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ON THE INVERSION FORMULAS FOR THE INTEGRAL REPRESENTATION OF $e^{\lambda x}$ – ANALYTIC FUNCTIONS AND THEIR APPLICATION

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In this paper, which is a continuation of [1], the inversion formulas for the integral representations of $e^{\lambda x}$ – analytic functions are established and applied to some boundary value problems for these functions and also to problems of filtration theory in nonhomogeneous medium.

1. INVERSION FORMULAS

Let G be an arbitrary simply connected region in the upper half-plane of the complex variable $z = x + iy$; $I_\nu(z)$ the modified Bessel's function of the first kind and ν – th order; $f(z) = u(x, y) + iv(x, y)$ an arbitrary analytic function in G. Assume that the boundary ∂G of G contains a segment L of the real axis Ox and $\operatorname{Im} f(z)|_L = 0$. Then, according to [1], the function $F(z) = U(x, y) + iV(x, y)$ defined by the formula

$$e^{\frac{\lambda}{2}x}[U(x,y)+C_1] + ie^{-\frac{\lambda}{2}x}[V(x,y)+C_2] =$$

$$= \int_{z_0}^{z} I_0\left(\frac{\lambda}{2}\sqrt{(z-t)(\overline{z}-t)}\right) f(t)dt + \tag{1}$$

$$+ \int_{\overline{z}_0}^{\overline{z}} \sqrt{\frac{\overline{z}-t}{z-t}}\, I_1\left(\frac{\lambda}{2}\sqrt{(z-t)(\overline{z}-t)}\right)\overline{f}(t)\, dt$$

is an $e^{\lambda x}$ — analytic function in G. Here λ is a non zero real constant, C_1 and C_2 are arbitrary real constants, $z_0 \in \partial G$, the integrals are taken over an arbitrary curve in G joining z_0, z ($\bar{z}_0$, $\bar{z}$ respectively), and a branch of arg $(z-t)$ $(\bar{z}-t)$ has been fixed.

If the region G contains a point at infinity, then the function $f(z)$ must satisfy an additional condition, namely:

$$f(z)I_\nu\left(\frac{\lambda}{2}z\right) = 0\left(|z|^{-1-\varepsilon}\right) \text{ as } z \to \infty, \tag{2}$$

where $\nu = 0{,}1$, and ε is a positive constant.

We shall show that if the unbounded region G contains an entire line segment joining two arbitrary points of G with the same absciss, then there exists an inversion formula for the integral representation (1).

At first, the function $f(z)$ is continued analytically from the region G to the region G^* across L, where G^* is the region symmetric to G with respect to the real axis, and for $z \in G^*$:

$$f(z) = \overline{f(\bar{z})}. \tag{3}$$

a) Let Γ_1 be a vertical line segment lying in $G + L + L^*$ and joining the points (x, a) and $(x, -a)$ $(a > 0,\ z_0 = x + i0 \equiv (x, 0))$. Let $z = x + iy$ and $t = x + i\eta$ belong to Γ_1. Assuming arg $(z-t)(\bar{z}-t) = 0$ for $0 < \eta < y$ or $0 > \eta > -y$, we obtain from (1)

$$e^{\frac{\lambda}{2}x}\left[U(x,y) + C_1\right] = -\int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) v(x,\gamma)\,d\gamma + \tag{4}$$

$$+\int_0^{-y} (y+\gamma)\,(y^2-\gamma^2)^{-\frac{1}{2}} I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u(x,-\gamma)\,d\gamma,$$

$$e^{-\frac{\lambda}{2}x}\left[V(x,y) + C_2\right] = \int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u(x,\gamma)\,d\gamma - \tag{5}$$

$$-\int_0^{-y} (y+\gamma)\,(y^2-\gamma^2)^{-\frac{1}{2}} I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) v(x,-\gamma)\,d\gamma.$$

Whence, it follows that

$$e^{\frac{\lambda}{2}x}\left[U(x,y) + C_1\right] = -\int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) v(x,\gamma)\,d\gamma - \tag{6}$$

$$-\int_0^y \sqrt{\frac{y-\gamma}{y+\gamma}}\, I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u(x,\gamma)\, d\gamma,$$

$$e^{-\frac{\lambda}{2}x}[V(x,y)+C_2] = \int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) u(x,\gamma)\, d\gamma + \tag{7}$$

$$+\int_0^y \sqrt{\frac{y-\gamma}{y+\gamma}}\, I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) v(x,\gamma)\, d\gamma.$$

To obtain the inversion formulas for the integral representations (6) and (7) we shall apply the Laplace-Carson transform:

$$\tilde{g}(p) = \mathcal{C}[g(\gamma)] = p\int_0^\infty e^{-p\gamma} g(\gamma)\, d\gamma,$$

where p is a complex number.

