A FREE BOUNDARY PROBLEM FOR THE NONLINEAR SECOND ORDER PARABOLIC EQUATIONS

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I — INTRODUCTION

The one-phase, one dimensional free boundary problems for nonlinear parabolic equations were studied in [1], [2], [3], [4]. In this paper we study the free boundary problem for nonlinear parabolic equation with nonlinear free boundary conditions which are different from those of [1] — [4].

The following problem is considered:

Problem I: Find a triple $(T, S(t), u(x, t))$ such that:

i) $T > 0$; $S(t) > 0 \ \forall t \in [0, T]$; $S(t) \in C^1 (0, T)$

ii) $u(x, t) \in C^1(\overline{D}_T)$; u_{xx} and u_t are continuous in D_T, where

$$D_T = \{(x, t) ; 0 < x < S(t) ; 0 < t < T\}$$

iii) The following equations are satisfied

$$u_t - a(x, t, u, S(t))\, u_{xx} = q(x, t, u, u_x, S(t), \dot{S}(t)) \quad \text{in } D_T ; \tag{1.1}$$

$$S(0) = b; \tag{1.2}$$

$$u(x, 0) = h(x), \ x \in (0, b); \tag{1.3}$$

$$u(0, t) = \varphi(t), \ t \in (0, T); \tag{1.4}$$

$$\dot{S}(t) = f\,[u(S(t), t)], \ t \in (0, T), \tag{1.5}$$

$$u_x(S(t), t) = -\dot{S}(t).\,[\lambda(\dot{S}(t), t) + u(S(t), t)], \quad t \in (0, T). \tag{1.6}$$

The free boundary conditions (1. 5), (1. 6) are obtained in the mathematical model proposed in [5] for the penetration of solvents into polymers. It is observed that if the solvent concentration exceeds some threshold value λ, then the solvent moves into the polymer, creating a layer, in which the solvent diffuses according to Fick's law represented by (1. 5). The condition (1. 6) represents the mass conservation.

In this paper the given functions are supposed to satisfy the following assumptions:

(A) $h(x) \in C^2[0,b]$.

(B) $\varphi(t)$ is continuously differentiable for $t \geqslant 0$.

(C) $a(x, t, u, S)$ is a function twice continuously differentiable with respect to x for $(x, t) \in \overline{D_T}$, $u \in R$, $S \in R^+$, Holder continuous with respect to x and t, Lipschitz continuous with respect to t, and satisfying

$$\mu_1^{-1}(|u|) \leqslant a(x, t, u, S) \leqslant \mu_1(|u|)$$

$$|Sa_x| \leqslant \mu_1(|u|)\ (1 + |S|)^2,$$

where $\mu_{-1}(\xi)$ is a positive function, non-decreasing for $\xi > 0$.

(D) $q(x, t, u, p, S, \sigma)$ is a continuously differentiable function for $(x, t) \in D_T$, $u \in R$, $p \in R$, $\sigma \in R$, $S \in R^+$ and

$$|q(x, t, u, p, S, \sigma)| \leqslant \mu_2(|u|, |p|),$$

where $\mu_2(\xi, \eta)$ is a positive function, non-decreasing with respect to ξ for $\xi > 0$, non-decreasing with respect to η for $\eta > 0$.

(E) $f \in C^2$, $f' > 0$, f has the inverse function $f^{-1} \equiv \Phi$ satisfying the conditions

$$|\Phi| \leqslant \Phi_0 . t; \quad |\Phi'| \leqslant \Phi_0 . t$$

where Φ_0 is a positive constant.

(F) $\lambda(x, t)$ is continuously differentiable for $x \geqslant 0$, $t \geqslant 0$ and

$$|\lambda(x, t)| \leqslant \lambda_0 . t,$$

where λ_0 is a positive constant.

