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REMARKS ON MEASURABLE FUNCTION SPACES

NGUYEN TO NHU

Let X be a metrizable space. By $M(X)$ we denote the space of all measurable functions from [0, 1] into X equipped with the topology of convergence in measure. We identify $f \equiv g$ iff

$$|\{t \in [0, 1] : f(t) \neq g(t)\}| = 0.$$

Here $|A|$ denotes the Lebesgue measure of A in [0, 1].

In this note we investigate the AR-property of certain subsets of $M(X)$. As shown by Torunczyk [5] the AR-property of a metrizable space is closely related to its topological structure. Therefore the study of the AR-property of metrizable spaces is of great importance. Our results in the sequel are similar to those in [2]—[4].

We shall say that a subset $D \subset M(X)$ is convex iff for any $f, g \in D$, for any $\alpha \in [0,1]$ and for any partition $\pi : a_0 = 0 < a_1 < \dots < a_n = 1$ of [0, 1] we have $\alpha_\pi(f, g) \in D$, $\alpha_\pi(f, g)$ is defined as follows

$$\alpha_\pi(f, g)(t) = \begin{cases} f \text{ if } t \in [a_i, a_i + \alpha(a_{i+1} - a_i)), \\ g \text{ if } t \in [a_i + \alpha(a_{i+1} - a_i), a_{i+1}). \end{cases}$$

We also use the following notation :

l_2 = the Hilbert space of all square summable sequences.

$l_2^f = \{x = (x_n) \in l_2 : x_n = 0$ for almost all $n\}$. Recall that a function $f : [0, 1] \to X$ is simple iff there exists a partition of [0,1] into subintervals such that the restriction of f to any subinterval is constant.

$$M_s(X) = \{f \in M(X) : f \text{ is a simple function}\}$$

For other notions we refer the reader to [1].

We shall establish the following results.

THEOREM 1. *Let D be a subset of $M(X)$. If there exists a convex set $C \subset D$ consisting of simple functions such that C is dense in D then D is an absolute retract.*

THEOREM 2. *Let X be a separable complete metrizable space having more than one point. Then for any countable dense subset $X_0 \subset X$ we have $(M(X), M_s(X_0)) \cong (l_2, l_2^f)$.*

We note the following special case of Theorem 1.

COROLLARY 1 [2]. *$M(X)$ and $M_s(X)$ are absolute retracts for any metric space X.*

1. Proof of Theorem 1.

Let $\{\mathcal{U}_n\}$ be a sequence of open covers of a metric space X. By $\mathcal{N}(\mathcal{U})$ we denote the nerve of $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$. We write $K < \{\mathcal{U}_n\}$ iff K is a subcomplex of $\mathcal{N}(\mathcal{U})$ and each simplex σ of K is contained in $\mathcal{U}_n \cup \mathcal{U}_{n+1}$ for some $n \in N$. We let

$$n(\sigma) = \max\{n : \sigma \subset \mathcal{U}_n \cup \mathcal{U}_{n+1}\}.$$

Let K^0 denote the O-skeleton of K, that is the set of all vertices of K and let K denote the simplex K with Whitehead topology. Our proof of Theorem 1 is based on the following

1-1. THEOREM [2] *A metric space $X \in ANR$ iff there exists a sequence of open covers $\{\mathcal{U}_n\}$ of X such that for each $K < \{\mathcal{U}_n\}$ and for each selection $f: K^0 \to X$ there exists a map $g: |K| \to X$ such that for any sequence σ_k with $n(\sigma_k) \to \infty$ we have*

$$\delta(\sigma_k) = \sup\{d(f(V), g(x)) : V \in \overset{\circ}{\sigma}_k, x \in \sigma_k\} \to 0.$$

We shall show that D satisfies the characterization condition of Theorem 1-1.

Without loss of generality we may assume that the topology of X is induced by a metric d bounded by 1. We put

$$\Delta(k, i) = [i2^{-k}, (i+1)2^{-k}) \text{ for } i = 0, \dots, 2^k - 1.$$

For each $f, g \in M(X)$, write

$$\omega(k, i)(f, g) = \sup_{x \in X} \left| \int_{\Delta(k,i)} d(f(t), x)\, dt - \int_{\Delta(k,i)} d(g(t), x)\, dt \right|,$$

$$d_k(f, g) = \sum_{i=0}^{2^k - 1} \omega(k, i)(f, g),$$

$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} d_k(f, g).$$

It is easy to see that d is a compatible metric on $M(X)$.

