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A FINITE ALGORITHM FOR GLOBALLY MINIMIZING A CONCAVE FUNCTION UNDER LINEAR CONSTRAINTS AND ITS APPLICATIONS

VU THIEN BAN

1. INTRODUCTION

The problem of minimizing a concave function $f(x)$ subject to linear constraints was first studied by H. Tuy [3]. Since then, this problem has attracted a great deal of attention from a number of researchers (see [2], where an extensive bibliography up to 1979 was given). This interest is motivated, on the one hand, by the practical origin of the problem and the importance of concave minimization as a typical case of global optimization and, on the other, by the fact that a large class of mathematical programming problems (including $0-1$ integer programming and bilinear programming can be reduced to a problem of concave minimization (see [2]).

Up to recently, however, most of the authors have concentrated only on the special (though very important) case where the feasible set of the problem is compact, i.e. is a polytope. The only papers that deal with the problem without the boundedness condition imposed upon the constraint set are, to my knowledge, [5] and [4], where the most general problem of minimizing a concave function over an arbitrary closed convex set was considered for the first time. Note that the algorithms given in these papers are in general infinite (though surely convergent), as might be expected. It is therefore of interest to have a *finite* and reasonably efficient algorithm for the minimization of a concave function over a *polyhedral* convex set which is *not necessarily bounded*. The purpose of the present paper is just to develop such an algorithm.

In Section 2, we shall describe the general idea of our method, which can be regarded as a further development of the cone splitting and the cone bisection procedures worked out in H. Tuy [3] and Ng. V. Thoai and H. Tuy [2], respectively. The novelty of our approach is the device of a bisection method which will guarantee the finiteness of the algorithm, taking account of the linear structure of the feasible set. This cone bisection method, along with the bounding operations involved in the algorithm, will be discussed in detail in

Sections 3 and 4. The finiteness of the algorithm will be derived in Section 5. Finally in Section 6 we shall discuss the applications of the present algorithm to the fixed charge, the bilinear programming and the linear complementarity problems.

2. OUTLINE OF THE METHOD

In precise terms the problem we are concerned with can be formulated as follows :

$$\text{Minimize } f(x) \text{ subject to } x \in D,$$

where D is a given polyhedral convex (not necessarily bounded) set in R^n, $f : R^n \to R$ is a given concave, upper semi-continuous function on R^n. We emphasize that we are seeking here a *global* minimum, i. e. a point $\bar{x} \in D$ such that $f(\bar{x}) \leqslant f(x)$ for all $x \in D$ (the problem would be a standard one if only a local minimum was required).

Without any loss of generality we may (by increasing if necessary the dimension of the underlying space) assume that D lies in the hyperplane $x_n = 1$ of R^n and is given by a system of form :

$$Ax = 0 \tag{1}$$

$$x \geqslant 0 \tag{2}$$

$$x_n = 1, \tag{3}$$

where A is an $m \times n$—matrix of rank m, with the m first columns $A_1, \ldots, A_m$ linearly independent.

Writing for each $k = m + 1, \ldots, n$

$$A_k = \sum_{j=1}^{m} \alpha_j^k A_j ,$$

and defining $z^k \in R^n$ by

$$z_j^k = -\alpha_j^k \ (j = 1, \ldots, m), \ z_k^k = 1, \ z_j^k = 0 \ (j \neq k \,; \, j = m + 1, \ldots, n),$$

we have $n - m$ vectors $z^{m+1}, \ldots, z^n$ such that $Az^k = 0$ ($k = m + 1, \ldots, n$). Denote by M^0 the cone vertexed at 0, generated by these vectors

$$M^0 = \text{cone } \{ z^{m+1}, \ldots, z^n \}.$$

LEMMA 1. *M^0 coincides with the set of all $x \in R^n$ satisfying*

$$Ax = 0, \ x_{m+1} = 0, \ldots, x_n = 0. \tag{4}$$

Proof. If $x \in M^0$ then $x = \sum_{k=m+1}^{n} \xi_k z^k$ with $\xi_k = 0$, hence $Ax = \Sigma \, \xi_k Ax_k = 0$, and $x_k = \xi_k \geqslant 0$ ($k = m + 1, \ldots, n$).

