + \mid p \mid & p \operatorname{sign} x_2 \end{bmatrix} \right\}$$

if $x_1 x_2 \neq 0$.

From Corollary 4.2 it follows that $(p_0, x_0) = (0, 0)$ is a stable point.

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REFERENCES

1. J—P. Aubin, *Comportement lipschitzien des solutions de problemes de minimisation convexes*, C.R. Acad. Sci. Paris, 295 (1982), 235—238.
2. J—P. Aubin, H, Frankowska, *On inverse function theorems for set-valued. maps*, IIASA, Laxenburg (Austria), Working Paper N^{0}48–68, September 1984.
3. C. Berge, *Topological spaces*, Oliver-Boyd, Edingburg and London, 1963.
4. F.H. Clarke, *On the inverse function theorem*, Pacific J. Math., v. 62 (1976), 97–102.
5. I. Ekeland, *On the variational principle*,, Journal Math. Anal. Appl., 47 (1974), 324–353.
6. H. Methlouthi, *Calcul differentiel multivoque*, Cahiers de Mathematiques de la Decision, Univ. Paris IX—Dauphine, Paris, 1977.
7. S. M, Robinson, *Stability theory for systems of inequalities* II. SIAM J. Numer. Anal., V. 13 (1976), 497—513.
8. P.H. Sach, *A support principle for a mathematical programming problem involving set-valued maps*. Inst. of Math., Hanoï, Preprint N^{0}1 (1985).
9. P.H. Sach, *A surjectivity theorem for set-valued maps*, Inst. of Math., Hanoi, Preprint N^{0}25 (1986).

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