Setting

$$A(\gamma) = e^{\frac{\lambda}{2}x}[U(x,\gamma)+C_1],\ \alpha(\gamma) = -u(x,\gamma),$$

$$B(\gamma) = e^{-\frac{\lambda}{2}x}[V(x,\gamma)+C_2],\ \beta(\gamma) = -v(x,\gamma), \tag{8}$$

and using the formulas (see [2])

$$\mathcal{C}\left[\int_0^\gamma I_0\left(\frac{\lambda}{2}\sqrt{\gamma^2-\eta^2}\right) h(\eta)\, d\eta\right] =$$

$$= p\left(p^2-\frac{\lambda^2}{4}\right)^{-1} \tilde{h}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right),$$

$$\mathcal{C}\left[\int_0^\gamma \sqrt{\frac{\gamma-\eta}{\gamma+\eta}}\, I_1\left(\frac{\lambda}{2}\sqrt{\gamma^2-\eta^2}\right) k(\eta)\, d\eta\right] =$$

$$= \frac{\lambda}{2}\, p\left(p^2-\frac{\lambda^2}{4}\right)^{-1}\left(\sqrt{p^2-\frac{\lambda^2}{4}}+p\right)^{-1} \tilde{k}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right),$$

we have from (6) and (7)

$$\frac{\lambda}{2}\, p\left(p^2-\frac{\lambda^2}{4}\right)^{-1}\left(\sqrt{p^2-\frac{\lambda^2}{4}}+p\right)^{-1}\tilde{\alpha}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right)+$$
$$+\,p\left(p^2-\frac{\lambda^2}{4}\right)^{-1}\tilde{\beta}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right)=\tilde{A}(p),$$

$$\frac{\lambda}{2}\, p\left(p^2-\frac{\lambda^2}{4}\right)^{-1}\left(\sqrt{p^2-\frac{\lambda^2}{4}}+p\right)^{-1}\tilde{\beta}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right)+ \qquad (9)$$
$$+\,p\left(p^2-\frac{\lambda^2}{4}\right)^{-1}\tilde{\alpha}\left(\sqrt{p^2-\frac{\lambda^2}{4}}\right)=-\tilde{B}(p).$$

Hence

$$-2\left(p^2+\frac{\lambda^2}{4}\right)^{-\frac{1}{2}}\tilde{\alpha}(p)\equiv\tilde{D}(p)=\frac{\lambda}{2}\,p\left(p^2+\frac{\lambda^2}{4}\right)^{-1}\tilde{A}\left(\sqrt{p^2+\frac{\lambda^2}{4}}\right)+$$
$$+\,p\left(\sqrt{p^2+\frac{\lambda^2}{4}}+p\right)\left(p^2+\frac{\lambda^2}{4}\right)^{-1}\tilde{B}\left(\sqrt{p^2+\frac{\lambda^2}{4}}\right) \qquad (10)$$

According to [2]

$$\mathcal{L}\left[\int_0^{\gamma}\left(\frac{\gamma-\eta}{\gamma+\eta}\right)^{\nu} J_{2\nu}\left(\frac{\lambda}{2}\sqrt{\gamma^2-\eta^2}\right) l(\eta)\, d\eta\right]=$$
$$=\left(\frac{\lambda}{2}\right)^{2\nu} p\left(p^2+\frac{\lambda^2}{4}\right)\left(\sqrt{p^2+\frac{\lambda^2}{4}}+p\right)^{-2\nu}\tilde{l}\left(\sqrt{p^2+\frac{\lambda^2}{4}}\right),$$
$$\left(\mathrm{Re}\,\nu>-\frac{1}{2}\right),$$

Where $J_\nu(.)$ is the Bessel's function of the first kind and ν – th order, from (10), we obtain

$$D(\gamma)=2e^{-\frac{\lambda}{2}x}\left[V(x,\gamma)+C_2\right]+ \qquad (11)$$
$$+\frac{\lambda}{2}\int_0^{\gamma} J_0\left(\frac{\lambda}{2}\sqrt{\gamma^2-\eta^2}\right)A(\eta)\,d\eta-$$
$$-\frac{\lambda}{2}\int_0^{\gamma}\sqrt{\frac{\gamma+\eta}{\gamma-\eta}}\,J_1\left(\frac{\lambda}{2}\sqrt{\gamma^2-\eta^2}\right)B(\eta)\,d\eta.$$

