ACTA MATHEMATICA VIETNAMICA
Vol 11 No 1 (1986)

ON A BOUNDARY VALUE PROBLEM OF HARMONIC FUNCTIONS

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Several properties of analytic functions: analyticity, expansion into series, Picard theorems etc... can be generalized to harmonic functions of n variables.

For harmonic functions and solutions of equations of elliptic type, it would be interesting to study models analogous to boundary value problems of analytic functions such as conditions of analytic continuation, jump problems, Riemann problems etc...

In this direction, some results on jump problems have been obtained in [1], [2] and on conjugation problems in bounded domains in [3], [4]. The conditions of harmonic continuation have been discussed in [5].

This paper adresses a counterpart of Riemann problem for harmonic vectors behaving at infinity as $O(|x|^m)$ (m is an integer $\geqslant -1$). The index of the problem will be given. In the case of jump problem when the solution vanishes at infinity, we shall recover the results obtained in [1].

In the sequel the following notations are adopted:

a) E_n: real euclidean space of dimension n, with points $x = (x_1, \dots, x_n)$.

b) $\Omega = \Omega^+$: bounded multiply connected domain with a Lipschitz boundary $S = \bigcup_{i=0}^{m} S_i$ where S_i, $i = 1, \dots, m$, are disjoint and S_0 encloses every S_i, $i = 1, \dots, m$, in its interior.

c) $\Omega^- = \bigcup_{i=0}^{m} \Omega_i^-$: complement of $\Omega^+ + S$ with respect to the entire space E_n. This set is composed of $m+1$ connected parts with respective boundaries S_i, $i = 0, \dots, m$. Ω_0^- is thus the exterior unbounded component of Ω^-.

d) $u^+(y)$ (resp., $u^-(y)$): limit value of the function $u(x)$ as x tends to $y \in S$, by values belonging to Ω^+ (resp., to Ω^-).

e) n_y^+ (resp., $n^-(y)$): inward (resp., outward) normal of S at the point $y \in S$, n_y: the corresponding non oriented normal.

f) $\frac{\partial u^{\pm}(y)}{\partial n_y^{+}}$ (resp., $\frac{\partial u^{\pm}(y)}{\partial n_y^{-}}$) limit of the derivative $\frac{\partial u(x)}{\partial n_y^{+}}$ (resp., $\frac{\partial u(x)}{\partial n_y^{-}}$)

as $x \in \Omega^{\pm}$ tends to $y \in S$ along the normal n_y.

g) $r_{xy} = | x - y |$: euclidean distance of two points x and y.

h) $| S_1 |$: surface measure of the unit sphere of E_n.

DEFINITION 1. *A function $u(x)$ defined in a neighbourhood of S, is said to have a regular normal derivative on S if there is a continuous function φ on S such that*

$$\lim_{\substack{x \to y \\ x \in n_y}} \frac{\partial u(x)}{\partial n_y} = \varphi(y) \ , \ \forall y \in S.$$

A function $u(x)$ defined in Ω^+ (resp., in Ω^-) is said to be of class $M(\Omega^+)$ (resp., $M(\Omega^-)$) if it is continuously extended from Ω^+ (resp., from Ω^-) up to S, and has a regular normal derivative on S.

DEFINITION 2. *A vector $u(x)$ of dimension N is said to be piecewise harmonic with surface of discontinuity S if its components $u_1(x), \dots u_N(x)$ are harmonic functions separately in Ω^+ and Ω^-, belonging to $M(\Omega^+)$ and $M(\Omega^-)$ respectively.*

Harmonic functions in Ω^-, under our consideration, may tend to infinity as $|x| \to +\infty$. If the equality $u(\infty) = 0$ holds, the harmonic function $u(x)$ is said to be regular at infinity. We state the following problem:

RIEMANN PROBLEM FOR HARMONIC VECTORS

Let $g_{ik}(y)$ be continuous square matrices of dimension $N \times N$ given on S and $f_i(y)$ continuous vectors of dimension N given on S.

