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AVERAGING OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. The paper presents a justification of the averaging method for a nonlinear integro-differential equation with impulses.

1. INTRODUCTION

Recently the differential equations with impulses have attracted the attention of a number of authors, see e.g. [1], [2], [3]. Beside the pure mathematical interest, a reason for such considerations seems to be the considerable importance of these equations in control theory, see [4]. For example, given a linear control system with relaxed controls

$$Dx = Ax + Du, \quad x \in R^n,$$

where D denotes the generalized derivative, one can define the closed-loop control law as follows: Let $\{\sigma_i\}$ be a family of hypersurfaces in R^{n+1}, $\sigma_i = \{(t, x), t = t_i(x)\}$, $i = 1, 2, \ldots$ Define $Du(\tau_i, x_i) = I_i(x_i)$, when $(x_i, \tau_i) \in \sigma_i$ and let $u(.)$ be smooth otherwise. Then the trajectory will have jumps when meeting the hypersurfaces σ_i with value $I_i(x_i^-)$ i. e. $x_i^+ = x_i^- + I_i(x_i^-)$. In this paper we concern ourselves with a special generalization of such a model.

Here we connect the concept of impulse equation with singular perturbation analysis. Singular perturbation is another branch of the qualitative theory of differential equations which has been developed intensively in the last years. The pathological behaviour of such systems is provided by a small or a large parameter in the derivatives which makes the dynamic of the system very fast or very slow. Consider the example

$$\frac{1}{\varepsilon} Dx = Ax + y + Du, \quad t \geqslant 0,$$

$$\varepsilon \dot{y} = By + x, \qquad y(0) = 0,$$

where ε is a small parameter. Here x represents the very slow variable and y the very fast variable. Solving the second equation and assuming the control law defined as above one comes to the equation

$$\frac{1}{\varepsilon}\dot{x} = Ax + \frac{1}{\varepsilon}\int_0^t e^{B\frac{t-s}{\varepsilon}} x(s)ds + f(t) \tag{1}$$

associated with the manifolds σ_i where the trajectory $x(.)$ has jumps $x_i^+ = x_i^- + \varepsilon I_i(x_i^-)$.

In as much as the singular perturbation analysis in control theory has numerous applications, the above discussion, in authors' opinion provides a reasonable motivation for investigations of general equations of the type(1). Here we justify the averaging method for solving a nonlinear integro-differential equation with impulses, which is formulated in Section 2. In Section 3 we present and prove the main result: under some assumptions the solution of the problem is convergent to the solution of an averaged problem associated with the original one.

A related result is published in [5] where a similar problem for an ordinary differential equation is considered.

2. STATEMENT OF THE PROBLEM

Consider the hypersurfaces

$$\sigma_i : t = t_i(x),\ i = 1, 2, \ldots$$

in the $n+1$-dimensional space (t, x), where x is a n-vector, which for $x \in G \subset R^n$ lie in the half-space $t > 0$ and satisfy the condition $t_i(x) < t_{i+1}(x)$, $i = 1, 2, \ldots$

Let a mapping point P_t with current coordinates $(t, x(t))$ move in the domain $\{t \geqslant 0,\ x \in G\}$. We assume that the motion law is described by

(i) the system of integro-differential equations

$$\dot{x}(t) = \nu X\left(t, x(t), \frac{1}{\varepsilon}\int_{-\infty}^{t} \Psi\left(\frac{t-s}{\varepsilon}, s, x(s), \varepsilon\right) ds\right),\ t > 0,\ t \neq t_i(x), \tag{2}$$

$x(t) = \varphi(t, \varepsilon)$, $t \leqslant 0$, where $\varepsilon > 0$ is a small parameter, $\Psi(t, s, x, \varepsilon) \in R^m$, $\nu = \nu(\varepsilon)$ is a function of ε tending to zero as $\varepsilon \to 0$ and $\varphi(t, \varepsilon)$ is an initial function defined for $t \leqslant 0$ and $\varepsilon \in (0, \mathscr{E})$, $\mathscr{E} = \text{const} > 0$.

(ii) the set of hypersurfaces σ_i, $i = 1, 2, \ldots$

(iii) the set of vector-functions $I_i(x)$, $i = 1, 2, \ldots$ defined over G.

We note that the velocity of the point P_t at the moment t depends on the initial function $\varphi(t, \varepsilon)$ and on the motion of P_t in the whole preceding interval $(0, t)$.