The condition (1.5) can be written as

$$u(S(t), t) = \phi(\dot{S}(t)).$$

II — AN AUXILIARY PROBLEM

Let T, β, R_0 be positive constants, $\beta < b$. We denote by $\mathcal{S}$ the class:

$$\{S(t) \in C^1[0, T];\ S(T) \geqslant \beta;\quad |\dot{S}(T)| \leqslant R_0\}.$$

Let $S(t)$ be a function $\in \mathcal{S}$. Performing the transformation:

$y = \frac{x}{S(t)}$; $V(y, t) = u(S(t) \cdot y, t)$; $H(y) = h(by)$ the equations (1. 1) — (1. 6) are reduced to

$$V_t - \frac{a(Sy, t, v, S)}{S^2(t)} V_{yy} = \frac{\dot{S}}{S} y \cdot V_y + q(Sy, t, v, \frac{V_y}{S}, S, \dot{S}).$$

in $D^T = \{ (y, t) : y \in (0,1), \ t \in (0, T) \}$; (2. 4)

$$V(y, 0) = H(y), \quad y \in (0,1); \tag{2. 5}$$

$$V(0, t) = \varphi(t), \quad t \in (0, T); \tag{2. 6}$$

$$V_y(1, t) = -\dot{S}(t) \cdot S(t) \cdot [\lambda (S(t), t) + \Phi (\dot{S}(t))] \ t \in (0, T); \tag{2. 7}$$

$$\dot{S}(t) = f [V(1, t)], \ t \in (0, T); \tag{2. 8}$$

$$S(0) = b \tag{2. 9}$$

For any function $S(t) \in \mathcal{S}$ we consider the following auxiliary problem.

Problem II. Find a solution $V(y, t)$ of the equation:

$$V_t - A(y, t, v) V_{yy} = Q(y, t, v, V_y) \ \text{in} \ D^T, \tag{2. 4)'$$

satisfying the following conditions:

$$V(y, 0) = H(y), \tag{2. 5)'$$

$$V(0, t) = \varphi(t), \tag{2. 6)'$$

$$V_y(1, t) = - \dot{S}(t) S(t) [\lambda (S(t), t) + \Phi(\dot{S}(t))], \tag{2. 7)'$$

where

$$A(y, t, v) = \frac{q(Sy, t, v, S)}{S^2(t)},$$

$$Q(y, t, v, V_y) = \frac{\dot{S}}{S} y V_y + q(S_y, t, v, \frac{V_y}{S}, S, \dot{S}).$$

It will be proved that the problem II has a unique solution $V(y, t)$ for sufficiently small T and that $V(y, t) \in C^{1+\delta}(\overline{D}^T)$ for some $\delta \in (0, 1)$.

Let X be the set of functions $V(y, t) \in C^{1+\delta}(\overline{D}^T)$ satisfying the following conditions:

$$\| V \|_0 \leqslant C_0 ; \tag{2. 10}$$

$$\| V \|_{C_{1+\delta}(\overline{D}^T)} \leqslant K_0, \ 0 < \delta < 1; \tag{2. 11}$$

$$\begin{aligned} &V(y, 0) = H(y), \quad y \in (0, 1); \\ &V(0, t) = \varphi(t), \quad t \in (0, T); \\ &V_y(1, t) = -\dot{S}(t) S(t) [\lambda (S(t), t) + \Phi (\dot{S}(t))]. \end{aligned} \tag{2. 12}$$

Note that the set X is closed in $C^{1+\delta}(\overline{D}^T)$.

For any $\widehat{V}(y, t) \in X$ let us consider the following problem.

Problem III. Find a solution $V(y, t)$ of the equation

$$V_t = \widetilde{A}(y, t) V_{yy} + \widetilde{Q}(y, t) \quad \text{in } (0, 1) \times (0, \widetilde{T}) = D^{\widetilde{T}}$$

satisfying the following conditions

$$V(y, 0) = H(y), \qquad y \in (0, 1);$$

$$V(0, t) = \varphi(t), \qquad t \in (0, T);$$

$$V_y(1, t) = -\dot{S}(t) \cdot S(t) \cdot [\lambda(S(t), t) + \Phi(\dot{S}(t))], \ t \in (0, T);$$

where

$$\widetilde{A}(y, t) = A(y, t, \widehat{V}(y, t), S)$$

$$Q(y, t) = Q(y, t, \widehat{V}(y, t), \widehat{V}_y(y, t), S, \dot{S})$$

and $\widetilde{T} \leqslant T$.