Formula (10) can also be written as

$$-2\tilde{\alpha}(p)=\frac{\lambda^2}{4}\,\frac{1}{p}\left[p\left(p^2+\frac{\lambda^2}{4}\right)^{-\frac{1}{2}}\right]\tilde{D}(p)+$$
$$+\,p^2\frac{1}{p}\left[p\left(p^2+\frac{\lambda^2}{4}\right)^{-\frac{1}{2}}\right]\tilde{D}(p). \qquad (12)$$

Taking into account the formulas (see [2])

$$\mathcal{L}\left[J_\nu(a\gamma)\right] = a^\nu\, p\,(p^2+a^2)^{-\frac{1}{2}}\left(\sqrt{p^2+a^2}+p\right)^{-\nu} \quad (\mathrm{Re}\,\nu > -1),$$

$$\mathcal{L}\left[\int_0^\gamma r(\gamma-\eta)\, s(\eta)\, d\eta\right] = \frac{1}{p}\,\widetilde{r}(p)\,\widetilde{s}(p)\,,$$

$$\mathcal{L}\left[g^{(n)}(\gamma)\right] = p^n\,\widetilde{g}(p) - p^n g(0) - \ldots - p g^{(n-1)}(0),$$

we deduce from (12) and (8)

$$u(x,y) = \frac{1}{2}\, D'(y) + \frac{\lambda}{4}\int_0^\lambda \frac{J_1\left(\frac{\lambda}{2}(y-\gamma)\right)}{y-\gamma}\, D(\gamma)\, d\gamma \tag{13}$$

where by (11) and (8)

$$D(y) = 2e^{-\frac{\lambda}{2}x}\left[V(x,y)+C_2\right] + \tag{14}$$

$$+\frac{\lambda}{2}\int_0^y J_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) e^{\frac{\lambda}{2}x}\left[U(x,\gamma)+C_1\right]d\gamma -$$

$$-\frac{\lambda}{2}\int_0^y \sqrt{\frac{y+\gamma}{y-\gamma}}\, J_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) e^{-\frac{\lambda}{2}x}\left[V(x,\gamma)+C_2\right]d\gamma,$$

$$C_2 = -V(x,o).$$

In a similar manner, from (9) it follows that

$$v(x,y) = \frac{1}{2}\, E'(y) + \frac{\lambda}{4}\int_0^y \frac{J_1\left(\frac{\lambda}{2}(y-\gamma)\right)}{y-\gamma}\, E(\gamma)\, d\gamma, \tag{15}$$

where

$$E(y) = 2e^{\frac{\lambda}{2}x}\left[U(x,y)+C_1\right] +$$

$$+\frac{\lambda}{2}\int_0^y J_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) e^{-\frac{\lambda}{2}x}\left[V(x,\gamma)+C_2\right]d\gamma -$$

$$-\frac{\lambda}{2}\int_0^y \sqrt{\frac{y+\gamma}{y-\gamma}}\, J_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) e^{\frac{\lambda}{2}x}\left[U(x,\gamma)+C_1\right]d\gamma,$$

$$C_1 = -U(x,0).$$

b) Let Γ_2 be a vertical line lying in G and joining the points (x, ∞) and (x, b) ($b > 0$). Denote by Γ^*_2 the line symmetric to Γ_2 with respect to the real axis. Let $z = x + iy$ and $t = x + i\eta$ be two points on Γ_2.