Take a sequence of open covers $\{\mathcal{U}_n\}$ of D such that $\operatorname{diam} U < 2^{-n}$ for each $U \in \mathcal{U}_n$.

Let $K < \{\mathcal{U}_n\}$ and let $f: K^0 \to D$ be a selection. Take a map $g_0: K^0 \to C \subset D$ such that

$$d(f(V), g_0(V)) < 2^{-n(V)} \text{ for each } V \in K^0$$

where

$$n(V) = \sup \{n : V \in \mathcal{U}_n\}.$$

Let $K^{(n)}$ denote the n-skeleton of K. We shall define inductively a sequence of maps $g_n : K^{(n)} \to M(X)$ with the following properties

(2) $g_n \mid K^{(n-1)} = g_{(n-1)}$ for each $n \geqslant 1$,

(3) For each $\sigma \in K^{(n)}$ there exists an $m(\sigma) \in N$ such that for each $x \in \sigma$ there exist intervals $A_1, \ldots, A_k$, $k \leqslant m(\sigma)$ such that $\bigcup_{i=1}^{k} A_i = [0,1]$, $A_i^{\circ} \cap A_j^{\circ} = \emptyset$ for $i \neq j$ and $g_n(x) \mid_{A_i} = \text{const}$ for each $i = 1, \ldots, k$.

(4) For each $\sigma \in K^{(n)}$, for each $x \in \sigma$ and for each $h \in M(X)$ we have

$$d(h, g_n(x)) \leqslant \max \left\{ d(h, g_0(V)) : V \in \sigma^{\circ} + (1-2^{-n}) \operatorname{diam} g_0(\sigma^{\circ}) \right\}.$$

Obviously g_0 satisfies the conditions (3), (4). Assume that g_{n-1} has been defined with the properties (2) – (4). Let us define $g_n : K^{(n)} \to C$. For each $\sigma \in K^{(n)}$ take $K(\sigma) \in N$ such that

(5) $K(\sigma) > n + 3 + \log_2 \dfrac{\max\{m(\sigma') : \sigma' \text{ is a face of } \sigma\}}{\operatorname{diam} g_0(\sigma^0)}$

Let c be an interior point of the simplex σ and put

$$g_n(c) = g_{n-1}(V_0) = g_0(V_0)$$

where V_0 is any vertex of σ.

Identifying each simplex $\sigma \in K$ with its associated geometric simplex we see that for each $x \in \sigma$ there exist unique $s \in [0,1]$ and $y \in \dot{\sigma}$ (the boundary of σ) such that $x = sc + (1-s)y$. We define $g_n(x)$ as follows: If $g_n(c)|_{\Delta(k(\sigma), i)}$ and $g_{n-1}(y)|_{\Delta(k(\sigma), i)}$ are constant then we put

$$(6) \quad g_n(x)(t) = \begin{cases} g_n(c)\ (t) & \text{if } t \in [i2^{-k(\sigma)}, (i+s)2^{-k(\sigma)}), \\ g_{n-1}(y)\ (t) & \text{if } t \in [(i+s)2^{-k(\sigma)}, (i+1)2^{-k(\sigma)}). \end{cases}$$

Otherwise we subdivide $\Delta(k(\sigma), i)$ into subintervals so that $g_n(c)|_{\Delta}$ and $g_{n-1}(y)|_{\Delta}$ are constant and that each $\Delta \in \{\Delta\}$ is maximal, (that is, if $\Delta' \supsetneq \Delta$ then either $g_n(c)|_{\Delta'}$ or $g_{n-1}(y)|_{\Delta'}$ is not constant). We define $g_n(x) \mid \Delta$ by the formula (6), where $\Delta(k(\sigma), i)$ is replaced by Δ. Obviously g_n satisfies the conditions (2) (3). Let us check (4).

Consider $x \in \sigma$ with $x = sc + (1-s)y$ for some $s \in [0,1]$ and $y \in \dot{\sigma}$. For each $k \leqslant k(\sigma)$, put

$\overline{\Delta}(k, i) = \cup \{ \Delta(k(\sigma), i) \subset \Delta(k, i) : g_{n-1}(z) \big|_{\Delta(k(\sigma), i)}$ is not constant for some $z \in \{c, y\}\}$

$$\widetilde{\Delta}(k, i) = \Delta(k, i) \setminus \overline{\Delta}(k, i).$$

Note that $\overline{\Delta}(k, i)$ and $\widetilde{\Delta}(k, i)$ depend on σ and y. For each $k \leqslant k(\sigma)$ we subdivide $\widetilde{\Delta}(k, i) = \Delta^*(k, i) \cup \Delta^{**}(k, i)$ so that $g^n(x)|_{\Delta^*(k,i)} = g_n(c)$ and $g_n(x)|_{\Delta^{**}(k,i)} = g_{n-1}(y)$.