LEMMA 1. *There exist positive constants N, K, $\overline{K}$, $б$ such that the solution $V(y, t)$ of the problem III satisfies the following estimations*

$$\| V \|_0 \leqslant N t^{\gamma} + K \quad \text{in} \quad (0,1) \times (0,б) = D^{б}, \quad б < T \qquad (2.13)$$

$$\| V \|_{C^{1+\delta}(\bar{D}^{б})} \leqslant N t^{\gamma} + \overline{K}, \qquad (2.14)$$

where K depends on the data, and $\sup |s(t)|$, $\overline{K}$ depends on K, $\gamma = \frac{1-\delta}{2}$, N depends on the data.

Proof. $V(y,t)$ can be written as the sum $V = V_1 + V_2$ where V_1 is solution of the problem

$$\begin{cases} V_{1t} = \widetilde{A} V_{1yy} + \widetilde{Q} \text{ in } D^T, \\ V_1(y,0) = 0, \\ V_1(0,t) = 0, \\ V_{1y}(1,t) = 0 \end{cases}$$

and V_2 is solution of the problem

$$\begin{cases} V_{2t} = \widetilde{A} V_{2yy} \text{ in } D^T, \\ V_2(y,0) = H(y), \\ V_2(0,t) = \varphi(t), \\ V_{2y}(1,t) = -\dot{S}(t) \cdot S(t) \cdot [\lambda(S(t), t) + \Phi(\dot{S}(t))] \cdot \end{cases}$$

Further, we can consider $V_1(y,t)$ as the restriction to $\overline{D}^T$ of the solution of the problem

$$\begin{cases} V_{1t} = \tilde{\tilde{A}}(y,t)\, V_{1yy} + \tilde{\tilde{Q}}(y,t) \text{ in } (0,2) \times (0,T), \\ V_1(y,0) = 0\,, \\ V_1(0,t) = 0\,, \\ V_1(2,t) = 0\,, \end{cases}$$

where

$$\tilde{\tilde{A}}(y,t) = \begin{cases} \tilde{A}(y,t)\,,\ y \in (0,1)\,; \\ \tilde{A}(2-y,t)\,, y \in (1,2)\,; \end{cases}$$

$$\tilde{\tilde{Q}}(y,t) = \begin{cases} \tilde{Q}(y,t)\,,\ y \in (0,1)\,; \\ \tilde{Q}(2-y,t)\,,\ y \in (1,2)\,. \end{cases}$$

According to Friedman's theorem (see [6], p 249) there exists a positive constant $\sigma < T$ such that in $D^\sigma = D^T \cap \{t < \sigma\}$

$$\| V_1 \|_{C^{1+\delta}(\overline{D}^\sigma)} \leqslant N_1 \,.\, \sigma^\gamma \tag{2.15}$$

with $\gamma = \dfrac{1-\delta}{2}$ $(0 < \delta < 1)$.

To estimate $V_2(y,t)$, we use the maximum principle (see [7])

$$| V_2 | \leqslant K \left(\max_{D^T} | \varphi(t) | + \max_{D^T} | H | + \max_{D^T} | \dot{S}S(\lambda + \Phi) | \right. \tag{2.16}$$

The estimations (2.15) and (2.16) imply (2.12).