Assuming that

$$\arg(z - t)(\bar{z} - t) = \begin{cases} -\pi & \text{for } 0 < y < \eta, \\ \pi & \text{for } 0 > -y > \eta, \end{cases} \tag{17}$$

we have from (1)

$$e^{\frac{\lambda}{2}x}[U(x, y) + C_1] = -\int_{\infty}^{y} I_o\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)v(x,\gamma)\,d\gamma - \tag{18}$$

$$-\int_{-\infty}^{-y}(y + \gamma)(\gamma^2 - y^2)^{-\frac{1}{2}}\, iI_1\left(\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)u(x, -\gamma)d\gamma,$$

$$e^{-\frac{\lambda}{2}x}[V(x,y) + C_2] = \int_{\infty}^{y} I_o\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)u(x,\gamma)\,d\gamma + \tag{19}$$

$$+\int_{-\infty}^{-y}(y + \gamma)(\gamma^2 - y^2)^{-\frac{1}{2}}\, iI_1\left(\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)v(x,-\gamma)\,d\gamma.$$

Whence

$$e^{\frac{\lambda}{2}x}[U(x, y) + C_1] = -\int_{\infty}^{y} I_o\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)v(x, \gamma)\,d\gamma - \tag{20}$$

$$-\int(y - \gamma)(\gamma^2 - y^2)^{-\frac{1}{2}}\, iI_1\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)u(x, \gamma)\,d\gamma,$$

$$e^{-\frac{\lambda}{2}x}[V(x,y) + C_2] = \int_{\infty}^{y} I_o\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)u(x, \gamma)\,d\gamma + \tag{21}$$

$$+\int_{\infty}^{y}(y - \gamma)(\gamma^2 - y^2)^{-\frac{1}{2}}\, iI_1\left(-\frac{\lambda}{2}i\sqrt{\gamma^2 - y^2}\right)v(x, \gamma)\,d\gamma.$$

c) Uniqueness of the integral representations.

Let G be an unbounded region and $f(z) = u(x,y) + iv(x,y)$ an **arbitrary analytic function in** G, **satisfying the conditions (2) and** $\operatorname{Im} f(z)\,|_L = 0$. **Then** the integral representations (6), (7) and (20), (21) determine a unique $e^{\lambda x}$ — analytic function $F(z) = U(x,y) + iV(x,y)$ in G (with definite boundary condition), that is

$$e^{\frac{\lambda}{2}x}\left[U(x,y)+C_1\right]+ie^{-\frac{\lambda}{2}x}\left[V(x,y)+C_2\right]=$$

$$=\int_0^y i I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right) f(x,\gamma)\,d\gamma +$$

$$+\int_0^{-y}\left(y+\gamma\right)\left(y^2-\gamma^2\right)^{-\frac{1}{2}} I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)\overline{f(x,\gamma)}\,d\gamma =$$

$$=\int_\infty^y i\, I_0\left(-\frac{\lambda}{2} i\sqrt{\gamma^2-\gamma^2}\right) f(x,\gamma)\,d\gamma -$$

$$-\int_{-\infty}^{-y}(y+\gamma)\,(\gamma^2-y^2)^{-\frac{1}{2}} I_1\left(\frac{\lambda}{2} i\sqrt{\gamma^2-y^2}\right)\overline{f(x,-\gamma)}\,d\gamma. \qquad (22)$$

2. APPLICATION TO SOME BOUNDARY-VALUE PROBLEMS FOR $e^{\lambda x}$—ANALYTIC FUNCTIONS

We now show how the inversion formulas established in section 1 can be used to solve in quadratures some boundary-value problems for $e^{\lambda x}$— analytic functions.

Problem 1. Let G be the first orthant $\{z=x+iy : x \geq 0, y > 0\}$. We wish to find an $e^{\lambda x}$—analytic function in G, $F(z)=U(x,y)+iV(x,y)$, whose real and imaginary parts are continuous together with their partial derivatives of first order on ∂G and satisfy the boundary conditions;

$$U(x,y)\Big|_{x=0}=E(y) \text{ for } 0\leqslant y<\infty, \qquad (23)$$

$$V(x,y)\Big|_{y=0}=0 \quad \text{for} \quad 0\leqslant x<\infty, \qquad (24)$$

where $E(y)$ is a given function, continuous together with its derivative of first order in the interval $[0, \infty)$.

In view of (6) and (7) we shall seek the solution $F(z)$ in the form

$$U(x,y)=-e^{-\frac{\lambda}{2}x}\left[\int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)v(x,\gamma)\,d\gamma+\right.$$

$$\left.+\int_0^y\sqrt{\frac{y-\gamma}{y+\gamma}}\,I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)u(x,\gamma)d\gamma\right]-C_1, \tag{25}$$

$$V(x,y)=e^{\frac{\lambda}{2}x}\left[\int_0^y I_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)u(x,\gamma)d\gamma+\right.$$

$$\left.+\int_0^y\sqrt{\frac{y-\gamma}{y+\gamma}}\,I_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)v(x,\gamma)\,d\gamma,\right. \tag{26}$$