Then for each $k \leqslant k(\sigma)$ and $z \in X$ we have

$$\int_{\Delta(k,\ i)} d(g_n(x)(t), z)dt = \int_{\Delta^*(k,\ i)} d(g_n(x)(t), z)dt +$$

$$+ \int_{\Delta^{**}(k,\ i)} d(g_n(x)(t), z)dt + \int_{\overline{\Delta}(k\ ,i)} d(g_n(x)(t), z)dt =$$

$$= s\int_{\widetilde{\Delta}(k,\ i)} d(g_n(c)(t), z)dt + (1-s)\int_{\widetilde{\Delta}(k,\ i)} d(g_{n-1}(y)(t), z)dt$$

$$+ \int_{\overline{\Delta}(k,\ i)} d(g_n(x)(t), z)dt =$$

$$= s\int_{\Delta(k,\ i)} d(g_n(c)(t), z)dt + (1-s)\int_{\Delta(k,\ i)} d(g_{n-1}(y)(t), z)dt$$

$$+ \int_{\Delta(k, i)} \{d(g_n(x)(t), z) - sd(g_n(c)(t), z) - (1-s)\, d(g_{n-1}(y)(t), z)\}\, dt.$$

Therefore for each $k \leqslant k(\sigma)$, $z \in X$ and $h \in M(X)$ we have

$$\left| \int_{\Delta(k, i)} d(g_n(x)(t), z)dt - \int_{\Delta(k, i)} d(h(t), z)dt \right|$$

$$\leqslant s\left| \int_{\Delta(k, i)} d(g_n(c)(t), z)dt - \int_{\Delta(k, i)} d(h(t), z)\, dt \right| +$$

$$+ (1-s)\left| \int_{\Delta(k,\ i)} d(g_{n-1}(y)(t), z)\, dt - \int_{\Delta(k, i)} d(h(t), z)\, dt \right| + 2|\overline{\Delta}(k, i)|.$$

Let $m(k,i)(\varphi)$ denote the number of intervals $\Delta(k(\sigma), i) \subset \Delta(k, i)$ on which φ is not constant, where $\varphi \in \{g_n(c), g_{n-1}(y)\}$. Then

$|\overline{\Delta}(k, i)| \leqslant 2^{-k(\sigma)}(m(k,i)(g_n(c)) + m(k,i)(g_{n-1}(y)))$. Consequently, for each $k \leqslant k(\sigma)$ we get $\omega(k, i)(g_n(x), h) \leqslant s\,\omega(k, i)(g_n(c), h) +$ $+ (1-s)\,\omega(k, i)(g_{n-1}(y), h) + 2^{-k(\sigma)+1}(m(k,i)(g_n(c)) + m(k, i)(g_{n-1}(y)))$

Hence, for each $k \leqslant k(\sigma)$ we can write

$$d_k(g_n(x), h) = \sum_{i=0}^{2^k-1} \omega(k, i)(g_n(x), h)$$

$$s \sum_{i=0}^{2^k-1} \omega(k, i)(g_n(c), h) + (1-s) \sum_{i=0}^{2^k-1} \omega(k,i)(g_{n-1}(y), h) +$$

$$+ 2^{-k(\sigma)+1} \sum_{i=0}^{2^k-1} (m(k, i)(g_n(c)) + m(k, i)(g_{n-1}(y)) \leqslant sd_k(g_n(c), h) +$$

$$+ (1-s)\, d_k(g_{n-1}(y), h) + 2^{-k(\sigma)+1} .\ 2 \max \{ m(\sigma') : \sigma' \text{ is a face of } \sigma \}$$

Therefore, from (5) we get for each $k \leqslant k(\sigma)$

$$d_k(g_n(x), h) \leqslant sd_k(g_n(c), h) + (1 - s) d_k(g_{n-1}(y), h) + 2^{-n-1} \operatorname{diam} g_0(\sigma^0)$$