Conversely, if x satisfies (4) then

$$\sum_{j=1}^{m} x_j A_j + \sum_{k=m+1}^{n} x_k \left(-\sum_{j=1}^{m} z_j^k A_j\right) = 0,$$

hence

$$x_j = \sum_{k=m+1}^{n} z_j^k x_k \quad (j = 1,\dots, m),$$

i. e. $x = \sum_{k=m+1}^{n} x_k z^k$ with $x_k \geqslant 0$ $(k = m + 1,\dots, n)$, or $x \in M^0$.

COROLLARY. *We have* $D \subset M^0$. (5)

Furthermore:

1) *A vector* $x \in M^0$ *with* $x_n = 1$ *is feasible if and only if* $x_1 \geqslant 0,\dots, x_m \geqslant 0$;

2) *A vector* $x \in M^0$ *with* $x_n = 0$ *is a direction of recession of the feasible set* D *if and only if* $x_1 \geqslant 0,\dots, x_m \geqslant 0$.

Now the method we propose for solving the problem proceeds according to the following scheme which is essentially the same as that of the method of Ng. V. Thoai and H. Tuy in [2].

We start with the cone M^0 and a feasible solution x^0.

At step $k = 0, 1,\dots$, we are presented with a collection $\mathfrak{R}^k$ of polyhedral convex cones vertexed at 0, each having at most $n - m$ edges (at step 0, $\mathfrak{R}^0 = \{M^0\}$). To each cone $M \in \mathfrak{R}^k$ is associated a number $\mu(M)$ which is an estimated lower bound for $f(x)$ in the feasible region contained in M. Moreover, a feasible solution x^k is available, which is the best feasible solution known up to this step. The collection $\mathfrak{R}^k$ is constructed in such a way that no feasible solution better than x^k exists in the feasible region outside $\cup \{ M : M \in \mathfrak{R}^k \}$, i. e. the feasible solutions x lying in the cones of $\mathfrak{R}^k$ are the only ones that remain to be explored. Therefore if it happens that $\mathfrak{R}^k = \phi$ then x^k must be an optimal solution of the problem, and the algorithm stops. Otherwise, we choose in the collection $\mathfrak{R}^k$ a cone M^k with $\mu(M^k) = \min \{\mu(M) : M \in \mathfrak{R}^k\}$, we perform some definite operations on M^k which result in replacing it by a subcone or splitting it into two subcones. Then we compute a lower bound $\mu(M)$ for each one of the newly obtained cones, update the current best feasible solution and pass to the next step, with a newly formed collection of cones, $\mathfrak{R}^{k+1}$, and a new current best feasible solution x^{k+1}.

For the convergence of the above procedure, the way of defining the operations to be performed on M^k and forming the new collection $\mathfrak{R}^{k+1}$, as well as the method of estimating a lower bound $\mu(M)$ for each cone M, are of crucial importance. In [2] it has been shown that convergence can be guaranteed if the splitting process is «exhaustive» and the bound estimation «consistent».

In the sequel we shall show that not only convergence, but even finiteness of the above procedure can be secured, provided one uses, along with a consistent bound estimation, a special cone splitting method exploiting the linear structure of the feasible set.

3. OPERATIONS ON CONES

First, we proceed to describe the operations to be performed on the cones of a given system $\mathfrak{R}^k$.

Consider an arbitrary cone $M \subset M^0$, with vertex at O and with at most $n - m$ edges. Denote by $p = p(M)$ the number of edges of M, and let $x^1, \ldots, x^p$ be the directions of these edges, so that

$$M = \text{cone}\,(x^1, \ldots, x^p).$$

The cone M can be characterized by the matrix:

$$\begin{matrix} x_1^1 & x_1^2 & \ldots & x_1^p \\ x_2^1 & x_2^2 & \ldots & x_2^p \\ \cdots & \cdots & \cdots & \cdots \\ x_n^1 & x_n^2 & \ldots & x_n^p. \end{matrix} \qquad (6)$$

We shall call this matrix the *matrix of M*. Sometimes we shall find it convenient to denote this matrix by the same symbol M.

Since $M \subset M^0$, the last row of the matrix (6) has no negative entry.

We now describe some operations which can be performed on a cone (i.e. on its matrix).

I. DELETION

LEMMA 2. *Suppose that either of the following conditions holds:*

1. The matrix M has one row with all negative entries;

2. The matrix M has its last row consisting entirely of zeros.

Then the cone M contains no feasible solution, i. e. $M \cap D = \varnothing$.