where the function $f(z)=u(x,y)+iv(x,y)$ is analytic in G, continuous and bounded on ∂G and satisfies the condition

$$v(x,y)\big|_{y=0}=0 \text{ for } 0\leqslant x<\infty. \tag{27}$$

It should be noted that, following [3], a function $F(z)=U(x,y)+iV(x,y)$ is said to be $e^{\lambda x}$ — analytic in G if its real and imaginary parts are continuous together with their partial derivatives of first order in this region and satisfy the system of partial differential equations

$$e^{\lambda x}\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y},\quad e^{\lambda x}\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}. \tag{28}$$

From these equations it follows that on the imaginary axis the values of an $e^{\lambda x}$ — analytic function $F(z)=U(x,y)+iV(x,y)$ coincide with those of the corresponding analytic function $G(z)=\alpha(x,y)+i\beta(x,y)$. Whence, taking into account (23), (24) and (4), we first solve the following boundary-value problem for the function $G(z)$:

$$\alpha(x,y)\big|_{x=0}=\begin{cases}E(y) \text{ for } 0\leqslant y<\infty,\\ E(-y) \text{ for } -\infty<y\leqslant 0.\end{cases} \tag{29}$$

From (29) the function $G(z)$ can be determined using Schwarz's formula for the right half-plane. Hence

$$V(x,y)\big|_{x=0}=\beta(x,y)\big|_{x=0}\equiv H(y),\ (0\leqslant y<\infty). \tag{30}$$

Applying the inversion formula (13) we obtain from (29) and (30)

$$u(x, y)\Big|_{x=0} = \frac{1}{2} D'(y) + \frac{\lambda}{4}\int_0^y \frac{J_1\left(\frac{\lambda}{2}(y-\gamma)\right)}{y-\gamma} D(\gamma)d\gamma, \qquad (31)$$

where

$$D(y) = \frac{\lambda}{2}\int_0^y J_0\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)\left[E(\gamma)+C_1\right]d\gamma -$$

$$-\frac{\lambda}{2}\int_0^y \sqrt{\frac{y+\gamma}{y-\gamma}}\; J_1\left(\frac{\lambda}{2}\sqrt{y^2-\gamma^2}\right)H(\gamma)d\gamma + 2H(y),\ C_1 = -E(0). \qquad (32)$$

From (27) it follows that the function $f(z)$ is continued analytically from the region G to the fourth orthant across the real half-axis Ox by the formula (3). Then, taking into account (31) and using Schwarz's formula for the right half-plane, we deduce

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(0, \gamma)d\gamma}{\gamma - iz}.$$

Finally, substituting the values of $f(z) = u(x, y) + iv(x, y)$ into (25) and (26) we derive the solution of the problem 1.

Problem 2. Let G be defined as in the preceding section. We wish to find an $e^{\lambda x}$ — analytic function in G, $F(z) = U(x, y) + iV(x, y)$ whose real and imaginary parts are continuous together with their partial derivatives of first order on ∂G (except at two points (0, 0) and (0, a) at which they may have a simple discontinuity) and satisfy the boundary conditions

$$U(x, y)\Big|_{x=0} = E(y) \text{ for } 0 \leqslant y \leqslant a, \qquad (33)$$

$$V(x, y)\Big|_{x=0} = H(y) \text{ for } a \leqslant y < \infty, \qquad (34)$$

$$V(x, y)\Big|_{y=0} = 0 \quad \text{ for } 0 \leqslant x < \infty, \qquad (35)$$

where $E(y)$ and $H(y)$ are two given functions, continuous together with their derivatives of first order in the intervals $0 \leqslant y \leqslant a$ and $a \leqslant y < \infty$, respectively.

In the same way as we did for the problem 1, first of all we consider the following mixed boundary value problem for the analytic function $G(z) = \alpha(x, y) + i\beta(x, y)$ corresponding to the function $F(z)$:

$$\alpha(x, y)\Big|_{x=0} = \begin{cases} E(y) & \text{for } \quad 0 \leqslant y \leqslant a, \\ E(-y) & \text{for } -a \leqslant y \leqslant 0, \end{cases} \qquad (36)$$

$$\beta(x, y)\Big|_{x=0} = \begin{cases} H(y) & \text{for } a \leqslant y < \infty, \\ -H(-y) & \text{for } -\infty < y \leqslant -a. \end{cases} \tag{37}$$

As shown in [4] the problem (36), (37) may be solved in quadratures. Moreover, the solution, which is bounded in the neighbourhoods of the points (0, 0), (0, a) and (0, $-a$), is uniquely determined. Whence

$$U(x, y)\Big|_{x=0} = \alpha(x, y)\Big|_{x=0} \quad \text{for } 0 \leqslant y < \infty. \tag{38}$$

Thus, the mixed problem (33) -- (37) has been reduced to the problem (38), (35).