Consequently,

$$d(g_n(x), h) = \sum_{k=1}^{\infty} 2^{-k} d_k(g_n(x), h) =$$

$$= \sum_{k=1}^{k(\sigma)} 2^{-k} d_k(g_n(x), h) + \sum_{k=k(\sigma)+1}^{\infty} 2^{-k} d_k(g_n(x), h)$$

$$\leqslant \sum_{k=1}^{k(\sigma)} 2^{-k}(sd_k(g_n(c), h) + (1-s) d_k(g_{n-1}(y), h) +$$

$$+ 2^{-n-1} \operatorname{diam} g_0(\sigma^0)) + \sum_{k=k(\sigma)+1} 2^{-k} d_k(g_n(x), h)$$

$$\leqslant d(g_n(c), h) + (1 - s)\, d(g_{n-1}(y), h) + 2^{-n-1} \operatorname{diam} g_0(\sigma^0) + 2^{-k(\sigma)}.$$

Note that from (5)

$$2^{-k(\sigma)} < 2^{-n-3} \operatorname{diam} g_0(\sigma^0).$$

Hence

$$d(g_n(x), h) \leqslant \max \{d(g_n(c), h), d(g_{n-1}(y), h)\} + 2^{-n} \operatorname{diam} g_0(\sigma^0).$$

Let $\sigma' \in K^{(n-1)}$ denote a face of σ containing y. Then by the inductive assumption we get for each $h \in M(X)$

$$d(h, g_{n-1}(y)) \leqslant \max \{d(h, g_0(V) : V \in \sigma^0\} + (1 - 2^{-n+1}) \operatorname{diam} g_0(\sigma^0)$$

$$\leqslant \max \{d(h, g_0(V)) : V \in \sigma^0\} + (1-2^{-n+1}) \operatorname{diam} g_0(\sigma^0).$$

Therefore

$$d(h, g_n(x)) \leqslant \max \{d(h, g_0(V)) : V \in \sigma^0\} + (1 - 2^{-n}) \operatorname{diam} g_0(\sigma^0).$$

Hence, the condition (4) holds.

Finally we define $g : K \to D$ by the formula

$$g(x) = \lim_{n\to\infty} g_n(x) \text{ for each } x \in K.$$

Then $g \mid K^0 = g_0$ and from (2) (4) we get for each $x \in \sigma \in K$ and $V \in \sigma^0$

$$d(g(x), f(V)) \leqslant d(g(x), g_0(V)) + d(g_0(V), f(V))$$
$$\leqslant 4 \cdot 2^{-n(\sigma)} + 2^{-n(\sigma)} = 5 \cdot 2^{-n(\sigma)}.$$

Hence

$$\delta(\sigma) = \sup \{ d(g(x), f(V)) : x \in \sigma, V \in \sigma^0 \} \leqslant 5 \cdot 2^{-n(\sigma)} .$$

Thus by Theorem 1–1 $D \in ANR$. It is easy to see that every convex set in $M(X)$ is contractible. Therefore $D \in AR$. This completes the proof of Theorem 1.

2. *Proof of Theorem 2.* We shall assume that X is infinite, though our argument is valid in the finite case as well.

2-1. LEMMA $M_s(X_0)$ *is the union of a sequence of finite dimensional compact sets, shortly* σ_{fd} *– compact.*

We say that $f: [0,1] \to X$ *is a* k*–step function iff there is a partition* $\pi: 0 = a_0 < a_1 < \ldots < a_k = 1$ *such that* $f \mid_{(a_i, a_{i+1})} =$ *const for each* $i = 1, \ldots, k-1$.

Let $M^k_s(X)$ denote the set of all k-step functions in $M(X)$.

Assume that $X_0 = \{x_1, \ldots, x_n, \ldots\}$. We put

$$X_n = \{x_1, \ldots, x_n\}.$$

Then

$$M_s(X_0) = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} M^k_s(X_n) = \bigcup_{n=1}^{\infty} M^n_s(X_n).$$

Therefore, to complete the proof of Lemma 2-1 it remains to prove that every $M^k_s(X_n)$ is a finite dimensional compact metric space.

Let $\{f_i\} \subset M^k_s(X_n)$ be a sequence. For each $i \in N$ let $\pi_i: a^i_0 = 0 < a^i_1 < \ldots < a^i_k = 1$ denote a partition of $[0,1]$ with respect to f_i, that is, $f_i \big|_{[a^i_j, a^i_{j+1})} =$ const for each $j = 0, \ldots, k = 1$.

Since $[0,1]$ is compact and X_n is finite, there exists a subsequence $f_{ij} \to f \in M^k_s(X_n)$. Thus $M^k_s(X_n)$ is compact. But it is easily seen that $\dim M^k_s(X_n) \leqslant 2k$ for every $k, n \in N$. This completes the proof of Lemma 2-1.