Find a vector $u(x)$ (of dimension N), piecewise harmonic in Ω^+ and Ω^- behaving at infinity as $O(|x|^m)$, (m being an integer $\geqslant -1$), and satisfying the following conditions:

$$\left.\begin{aligned} u^+(y) &= g_{11}(y)u^-(y) + g_{12}(y)\frac{\partial u^-(y)}{\partial n_y^-} + f_1(y) \\ \frac{\partial u^+(y)}{\partial u_y^+} &= g_{21}(y)u^-(y) + g_{22}(y)\frac{\partial u^-(y)}{\partial n_y^-} + f_2(y) \end{aligned}\right\} \forall y \in S \qquad (1)$$

For the sake of convenience, we denote

$$u^+(y)=\mu^+(y),\qquad \frac{\partial u^+(y)}{\partial n_y^+}=\nu^+(y),$$ (2)

$$u^-(y)=\mu^-(y),\qquad \frac{\partial u^-(y)}{\partial n_y^-}=\nu^-(y),$$

$$\varphi^+(y)=\begin{pmatrix}\mu^+(y)\\ \nu^+(y)\end{pmatrix},\qquad \varphi^-(y)=\begin{pmatrix}\mu^-(y)\\ \nu^-(y)\end{pmatrix},$$

$$G(y)=\begin{pmatrix}g_{11}(y) & g_{12}(y)\\ g_{21}(y) & g_{22}(y)\end{pmatrix},\qquad H(y)=\begin{pmatrix}f_1(y)\\ f_2(y)\end{pmatrix}.$$

Problem (1) has the same form as Riemann problem of analytic functions

$$\varphi^+(y)=G(y)\,\varphi^-(y)+H(y) \tag{1'}$$

whence the terminology « Riemann problems for harmonic vectors ».

We confine ourselves to the case $n\geqslant 3$. The plane case can be investigated in the same manner by using the corresponding theorems of harmonic continuation in the plane [5].

Under the stated assumptions, it is easily seen, by the method given in [5], that the conditions, necessary and sufficient for validity the equalities (2) are:

$$\left.\begin{aligned}
&-\frac{1}{2}\mu^+(y_0)+\frac{1}{(n-2)\,|S_1|}\int_S\left[\mu^+(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right)-\frac{1}{r_{y_0y}^{n-2}}\nu^+(y)\right]dS_y=0,\\
&\frac{1}{2}\mu^-(y_0)+\frac{1}{(n-2)\,|S_1|}\int_S\left[\mu^-(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right)+\frac{1}{r_{y_0y}^{n-2}}\nu^-(y)\right]dS_y=P_m(y_0).
\end{aligned}\right\} \tag{3}$$

Here, $P_m(y_0)$ is the vector whose components are harmonic polynomials of order $\leqslant m$.

Substituting (1) and (2) into relations (3) yields an integral equation in the matrix form

$$A(y_0)\,\psi(y_0)+\int_S K(y_0,\,y)\,\psi(y)\,dS_y=F(y_0), \tag{4}$$

where

$$A(y_0)=\begin{pmatrix}g_{11}(y_0) & g_{12}(y_0)\\ E & 0\end{pmatrix},\qquad \psi(y)=\begin{pmatrix}\mu^-(y)\\ \nu^-(y)\end{pmatrix},$$

$$K(y_0,y)=$$

$$=\frac{2}{(n-2)\,|S_1|}\begin{pmatrix}-\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right)g_{11}(y)+\frac{1}{r_{y_0y}^{n-2}}g_{21}(y) & -\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right)g_{12}(y)+\frac{1}{r_{y_0y}^{n-2}}g_{22}(y)\\ \frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right) & \frac{1}{r_{y_0y}^{n-2}}\end{pmatrix}$$

$$F(y_0)=\begin{bmatrix}-f_1(y_0)+\frac{2}{(n-2)|S_1|}\int_S\left[\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right)f_1(y)-\frac{1}{r_{y_0y}^{n-2}}f_2(y)\right]dS_y\\ P_m(y_0)\end{bmatrix}=\begin{bmatrix}F_1(y_0)\\ P_m(y_0)\end{bmatrix}.$$