Proof. If for some $j : x_j^k < 0$ $(k = 1, \ldots, p)$, then every edge of M lies in the halfspace $\{x \in R^n : x_j < 0\}$. Hence M lies in this halfspace, and so $M \cap D = \varnothing$. If $x_n^k = 0$ $(k = 1, \ldots, p)$ then every edge of M lies in the « horizontal » hyperplane $x_n = 0$. Hence M lies in this hyperplane, and so $M \cap D = \varnothing$ because $D = \{x_n = 1\}$.

Thus if either of the above conditions holds, the cone M offers no interest for our purpose and can simply be *deleted* as irrelevant.

Suppose now that the matrix M has at least one negative entry. We shall call the first row of M that contains a negative entry the *test row*. Let s denote the index of the test row of M.

LEMMA 3. *Suppose that the test row s of M contains at most one positive entry. Denote by M' the matrix that is obtained from M by the following operation:*

1) If the row s has just one positive entry x^i, *then replace every* x^j *such that* $x^j_s < 0$ *by*

$$y^j = x^i_s x^j - x^j_s x^i; \tag{7}$$

2. If the row s has no positive entry, then delete every x^j *such that* $x^j_s < 0$.

Then $M' \supset \{x \in M : x_s \geqslant 0\}$. (8)

Proof. Suppose $x^i_s > 0$, while $x^j_s \leqslant 0$ $(i \neq j)$, and let $x \in M$ with $x_s \geqslant 0$. Then $x = \sum_{k=1}^{p} \alpha_k x^k$ with $\alpha_k \geqslant 0$. Denote $J = \{j : x^j_s < 0\}$. Replacing each x^j $(j \in J)$ by $(y^j + x^j_s x^i) / x^i_s$, we get

$$x = \sum_{k \notin J \cup \{i\}} \alpha_k x^k + \Big(\sum_{j \in J} \alpha_j x^j_s / x^i_s + \alpha_i\Big) x^i + \sum_{j \in J} (\alpha_j / x^i_s)\, y^j,$$

where $\sum_{j \in J} \alpha_j x^j_s + \alpha_i x^i_s = x_s \geqslant 0$ (note that $x^k_s = 0$ for $k \notin J \cup \{i\}$). Therefore $x \in M'$, proving (8).

If the row s has no positive entry, i.e. $x^j_s \leqslant 0 (\forall j)$, then obviously $x_s \leqslant 0$ for all $x \in M$, and M' is nothing but the intersection of M with the hyperplane $x_s = 0$. Hence (8) must hold.

Thus if the matrix M satisfies the condition of Lemma 3, then $M \setminus M'$ contains no feasible solution (since $x_s < 0$ for every $x \in M \setminus M'$). Therefore in this case, we can *reduce M* by replacing it with its subcone M'.

III. SPLITTING

If a matrix M has a test row s, but cannot be reduced according to the previous Lemma, there are two entries $x^i_s > 0$ and $x^j_s < 0$ in this test row. We can then split the cone M into two subcones in the following way. Let

$$y = x^i_s x^j - x^j_s x^i \tag{9}$$

and denote by M_1 (M_2, resp.) the matrix that is obtained from M by replacing x^i (x^j, resp.) with y.

LEMMA 4. *We have* $M = M_1 \cup M_2$.

Proof. Since y is a positive combination of x^i and x^j, the inclusion $M_1 \cup M_2 \subset M$ is obvious. To prove the converse inclusion, let $x \in M$ i.e. $x = \sum_{k=1}^{p} \alpha_k x^k$ with $\alpha_k \geqslant 0$. If $\eta = \alpha_i x_s^i + \alpha_j x_s^j \geqslant 0$, then substituting $x^i = (x_s^i x^j + y)/x_s^j$ we can write

$$x = \sum_{\substack{k \neq i \\ k \neq j}} \alpha_k x^k + (\alpha_i / x_s^j)\, y + (\eta / x_s^j)\, x^j,$$

i. e. $x \in M_1$. Similarly, if $\eta \leqslant 0$ then $x \in M_2$. Therefore $M \subset M_1 \cup M_2$.

The reason why the test row has been taken to be the *first* row that contains negative entries can be seen from the following proposition.