3. APPLICATION TO SOME PROBLEMS OF FILTRATION

Using the results obtained in the preceding section we can find the solutions in quadratures of certain problems of filtration theory in nonhomogenous medium.

a) *Sheet pile in a filter layer of infinite depth.* Let us consider the two-dimensional flow of an incompressible fluid around a sheet pile in the nonhomogeneous medium with the permeability coefficient $\chi = e^{-\lambda|x|}$ ($-\infty < x < \infty$, $-\infty < y < 0$, fig. 1). We wish to find the complex potential of the pressure filtration.

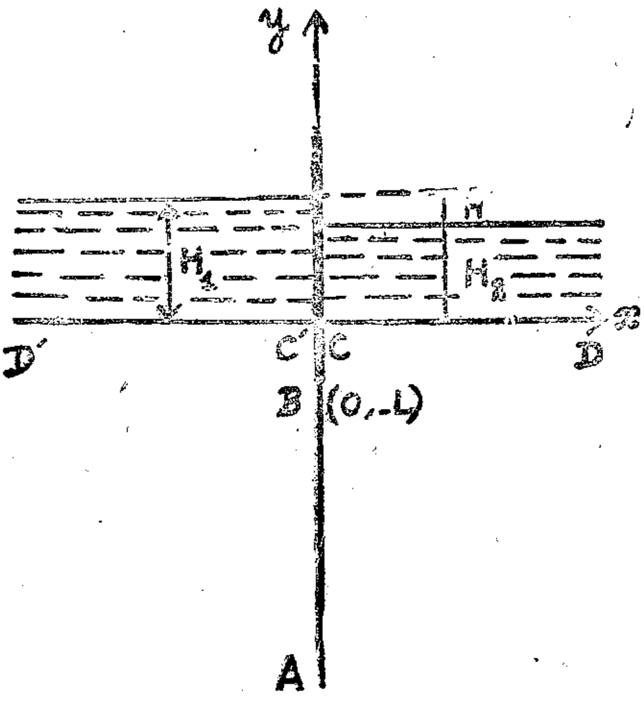

Fig. 1

As shown in [5] the boundary conditions of the complex potential $w(z) = \varphi(x, y) + i\psi(x, y)$ can be written in the form:

$$\varphi = \frac{H}{2} \text{ on } C'D', \ \varphi = -\frac{H}{2} \text{ on } CD,$$
$$\psi = -E \text{ on } C' \ BC, \ \varphi = 0 \text{ on } AB, \qquad (39)$$

where $H \equiv H_1 - H_2$ is the effective pressure, E is a real constant.

We shall consider the filtration only in the fourth orthant. The filtration in the region $\{z = x + iy : -\infty < x \leqslant 0, -\infty < y < 0\}$ is treated similarly. According to [3], in the case under consideration, the complex potential $w(z) = \varphi(x, y) + i\,\psi(x, y)$ satisfies the system of equations

$$e^{-\lambda x} \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad e^{-\lambda x} \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad (40)$$
$$(0 \leqslant x < \infty, -\infty < y < 0).$$

Let us introduce the new function $F(\zeta)$ and variable ζ instead of $w(z)$ and z, respectively:

$$F(\zeta) = U(\xi, \eta) + iV(\xi, \eta) \equiv i\overline{w(z)} + i\frac{H}{2}, \qquad (41)$$
$$\zeta = \xi + i\eta \equiv \bar{z}.$$

From (40) and (41) we obtain

$$e^{\lambda\xi} \frac{\partial U}{\partial \xi} = \frac{\partial V}{\partial \eta}, \ e^{\lambda\xi} \frac{\partial U}{\partial \eta} = -\frac{\partial V}{\partial \xi}. \qquad (42)$$