Assume that

$$\det g_{12}(y_0) \neq 0, \qquad y_0 \in S. \tag{5}$$

Since S is a Lipschitz surface, (hence a Liapounov surface) $K(y_0, y)$ is a fredholmian kernel and under condition (5), equation (4) is a Fredholm integral equation.

Suppose problem (1) is solvable. Then, as seen above, equation (4) is solvable, too. Conversely, assume equation (4) is solvable and has a solution $\psi(y) = \begin{pmatrix}\mu^-(y)\\ \nu^-(y)\end{pmatrix}$. By virtue of the second equation of (3), there exists a harmonic vector $u^-(x) \in M(\Omega^-)$ behaving at infinity as $O(|x|^m)$ such that

$$u^-(y) = \mu^-(y), \qquad \frac{\partial u^-(y)}{\partial n_y^-} = \nu^-(y).$$

Furthermore, the right-hand side of (1), denoted by $(\mu^+(y), \nu^+(y))$, satisfies the first equation of (3), so that there exists a harmonic vector $u^+(x)$ satisfying

$$u^+(y) = \mu^+(y), \qquad \frac{\partial u^+(y)}{\partial n_y^+} = \nu^+(y).$$

The pair of vectors $u^+(x)$, $u^-(x)$, obtained in this way satisfies relations (1), i. e. problem (1) is solvable. Consequently, problem (1) and equation (4) are simultaneously solvable or not.

Denote by $(1)_0$ and $(4)_0$ the homogeneous problem and equation corresponding to (1) and (4) respectively. If we confine ourselves to these homogeneous problem and equation, it is readily seen that the trivial solution of one corresponds to the trivial solution of the other. Besides, an arbitrary linear combination of two linearly independent solutions of one corresponds to the same combination of two corresponding solutions of the other. Therefore, the numbers of linearly independent solutions of the homogeneous problem $(1)_0$ and the homogeneous equation $(4)_0$ are equal.

In this manner, the equivalence of problem (1) and equation (4) is established. Denoting by $V_k(x)$, $k = 1, 2, \ldots, s$, the fundamental harmonic polynomials of order $\leqslant m$, we can write the right hand side of equation (4) in the form

$$F(y_0) = \begin{pmatrix} F_1(y_0) \\ P_m(y_0) \end{pmatrix} = J(y_0) + H(y_0) = J(y_0) + \sum_{k=1}^{s} \sum_{j=1}^{N} C_{kj} J_{kj}(y_0), \tag{6}$$

where

$$J(y_0) = \begin{pmatrix} F_1(y_0) \\ \\ \\ 0 \end{pmatrix}, \qquad J_{kj}(y_0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ V_k(y_0) \\ \vdots \\ 0 \end{pmatrix} \ldots (N+j)^{th} \text{ line.}$$

For the sake of convenience, equation (6) is rewritten as follows

$$A(y_0)\,\psi(y_0) + \int_S K(y_0, y)\,\psi(y)\,dS_y = J(y_0) + \sum_{i=1}^{q} d_i L_i(y_0), \tag{7}$$

where

$$q = sN, \; L_1(y_0) = J_{11}(y_0), \; L_2(y_0) = J_{12}(y_0), \ldots, \; L_q(y_0) = J_{sN}(y_0).$$