2-2. LEMMA [1]. *Let* Y be *a metric space homeomorphic to* l_2. *Let* $\{\Psi_n\}$ *be a sequence of finite dimensional compact subsets of* Ψ *such that* $\Psi_0 = \bigcup_{n=1}^{\infty} \Psi_n$ *is dense* *in* Ψ. *Then* $(\Psi, \Psi_0) \cong (l_2, l_2^f)$ *iff the following condition holds*

(*) *For each finite dimensional compact set* $K \subset \Psi$, *for each* $\varepsilon > 0$ *and for each* $n \in N$ *there exists an embedding* f *of* K *into* Ψ_m *for some* $m > n$ *such that* $f|_{K \cap X_n} = \mathrm{id}$ *and* $d(f(x), x) < \varepsilon$ *for each* $x \in K$.

2-3. LEMMA [1]. *Let* A *be a proper closed subset of a metric space* Ψ. *Then there exists an indexed family* $\{U_j, c_j\}$ *(called a Dugundji system for* $X \setminus A$*) such that*

(i) $\mathcal{U}_j \subset \Psi \setminus A$ *and* $c_j \in A$ *for each* $j \in J$,

(ii) $\mathcal{U} = \{U_j\}_{j \in J}$ *is a locally finite open cover of* $\Psi \setminus A$,

(iii) *If* $x \in U_j$ *then* $d(x, c_j) \leqslant 2d(x, A)$ *for each* $j \in J$.

Now using Lemmas 2-1—2-3 we shall complete the proof of Theorem 2. Let us check the condition (*) of Lemma 2-2 for $(M(X), M_s(X_0))$. Note that by Theorem 1 we have $M(X) \cong l_2$.

Let $K \subset M(X)$ be a finite dimensional compact set, $\varepsilon > 0$ and $n \in N$. We take $p > n$ such that $M_s^p(X_p)$ is an $\frac{1}{2}\varepsilon$-net for K. Let $\{U_j, c_j\}_{j \in J}$ be a Dugundji system for $K \setminus M_s^p(X_p)$. Let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U} = \{U_j\}_{j \in J}$ and let $u: K \setminus M_s^p(X_p) \to \mathcal{N}(\mathcal{U})$ denote a canonical map. Using the proof of Theorem 1 we get a map $v: \mathcal{N}(\mathcal{U}) \to M_s^p(X_p)$ such that, denoting

$$g(x) = \begin{cases} x & \text{if } x \in K \cap M_s^p(X_p) \\ vu(x) & \text{if } x \in K \setminus M_s^p(X_p) \end{cases}$$

we obtain a continuous map $g: K \to M_s^p(X_p)$ such that $g|K \cap M_s^n(X_n) = \mathrm{id}$ and $d(g(x), x) < \frac{1}{2}\varepsilon$ for each $x \in K$.

Let us approximate g by an embedding. Denote $k = \dim K$ and let $\varphi: K \to I^{2k+1}$ be an embedding. For each $x \in K$ let $n(x)$ be the smallest natural number such that there exists a partition $\pi(x)$: $a_0 = 0 < a_1 < \ldots < a_{n(x)} = 1$ with respect to $g(x)$. Assume that

$$g(x)|_{[a_i, a_{i+1})} = y_i \quad \text{for } i = 1, \ldots, n(x) - 1$$

We define $f(x)|_{[a_i, a_{i+1})}$ as follows. Put

$$b_0 = a_i$$

$$b_j = b_{j-1} + \delta d(x, K \cap M_s^n(X_n))\varphi_j(x)$$
$$\text{for } j = 1, \ldots, 2k+1.$$
$$b_{2k+2} = b_{2k+1} + \delta d(x, K \cap M_s^n(X_n))$$

and

$$f(x)\big|_{[b_j, b_{j+1})} = x_{p+j+1} \text{ for } j = 0, \ldots, 2k+1$$
$$f(x)\big|_{[b_{2k+2}, a_{i+1})} = y_i.$$

Obviously $f\big|_{K \cap M_s^n(X_n)} = g\big|_{K \cap M_s^n(X_n)} = \mathrm{id}$ and if δ is chosen to be sufficiently small then $d(f(x), g(x)) < \frac{1}{2}\varepsilon$ for each $x \in K$. It is easily seen that f is an embedding and $f(k) \subset M_s^m(X_m)$ where $m = p(2k+2)$.

This completes the proof of Theorem 2.

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INSTITUTE OF MATHEMATICS, P. O. BOX 631, BO HO 10.000 HANOI, VIETNAM