LEMMA 5. *If M' is obtained from M by a reduction, or a splitting operation, then either the test row of M' has more zero than that of M or it has a greater index.*

Proof. It suffices to consider for instance the case of a reduction (because the argument in the case of a splitting operation is similar). Let j be an index such that $x_s^j < 0$. We have from (7), $y_s^i = 0$, while $y_h^j \geqslant 0$ for all $h < s$, (because $x_h^j \geqslant 0$, $x_h^i \geqslant 0$). Therefore the index of the test row of M' is at least equal to s, and if it is equal to s then $y_s^j = 0$ is a new zero (because $x_s^j < 0$).

As will soon become apparent this Lemma provides a basis for a proof of the finiteness of the algorithm.

4. BOUND ESTIMATION

Let M be any cone with matrix (6) (we assume of course $M \subset M_0$). Denoting by Q the hyperplane $x_n = 1$ we have obviously

$$M \cap D = M \cap Q \cap R_+^n.$$

Therefore the number

$$\mu(M) = \min \{f(x) : x \in M \cap Q\}$$

always provides a lower bound for f over $M \cap D$. To compute $\mu(M)$ we use the following

LEMMA 6. *$M \cap Q$ is a polyhedral convex set with at most $n - m$ extreme elements (points or rays), the extreme points being $u^i = x^i / x^i_n$ ($i \in I_+$), the extreme rays being x^j ($j \in I_0$), where $I_+ = \{ i : x^i_n > 0 \}$, $I_0 = \{ i : x^i_n = 0 \}$. (Note that $I_+ \neq \phi$, otherwise M would be deleted according to Lemma 2).*

Proof. Let Δ be the polyhedral convex set with extreme points u^i ($i \in I_+$) and extreme rays x^j ($j \in I_0$), i. e. the set of all x such that

$$x = \sum_{i \in I_+} \alpha_i u^i + \sum_{j \in I_0} \beta_j x^j,$$
$$\alpha_i \geq 0,\ \Sigma \alpha_i = 1,\ \beta_j \geq 0.$$

Then obviously $x_n = 1$ (because $u^i_n = 1$, $x^j_n = 0$), and $x \in M$, (because $x = \sum_{i \in I_+} \frac{\alpha_i}{x^i_n} x^i + \sum_{j \in I_0} \beta_j x^j$). Hence

$$\Delta \subset M \cap Q.$$

To prove the converse, let $x \in M \cap Q$, so that

$$x_n = 1,\ x = \sum_{i \in I_+} \alpha_i x^i + \sum_{j \in I_0} \beta_j x^j,\ \alpha_i,\ \beta_j \geq 0.$$

Setting $\lambda_i = \alpha_i / x^i_n$ we have

$$x = \sum_{i \in I_+} \lambda_i u^i + \sum_{j \in I_0} \beta_j x^j$$

with $\lambda_i = 0$, $\beta_j = 0$. Since $1 = x_n = \sum_{i \in I_+} \alpha_i / x^i_n = \sum_{i \in I_+} \lambda_i$, it follows that $x \in \Delta$, completing the proof.

COROLLARY. *Let i_* be an arbitrary (fixed) element of I_+. If there exist $j \in I_0$ and $\alpha > 0$ such that $f(u^i + \alpha x^j) < f(u^{i_*})$ then $\mu(M) = -\infty$; otherwise*

$$\mu(M) = \min \{ f(u^i) : i \in I_+ \}. \tag{10}$$

Proof. It is known that a concave function either is unbounded below over a ray, or attains its minimum at the origin of this ray (see e. g. [1]). Therefore if for some $j \in I_0$ and $\alpha > 0$ we have $f(u^i + \alpha x^j) < f(u^{i*})$ then f is unbounded below over the extreme ray of $M \cap Q$ emanating from u^{i*} in the direction x^j. In the contrary case, $\gamma = f(u^{i_*})$ is the minimum of $f(x)$ over each one of the rays emanating from u^{i*} in the directions x^j ($j \in I_0$). Since the set of all $x \in M \cap Q$ where $f(x) \geq \gamma$ is convex and closed, since each ray emanating from u^i in the direction $x_j (j \in I_0)$ is contained in this set, it follows that each ray emanating from any extreme point u of $M \cap Q$ in the direction x^j is also contained in this set (see e. g. [2]). Therefore f cannot be unbounded below in any extreme ray of $M \cap Q$. But then f attains its minimum over $M \cap Q$ at some extreme point. Hence (10) follows.