Let $\{\psi_i(y_0)\}$, $i=1,\dots, p$, and $\{\theta_i(y_0)\}$, $i=1, 2,\dots, p$, be linearly independen. solutions of the homogeneous equation

$$A(y_0)\,\psi(y_0)+\int_S K(y_0, y)\,\psi(y)\,dS_y=0 \tag{4$_0$}$$

and of the homogeneous adjoint equation

$$A^*(y_0)\,\theta(y_0)+\int_S K^*(y, y_0)\,\theta(y)\,dS_y=0, \tag{4$_0^*$}$$

respectively. (The sign* denotes the transposition of matrices). Equation (7) is solvable if and only if the constants d_i are chosen so that

$$\sum_{i=1}^{q} d_i\,(\theta_j(y), L_i(y))=-(\theta_j(y), J(y)),\quad j=1,\dots, p. \tag{8}$$

Here, the notation (u, v) for two vectors $u=(u_1,\dots, u_N)$, $v=(v_1,\dots, v_N)$ designate the quantity

$$\int_S \sum_{i=1}^{N} u_i(y)\,v_i(y)\,dS.$$

Let r be the rank of the matrix $\|(\theta_j(y), L_i(y))\|$, $j=1,\dots, p$; $i=1,\dots, q$.

Then the homogeneous algebraic system adjoint to (8) has $p-r$ linearly independent solutions

$$h^{(l)}=\begin{pmatrix} h_1^{(l)}\\ h_2^{(l)}\\ \vdots\\ h_p^{(l)}\end{pmatrix},\quad l=1, 2,\dots, p-r.$$

The conditions, necessary and sufficient for equation (8) to be solvable are

$$\sum_{j=1}^{p} h_j^{(l)}\,(\theta_j(y), J(y))=0,\quad l=1, 2,\dots, p-r$$

or

$$(\chi_l(y), J(y))=0,\quad l=1, 2,\dots, p-r, \tag{9}$$

where

$$\chi_l(y)=\sum_{j=1}^{p} h_j^{(l)}\,\theta_j(y).$$

The homogeneous equation corresponding to equation (8) has $q-r$ linearly independent solutions

$$d^{(m)}=\begin{pmatrix} d_1^{(m)}\\ d_2^{(m)}\\ \vdots\\ d_q^{(m)}\end{pmatrix},\quad m=1,\dots, q-r.$$

Therefore, under conditions (9) the general solution of (8) has the form

$$d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_q \end{pmatrix} = d^* + \sum_{m=1}^{q-r} \alpha_m d^{(m)} = \begin{pmatrix} d_1^* \\ d_2^* \\ \vdots \\ d_q^* \end{pmatrix} + \sum_{m=1}^{q-r} \alpha_m \begin{pmatrix} d_1^{(m)} \\ d_2^{(m)} \\ \vdots \\ d_q^{(m)} \end{pmatrix}$$

and so, we can write equation (7) as follows

$$A(y_0)\,\psi(y_0) + \int_S K(y_0, y)\,\psi(y)dS_y = J^*(y_0) + \sum_{m=1}^{q-r} \alpha_m M_m(y_0), \tag{10}$$

where

$$J^*(y_0) = J(y_0) + \sum_{i=1}^{q} d_1^* L_i(y_0), \quad M_m(y_0) = \sum_{i=1}^{q} d_i^{(m)} L_i(y_0).$$

Equation (10) is solvable with $q-r$ arbitrary constants α_m, as long as conditions (9) are fulfilled. The general solution of equation (7), or, which is the same, the general solution of equation (10) takes on the form

$$\psi(y_0) = \psi^*(y_0) + \sum_{i=1}^{p} \beta_i \psi_i(y_0) + \sum_{m=1}^{q-r} \alpha_m \sigma_m(y_0).$$

It is easily seen that the system $\{\psi_i(y_0),\ \sigma_m(y_0)\}$, $i = 1,\dots, p$; $m = 1,\dots, q\text{-}r$, are linearly independent, so that the homogeneous problem $(1)_0$ has exactly $p+q-r$ solutions and the non-homogeneous problem (1) is solvable if and only if $p-r$ conditions (9) are satisfied. These conditions are linearly independent, too. Hence the index of problem (1) is

$$\mathcal{X} = (p+q-r) - (p-r) = q.$$

Thus we have

THEOREM. *The Riemann problem (1) for harmonic vectors is noetherian and its index is equal to*

$$\mathcal{X} = q.$$

In particular, if the solutions are to be sought in the class of regular harmonic vectors ($u(\infty) = 0$), then the problem is fredhomian ($q = 0$).