For proving (2.13), we denote $\overline{V} = V_{2y}$ and identify $\overline{V}$ with the restriction to $\overline{D}^T$ of the solution of the problem

$$\begin{cases} \overline{V}_t = (\overline{A}.\, \overline{V}_y)_y\,, \\ \overline{V}(y,0) = \overline{H'}(y)\,, \\ \overline{V}(1,t) = -\dot{S}(t)\,.\, S(t)[\lambda\,(s(t),t) + \Phi\, \dot{S}(t)] \\ \overline{V}(-1,t) = \varphi(t)\,, \end{cases}$$

where

$$\overline{A}(y,t) = \begin{cases} A(y,t) & y \in (0,1); \\ A(-y,t) & y \in (-1,0); \end{cases}$$

$$\overline{H}'(y) = \begin{cases} H'(y), & y \in (0,1); \\ H'(-y) & y \in (-1,0) \end{cases}$$

From a result in [7] (Theorem 1.1, page 590) there exists a positive constant K such that

$$\| \overline{V} \|_{C^\delta(\overline{D}^T)} \leqslant K \,. \tag{2.17}$$

Finally, (2.13) follows from (2.15) and (2.17).

LEMMA 2. *The auxiliary problem II has a unique solution in the class* $C^{1+\delta}(\bar{D}_T)$ *for sufficiently small* T.

Proof. Let us choose $T_0 \in (0,\delta)$ such that $NT_0^{\gamma} < K$ and choose $C_0 \geqslant 2K$, $K + \bar{K} \leqslant K_0$. Then

$$\| V \|_0 \leqslant Nt^{\gamma} + K \leqslant C_0 ,$$

$$\| V \|_{C^{1+\delta}(\bar{D}_{T_0})} \leqslant Nt^{\gamma} + \bar{K} \leqslant K + \bar{K} \leqslant K_0 .$$

Now choose two functions $\widehat{V_1}$, $\widehat{V_2}$ satisfying the conditions (2.10) — (2.12). Denote by V_1, V_2 the solutions of the problem III corresponding to $\widehat{V_1}$, $\widehat{V_2}$. Then the difference $V = V_1 - V_2$ is solution of the following problem:

$$\begin{cases} V_t = \widetilde{A}_1(y,t)\, V_{yy} + \Delta \widetilde{Q}(y,t) & \text{in } D^{T_0}; \\ V(y,0) = 0 , & y \in (0,1); \\ V(0,t) = 0 , & 0 < t < T_0; \\ V_y(1,t) = 0 , & 0 < t < T_0 ; \end{cases} \tag{2.18}$$

where

$$\widetilde{A}_1(y,t) = A(y,t, \widehat{V}_1(y,t), s),$$

$$\Delta \widetilde{Q}(y,t) = \Delta Q(y,t) - \frac{1}{S^2} \Delta A(y,t)\, V_{2yy}$$

with

$$\Delta Q(y,t) = Q(y,t), \widehat{V}_1(y,t), \widehat{V}_{1y}(y,t), S, \dot{S}) -$$
$$- Q(y,t), \widehat{V}_2(y,t, \widehat{V}_{2y}(y,t), S, \dot{S}),$$

$$\Delta A(y,t) = A(y,t)\, \widehat{V}_1(y,t, S) - A(y,t, \widehat{V}_2(y,t), S).$$

We consider $V(y,t)$ as the restriction to $(0,1) \times (0,T_0)$ of the solution of the following boundary problem:

$$\begin{cases} V_t = A^*(y,t)\, V_{yy} + \Delta Q^* & \text{in } [0,2] \times [0,T_0], \\ V(y,t) = 0 , & y \in (0,2); \\ V(0,t) = 0 , & t \in (0,T_0); \\ V(2,t) = 0 , & t \in (0,T_0); \end{cases}$$

where

$$A^*(y,t) = \begin{cases} \widetilde{A}(y,t) ; & y \in (0,1); \\ \widetilde{A}(2-y,t), & y \in (1,2); \end{cases}$$

$$\Delta Q^*(y,t) = \begin{cases} \Delta \widetilde{Q}(y,t) ; & y \in (0,1); \\ \Delta \widetilde{Q}(2-y,t) ; & y \in (1,2). \end{cases}$$