Thus, $F(\zeta)$ is an $e^{\lambda\xi}$ — analytic function of the complex variable $\zeta = \xi + i\eta$. From (39) and (41) the boundary conditions of the function $F(\zeta)$ are:

$$U(\xi, \eta)\Big|_{\xi=0} = E \ for \ 0 \leqslant \eta \leqslant L,$$
$$V(\xi, \eta)\Big|_{\xi=0} = \frac{H}{2} \ for \ L \leqslant \eta < \infty, \qquad (43)$$
$$V(\xi, \eta)\Big|_{\eta=0} = 0 \quad for \ 0 \leqslant \xi < \infty.$$

The problem (42), (43) is a special case of the problem 2 considered in section 2.

b) Concrete dam in a layer of infinite depth. We consider the flow of an incompressible fluid around a concrete dam in the porous medium with the permeability coefficient $\chi = e^{\lambda y}$ $(-\infty < y < 0$, fig. 2).

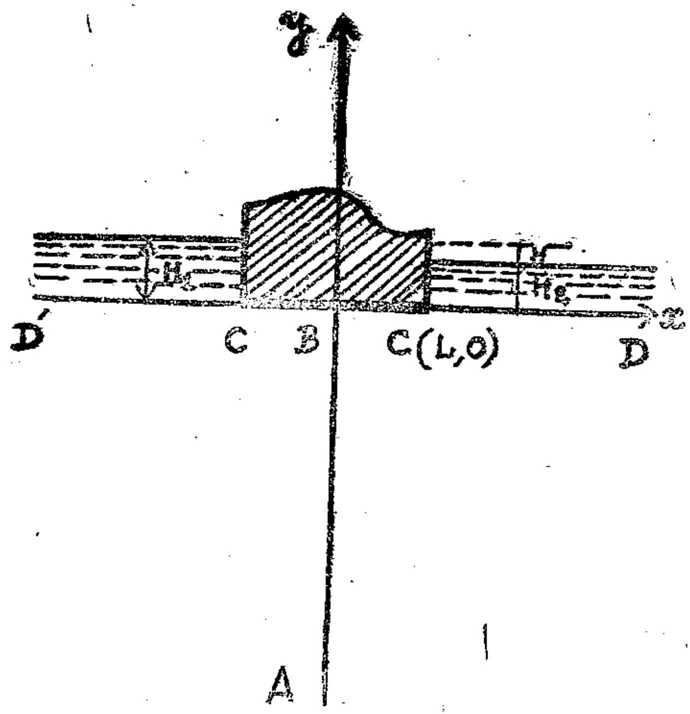

Fig 2

— The complex potential $w(z)$ we are seeking satisfies the system of equations

$$e^{\lambda y}\frac{\partial\varphi}{\partial x}=\frac{\partial\Psi}{\partial y},\ e^{\lambda y}\frac{\partial\varphi}{\partial y}=-\frac{\partial\Psi}{\partial x}, \tag{44}$$

and the boundary conditions

$$\begin{aligned}
&\varphi(x,y)\Big|_{y=0}=-\frac{H}{2}\quad \text{for } -\infty<x\leqslant L,\\
&\varphi(x,y)\Big|_{y=0}=\frac{H}{2}\quad \text{for } L\leqslant x<\infty,\\
&\Psi(x,y)\Big|_{y=0}=-E\quad \text{for } -L\leqslant x\leqslant L,\\
&\varphi(x,y)\Big|_{x=0}=0 \text{ for } -\infty<y\leqslant 0,
\end{aligned} \tag{45}$$

where E is an arbitrary consta t. Let us put

$$F(\zeta)=U(\xi,\eta)+iV(\xi,\eta)=iw(z),$$
$$\zeta=\xi+i\eta \quad iz. \tag{46}$$

From (44) — (46) it follows that

$$e^{\lambda\xi}\frac{\partial U}{\partial\xi}=\frac{\partial V}{\partial\eta},\ e^{\lambda\xi}\frac{\partial U}{\partial\eta}=-\frac{\partial V}{\partial\xi}, \tag{47}$$

$$U(\xi, \eta)\Big|_{\xi=0} = E \quad \text{for } 0 \leqslant \eta \leqslant L,$$
$$V(\xi, \eta)\Big|_{\xi=0} = \frac{H}{2} \text{ for } L \leqslant \eta < \infty \tag{48}$$
$$V(\xi, \eta)\Big|_{\eta=0} = 0 \quad \text{for } 0 \leqslant \xi < \infty.$$

The problem (47), (48) is a special case of the problem 2 considered in Section 2.

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