Remark.

If $\det g_{12}(y) = 0$, but $\det G(y) \neq 0$, $\det g_{21}(y) \neq 0$, then by multiplying (1) by $G^{-1}(y)$ from the left, we reduce (1)' to the following problem

$$\varphi^-(y) = G^{-1}(y)\varphi^+(y) + H'(y),$$

which can be investigate just by the same method.

Special degenerate case.

To close the paper, let us examine the important special case where

$$u^+(y) = u^-(y) + f_1(y)$$
$$\frac{\partial u^+(y)}{\partial n_y^+} = -\frac{\partial u^-(y)}{\partial n_y^-} + f_2(y) \tag{12}$$

This problem is degenerate, since $\det g_{12}(y) = 0$ (and also $\det g_{21}(y) = 0$).

By writing it in the form

$$u^+(y) - u^-(y) = f_1(y),$$
$$\frac{\partial u^+(y)}{\partial n_y^+} - \frac{\partial u^-(y)}{\partial n_y^+} = f_2(y),$$

we see that this problem is the counterpart of the jump problem of analytic functions for harmonic functions. If the solutions are to be sought in the class of regular vectors ($u(\infty) = 0$), the problem has been investigated by another method in [1].

Taking conditions (12) in to account, we can write relations (3) in the form:

$$-\frac{1}{2}\mu^-(y_0) + \frac{1}{(n-2)|S_1|}\int_S \left[\mu^-(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right) + \frac{1}{r_{y_0y}^{n-2}}\nu^-(y)\right]dS_y =$$
$$= \frac{1}{2}f_1(y_0) - \frac{1}{(n-2)\,|S_1|}\int_S \left[f_1(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right) - \frac{1}{r_{y_0y}^{n-2}}f_2(y)\right]dS_y,$$
$$\frac{1}{2}\mu^-(y_0) + \frac{1}{(n-2)\,|S_1|}\int_S \left[\mu^-(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right) + \frac{1}{r_{y_0y}^{n-2}}\nu^-(y)\right]dS_y = P_m(y_0),$$

whence by substracting:

$$\mu^-(y_0) = -\frac{1}{2}f_1(y_0) + \frac{1}{(n-2)\,|S_1|}\int_S \left[f_1(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{y_0y}^{n-2}}\right) - \right.$$
$$\left. - \frac{1}{r_{y_0y}^{n-2}}f_2(y)\right]dS_y + P_m(y_0) =$$
$$= \lim_{\substack{x\to y_0\\ x\in\Omega^-}} \left\{\frac{1}{(n-2)\,|S_1|}\int_S \left[f_1(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{xy}^{n-2}}\right) - \right.\right.$$
$$\left.\left. - \frac{1}{r_{xy}^{n-2}}f_2(y)\right]dS_y + P_m(x)\right\}.$$

Therefore,

$$u(x) = \frac{1}{(n-2)\,|S_1|}\int_S \left[f_1(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{xy}^{n-2}}\right) - \frac{1}{r_{xy}^{n-2}}f_2(y)\right]dS_y + P_m(x).$$

If $u(\infty) = 0$ (regular solution) then

$$u(x) = \frac{1}{(n-2)\,|S_1|}\int_S \left[f_1(y)\frac{\partial}{\partial n_y^+}\left(\frac{1}{r_{xy}^{n-2}}\right) - \frac{1}{r_{xy}^{n-2}}f_2(y)\right]dS_y$$

and we thus recover a result in [1].

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Received May 25, 1985

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