Remark 1. The above bound estimation is «*consistent*» in the following sense: whenever a cone M cannot be further splitted, then $\mu(M)$ equals the exact minimum of $f(x)$ over $M \cap D$. Indeed, M cannot be further splitted only if its matrix has no negative entry. Then, by Corollary of Lemma 1, every u^i $(i \in I_+)$ is a feasible point, while every x^j $(j \in I_0)$ is a direction of recession of D. Therefore, $\mu(M) = \min \{ f(x) ; x \in M \cap D \}$.

5. ALGORITHM

The above development leads to the following algorithm.

Initilization. Let M^0 be the initial cone spanned by $z^{m+1}, \ldots, z^n$.

Reduce M^0 if possible. Compute $\mu(M^0)$. Set $\mathcal{R}^0 = \{ M^0 \}$. Set x^0 = any available feasible solution.

Step $k = 0,1,\ldots$ At this step there are available a current best feasible solution x^k, a collection of cones $\mathcal{R}^k$, and for each cone $M \in \mathcal{R}^k$ a lower bound $\mu(M)$ for $f(x)$ over $M \cap D$.

a) If $\mathcal{R}^k = \varnothing$, stop ; x^k is an optimal solution.

b) Otherwise, choose $M^k = \arg\min \{ \mu(M) : M \in \mathcal{R}^k \}$. Split M^k into two subcones M_1^k, M_2^k. Delete any of these new cones that can be deleted according to Lemma 2. Reduce any of these cones that can be reduced according to Lemma 3. For each resulting cone M, compute $\mu(M)$ as indicated in the previous section. If in computing $\mu(M)$, an infinite edge of $M \cap D$ is discovered such that $f(x)$ is unbounded below on this edge (see Corollary of Lemma 6), stop; the problem has no finite optimal solution. Otherwise, set x^{k+1} equal to the new current best feasible solution (i.e. the best among x^k and all feasible solutions newly generated in the present step). Delete all cones M such that

$$\mu(M) \geqslant f(x^k). \tag{11}$$

Set $\mathcal{R}^{k+1}$ equal to the collection of all remaining cones and go to sep $k+1$.

Remark 2. According to the formation of $\mathcal{R}^k$ as above, M^k has always at least one negative entry in its matrix and hence can be splitted by the procedure indicated in Lemma 4. Indeed otherwise, by Remark 1, $M^k \cap D = M^k \cap Q$ and $\mu(M^k) = \min \{ f(x) ; x \in M^k \cap D \}$. Moreover this minimum must be finite (otherwise the algorithm would have stopped). Therefore $\mu(M^k) = f(u)$ for some extreme point u of $M^k \cap D$, and by the definition of x^k, we must have $f(x^k) \leqslant f(u) = \mu(M^k)$. So M would have been deleted.

Remark 3. Suppose that the problem has been originally set in R^{n-1}, with $D \subset R^{n-1}$ given by the system

$$\sum_{j=1}^{n-1} x_j A_j = b \tag{12}$$
$$x_j \geqslant 0 \ (j = 1. \ldots, n-1).$$

Introducing an additional variable x_n and setting $A_n = -b$ we can write this system into the form described in Section 2, namely

$$\sum_{j=1}^{n} x_j A_j = 0$$
$$x_j \geqslant 0 \ (j = 1, \ldots, n)$$
$$x_n = 1.$$

Therefore, if $\{A_1, \ldots, A_m\}$ is a feasible basis for (12), the Algorithm can star with $M^0 = \text{cone} \{z^{m+1}, \ldots, z^n\}$ and $x^0 = z^n$.

THEOREM. *The above Algorithm terminates after finitely many steps.*

Proof. First observe that the number of descendants of a cone M is always finite, where by « descendant » of a cone we mean any cone which can be derived from it by a finite sequence of reductions and splitting operations as described in Section 3. Indeed, by Lemma 5 one such operation increases either the index of the test row or the number of zeros in the test row. Since the matrix M has only n rows and at most $n-m$ columns, there exists a number N (independent of M) such that after at most N reduction and splitting operations we shall arrive at a cone which can not be further reduced, nor splitted. Now let us associate to the Algorithm a tree rooted at M^0, whose nodes are the cones generated during the procedure, and where there is an are from a node M to a node M' if and only if M' is obtained from M by a single reduction or splitting operation. Then by the above observation, any path in this tree starting from the root, is bounded in length by N. Therefore, the tree must be finite, which implies that the Algorithm itself must be finite.