From a theorem of Friedman [6] there exists a constant $\sigma > 0$ such that

$$\| V \|_{C^{1+\sigma}(\bar{D}^{\sigma})} \leqslant C . \| \Delta Q^* \|_0 . \sigma^{\gamma} ;$$

where $D^{\sigma} = D_{T_0} \cap \{t < \sigma\}$; $\gamma = \frac{1-\delta}{2}$. We have

$$\| \Delta Q^* \|_0 \leqslant L_1 \| \Delta Q \|_0 + L_2 \| \Delta A \|_0 \, | V_{2yy} |$$

and

$$\| \Delta Q \|_0 \leqslant L_3 \| \widehat{V}_1 - \widehat{V}_2 \|_{C_1}$$

$$\| \Delta A \|_0 \leqslant L_4 \| \widehat{V}_1 - \widehat{V}_2 \|_0 ,$$

$$| V_{2yy} | \leqslant \text{const} \quad \text{(see [1])}.$$

Consequently.

$$\| V_1 - V_2 \|_{C^{1+\delta}} \leqslant C . t^{\gamma} . \| \widehat{V}_1 - \widehat{V}_2 \|_{C^{1+\delta}} .$$

By reducing T_0, if necessary, we conclude that the mapping carrying $\widehat{V}(y, t) \in X$ to the corresponding solution $V(y, t) \in X$ of the problem III is a contraction. It remains then to apply Banach's fixed point theorem to complete the proof.

III — EXISTENCE AND UNIQUENESS OF THE SOLUTION

THEOREM. *Under the asssumptions (A) — (F), Problem I admits a unique solution.*

Proof. For any $S(t) \in \mathcal{S}$, let $V(y, t)$ be the solution of the auxiliary problem II. Denote by $\widetilde{S}(t)$ the solution of the problem:

$$\begin{cases} \dot{\widetilde{S}}(t) = f\,[V(1, t)] \\ \widetilde{S}(0) = b. \end{cases}$$

It suffices to prove that the mapping A: $S(t) \mapsto \widetilde{S}(t)$ is a contraction from $\mathcal{S}$ into itself.

First, to show that A is a contractive mapping from $\mathcal{S}$ into itself, we note that

$$| \dot{\widetilde{S}}(t) | = | f(V(1, t)) | \leqslant f(c_0) = R_0 .$$

Hence

$$| \widetilde{S}(t) - \widetilde{S}(0) | \leqslant R_0 T \text{ or } b - R_0 T \leqslant \widetilde{S}(t).$$

Thus we have $\breve{S}(t) \geqslant \beta$ if $T \leqslant \frac{b-\beta}{R_0}$. Therefore we can choose $\bar{T}$ such that A maps $\mathcal{S}$ into itself.

Denote by V_1, V_2 the solutions of the problem (2.4) — (2.7) corresponding to $S_1, S_2 \in \mathcal{S}$. Let $W = V_1 - V_2$,

$$\delta(t) = S_1(t) - S_2(t).$$

Then W is the solution of the problem:

$$\begin{cases} W_t = A(y,t)\, W_{yy} + B(y,t)\, W_y + c(y,t)\, W + F_0(y,t) \\ \qquad \text{in } D^{\bar{T}} = (0,1) \times (0,\bar{T}) & (3.5) \\ W(y,0) = 0, \quad y \in (0,1); & (3.6) \\ W(0,t) = 0, \quad t \in (0,\bar{T}); & (3.7) \\ W_y(1,t) = -\dot{S}_1(t)\, S_1(t)\, [\lambda(S_1(t),t) + \Phi(\dot{S}_1(t))] + \\ \qquad + \dot{S}_2(t)\, S_2(t)\, [\lambda(S_2(t),t) + \Phi(\dot{S}_2(t))], \\ \qquad t \in (0,\bar{T}); & (3.8) \end{cases}$$