Remark 4. The Algorithm can be started, even if no feasible solution x^0 is available at the beginning. In fact, it suffices to replace in step $k = 0,1,\ldots,$ the condition (11) by

$$\mu(M) \geqslant \gamma^k ,$$

where $\gamma^k = +\infty$ if no feasible solution has been available yet at step k, and $\gamma^k = f(x^k)$ if a feasible solution is already available and x^k is the best feasible solution so obtained.

Remark 5. In the above discussion it is not excluded that $f(x) = -\infty$ at some points outside D (provided the function is upper semi-continuous). However, if $f(x) = -\infty$ everywhere outside D then it can be seen that a cone M can be deleted only if $M \cap Q = M \cap D$ and so the Algorithm will necessarily generate all the extreme points and extreme rays of D. More generally, this is true if $f(x) < \min \{ f(y) : y \in D\}$ for all $x \notin D$.

Remark 6. If the feasible set D is compact, the Algorithm can be applied even if the function f is only quasiconcave and not necessarily upper semi-continuous. Indeed, in that case we can choose the initial cone M^o so that its intersection with the hyperplane $Q = \{ x_n = 1 \}$ is compact (and contains D). Then for every cone $M \subset M^o$ the intersection $M \cap Q$ is always compact, so that we can always take $\mu(M)$ to be equal to the minimum of f at the extreme points of $M \cap Q$.

6. APPLICATIONS

The above Algorithm can also be applied to some important problems of mathematical programming.

I – THE FIXED CHARGE PROBLEM

$$\text{Minimize } f(x), \text{ subject to}$$
$$Ax \geqslant b$$
$$x \geqslant 0,$$

where A is an $m \times n$ matrix, b an m-vector, x an n-vector and $f(x) = \sum_{i=1}^{n} f_i(x_i)$, with $f_i(t)$ a concave function such that $f_i(t) > 0$ $(t > 0)$; $f_i(+0) = d_i > 0$, $f_i(0) = 0$ (so the function $f_i(.)$ is discontinuous at 0). If we extend $f_i(t)$ over the whole line $(-\infty, \infty)$ by setting $f_i(t) = 0$ for $t < 0$ then the function f is quasiconcave. Hence, by Remark 6, the above Algorithm can be applied, provided the feasible set is compact.

II – THE BILINEAR PROGRAMMING PROBLEM

Minimize $f(x, y)$ subject to

$$x \in D, \; y \in E,$$

where D is a polytope in R^p, E a polyhedral convex set in R^q, and $f(x, y)$ is affine in x for y fixed, affine in y for x fixed. Setting

$$\varphi(x) = \min \{f(x, y) : y \in E\}$$

we have a concave function φ defined on R^p, and the problem is reduced to minimizing φ over D. Since for every x the value of $\varphi(x)$ can be easily computed by minimizing a linear function (depending upon x) over a fixed polytope, the above Algorithm applies and yields a finite procedure for solving the bilinear programming problem.

With a bit more effort the method can be extended to the case where D is unbounded. The computation of $\mu(M)$ in this case involves solving parametric linear programs of the form

$$\min \{ f(u + \theta x, y) : y \in E \}, \; 0 \leqslant \theta < \infty.$$

III – THE LINEAR COMPLEMENTARITY PROBLEM

Find $x \in R^n$, $y \in R^n$ such that

$$y = Ax + b \geqslant 0, \quad x \geqslant 0, \quad y^T x = 0.$$

As is known [6], this problem reduces to the concave program

Minimize $f(x)$ subject to

$$Ax + b \geqslant 0, \; x \geqslant 0$$

with $f(x) = \sum_{i=1}^{n} \min \{x_i, \sum_j a_{ij} x_j + b_i\}$. We can apply the above Algorithm to solve this concave program: if an optimal solution $\overline{x}$ exists such that $f(\overline{x}) = 0$, it is a solution to the linear complementarity problem; otherwise the linear complementarity problem has no solution.

7. COMPUTATIONAL EXPERIENCE

The above Algorithm was coded in Fortran IV and has been run on a Minsk-32 and an IBM/360-50. Preliminary results have shown that the Algorithm should be efficient for problems with $n = 100$ variables (approximately).

To avoid storing a large number of cones, a procedure has been devised to permit to store only the tree associated with the Algorithm as described in the proof of the convergence theorem. The cones needed in the course of computations are reconstructed from their positions on this tree.

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