where

$$A(y,t) = a(S_1 y, t, v_1, S_1)\, S_1^{-2}, \tag{3.9}$$

$$B(y,t) = \dot{S}_1 S_1^{-1} y + \bar{q}_p S_1^{-1}, \tag{3.10}$$

$$C(y,t) = \bar{q}_u + \bar{a}_u S_1^{-2} V_{2yy}, \tag{3.11}$$

$$F_0(y,t) = \delta \{\bar{q}_x \cdot y - \bar{q}_p S_1^{-2} S_2^{-1} V_{2y} + V_{2yy} S_1^{-2} (\bar{a}_x y + \bar{a}_s) +$$
$$+ V_{2yy} S_1^{-2} S_2^{-2} a(S_2 y, t, v_2, S_2)(S_1 + S_2) + \dot{S}_1 S_1^{-1} S_2^{-1} y\, v_{2y} +$$
$$+ \dot{\delta} \{ y S_2^{-1} V_{2y} + \bar{q}_\sigma \}.$$

Denote by $\bar{a}_x, \bar{a}_u, \bar{a}_s, \bar{q}_x, \bar{q}_u, \bar{q}_s, \bar{q}_\sigma$ the mean value of $a_x, a_u, a_s, q_x, q_u, q_s, q_\sigma$, respectively.

We write $W = W_1 + W_2$, where W_1 is solution of the problem

$$\begin{cases} W_{1t} = A(y,t)\, W_{1yy} \quad \text{in } D^{\bar{T}}; & (3.12) \\ W_1(y,0) = 0, \quad y \in (0,1); & (3.13) \\ W_1(0,t) = 0, \quad t \in (0,\bar{T}); & (3.14) \end{cases}$$

$$W_{1y}(1,t) = -\dot{S}_1(t) S_1(t)\, [\lambda(S_1(t),t) + \Phi(\dot{S}_1(t))]$$
$$+ \dot{S}_2(t) S_2(t)\, [\lambda(S_2(t),t) + \Phi(\dot{S}_2(t))] \equiv G(t)$$
$$t \in (0,\bar{T}). \tag{3.15}$$

and $\bar{W}_2$ is solution of the problem

$$\begin{cases} W_{2t} = A(y,t)W_{2yy} + B(y,t)W_{2y} + c(y,t)W_2 + F_1(y,t) \\ \qquad\qquad \text{in } D^{\bar{T}}; & (3.16) \\ W_2(y,0) = 0, \quad y \in (0,1); & (3.17) \\ W_2(0,t) = 0, \quad t \in (0,\bar{T}); & (3.18) \\ W_{2y}(1,t) = 0, \quad t \in (0,\bar{T}). & (3.19) \end{cases}$$

By applying the maximum principle to the problem (3.12) — (3.15) we get

$$|W_1| \leqslant \text{const}.\max_{D^{\bar{T}}} |G(t)|.$$

Using the inequality $\|\delta\|_t \leqslant t\|\dot{\delta}\|_t$ and the assumptions (D), (E) we have

$$\begin{aligned} &-\dot{S}_1(t)S_1(t)\lambda(S_1(t),t) + \dot{S}_2(t)S_2(t)\lambda(S_2(t),t) = \\ &= [S_2(t) - S_1(t)] \times [\dot{S}_1(t)\lambda(S_1(t),t)] + \\ &+ S_2(t)[\dot{S}_2(t)\lambda(S_2(t),t) - \dot{S}_1(t)\lambda(S_1(t),t)] \\ &\leqslant \text{const}\,|S_2 - S_1| + |S_2(t)|\cdot|\{(\dot{S}_2 - \dot{S}_1)\lambda(S_2(t),t) + \\ &+ \dot{S}_1[\lambda(S_2(t),t) - \lambda(S_1(t),t)]\}| \\ &\leqslant \text{const}\,|S_2 - S_1| + |S_2|.|\lambda(S_2(t),t)|.|\dot{S}_2 - \dot{S}_1| + \\ &+ |S_2\dot{S}_1|\,|\bar{\lambda}_x|\,|S_2 - S_1| \leqslant \text{const}.\,t.\,\|\dot{S}_2 - \dot{S}_1\|_t. \end{aligned}$$

Similarly, we get

$$|-\dot{S}_1(t)S_1(t)\Phi(\dot{S}_1(t)) + \dot{S}_2(t)S_2(t)\Phi(\dot{S}_2(t)| \leqslant \text{const}.\,t.\,\|\dot{\delta}\|_t.$$

Hence

$$|W_1(y,t)| \leqslant \text{const}.\,t.\,\|\dot{\delta}\|_t. \qquad (3.20)$$

To estimate $W_2(y,t)$, we identify it with the restriction to $D^{\bar{T}}$ of the solution of the following problem

$$\begin{cases} W_{2t} = \bar{A}(y,t)W_{2yy} + \bar{B}(y,t)W_{2y} + \bar{C}(y,t)W_2 + \bar{F}_1(y,t) \\ \qquad\qquad \text{in } (0,2)\times(0,\bar{T}); & (3.21) \\ W_2(y,0) = 0, \quad y \in (0,2); & (3.22) \\ W_2(0,t) = 0, \quad t \in (0,\bar{T}); & (3.23) \\ W_2(2,t) = 0, \quad t \in (0,\bar{T}); & (3.24) \end{cases}$$

where

$$\bar{A}(y,t) = \begin{cases} A(y,t), & y \in (0,1); \\ A(2-y,t), & y \in (1,2); \end{cases}$$

$$\overline{\overline{B}}(y,t) = \begin{cases} B(y,t) & , \ y \in (0, 1) \ ; \\ B(2-y, t) & , \ y \in (1, 2) \ ; \end{cases}$$

$$\overline{C}(y,t) = \begin{cases} C(y,t) & , \ y \in (0, 1) \ ; \\ C(2-y, t) & , \ y \in (1, 2) \ ; \end{cases}$$

$$\overline{F}(y,t) = \begin{cases} F_1(y,t) & , \ y \in (0, 1) \ ; \\ F_1(2-y, t) & , \ y \in (1, 2) \ ; \end{cases}$$

Using the method of [1] and Gronwall's lemma we obtain

$$| W_2(y,t) | \leqslant \int_0^t \max \overline{C}(\eta, \tau) W_2(y, \tau) d\tau + \int_0^t \max F_0(y, \tau) d\tau \ ; \ \eta \in [0, 1].$$

Taking account of the estimate $| V_{2_{yy}} | \leqslant \text{const } \tau^{-\frac{1}{2}}$, (see [1 appendix 3]) and remembering the definition of $F_0, \overline{C}$, we have

$$\max | W_2(y,t) | \leqslant \int_0^t \max \overline{C}(\eta, \tau) W_2(y, \tau) d\tau + \text{const. } t. \ \| \dot{\delta} \|_t$$

$$\leqslant L_0 \left\{ \int_0^t \tau^{-\frac{1}{2}} \max | W_2(y, \tau) | d\tau + t. \ \| \dot{\delta} \|_t \right\}. \tag{3.25}$$

By virtue of (3.25) and Gronwall's lemma we can write

$$\max_{y \in [0,1]} | W_2(y,t) | \leqslant L_0 \ . \ t. \ \| \dot{\delta} \|_t \ .$$

Finally,

$$| \dot{\widetilde{S}}_1 - \dot{\widetilde{S}}_2 | = | f(V_1(1,t)) - f(V_2(1,t)) | = | f'(v) | . | V_1(1,t) - V_2(1,t) |$$
$$\leqslant M \ . \ t \ . \| \dot{\delta} \|_t \ .$$

Let us choose $\overline{T}$ so small that $M \, . \, t = \omega < 1, \ \forall \ t \in [0, \overline{T}]$. Then we have

$$\left| \dot{\widetilde{S}} - \dot{\widetilde{S}}_2 \right| \leqslant \omega \ | \dot{S}_1 - \dot{S}_2 | \quad \text{with } 0 < \omega < 1$$

and finally

$$\| \widetilde{S}_1 - \widetilde{S}_2 \|_{C_1} \leqslant \omega \ . \| S_1 - S_2 \|_{C_1}$$

This concludes the proof of the theorem.

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