The motion itself can be described as follows. Departing from the position ($\tau_0 = 0$, $x_0 = \varphi(0, \varepsilon)$), the point moves along the trajectory $(t, x(t))$ determined by the solution $x(t)$ of (2) until the moment $\tau_1 > 0$ at which it meets the hypersurface σ_1.

Then the point P_t instantly moves from the position $(\tau_1, x_1^- = x(\tau_1))$ to the position $(\tau_1, x_1^+ = x_1^- + \varepsilon I_1(x_1^-))$. Further on it goes along the trajectory $(t, x(t))$ described by the solution $x(t)$ of the system (2) until meeting the second hypersurface σ_2 etc.

The relations (i), (ii) and (iii) characterizing the motion of the point P_t are called a system of integrodifferential equations (2) with impulses. The curve described by the motion of the point P_t is said to be the trajectory of the system in the space $(t, \dot{x})$.

Thus, the solution of the system of integro-differential equations (2) with impulses is a function which satisfies (2) for given hypersurfaces σ_i, $i = 1, 2, \ldots$ and has instantaneous jumps

$$x_i^+ = x_i^- + \varepsilon I_i(x_i^-), \quad i = 1, 2, \ldots \tag{3}$$

when meeting the hypersurfaces σ_i, $i = 1, 2, \ldots$ Note that the point (τ_i, x_i^+) does not necessarily belong to the hypersurface σ_i.

Suppose that the following limits exist

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} X\left(\theta, x, \int_0^\infty \Psi(\sigma, \theta, x, \varepsilon)\, d\sigma\right) d\theta = X_0(x, \varepsilon), \\ &\lim_{T\to\infty} \frac{1}{T} \sum_{t < t_i < t+T} I_i(x) = I_0(x). \end{aligned} \tag{4}$$

We introduce the following averaged system of ordinary differential equations associated with the integro-differential system

$$\dot{\bar{x}}(t) = \nu X_0(\bar{x}(t), \varepsilon) + \varepsilon I_0(\bar{x}(t)) \tag{5}$$

$$\bar{x}(0) = x_0. \tag{6}$$

We shall use the following notation: if $x = (x_1, \ldots, x_n)$ and $A = (a_{ij})_{nm}$ then

$$\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{\frac{1}{2}}, \quad \|A\| = \left[\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2\right]^{\frac{1}{2}}.$$

By $\overline{1, n}$ we denote the set of positive integers $\{1, 2, \ldots, n\}$.

3. MAIN RESULT

The following theorem which deals with the closeness of the integro-differential equation (2) to the averaged system (5) (6) is the main result of this paper.

THEOREM 1. *Let the following assumptions be fulfilled*

(i) *The functions* $X(t, x, y)$, $\psi(t, s, x, \varepsilon)$ *and* $I_i(x)$; $i = 1, 2, \ldots$, *are continuous on the appropriate projections of the domain*

$$\{t \geqslant 0,\ |s| < \infty,\ x \in G \subset R^n,\ y \in G_1 \subset R^m,\ \varepsilon \in (0, \mathscr{E}],\ \mathscr{E} = const > 0\},$$

the function $\varphi(t, \varepsilon)$ *is continuous on* $\{t \leqslant 0, \varepsilon \in (0, \mathscr{E}]\}$, *the functions* $t_i(x)$, $i = 1, 2, \ldots$ *are twice continuously differentiable on* G *; the function* $\nu(\varepsilon)$ *is defined for* $\varepsilon \in (0, \mathscr{E}]$ *and tends to zero when* $\varepsilon \to 0$.

(ii) *There exist positive constants* λ, μ, β, ρ, M, C, C_1, C_2 *and a function* $\Pi(\varepsilon)$ *such that*

$$\left\| \frac{\partial t_i(x)}{\partial x} \right\| + \| X(t, x, y) \| + \| I_i(x) \| \leqslant M, \qquad \left\| \frac{\partial^2 t_i(x)}{\partial x^2} \right\| \leqslant C,$$

$$\| X(t, x, y) - X(t, x', y') \| \leqslant \lambda (\| x - x' \| + \| y - y' \|),$$

$$\| I_i(x) - I_i(x') \| \leqslant \lambda \| x - x' \|,$$

$$\| \psi(t, s, x, \varepsilon) - \psi(t, s, x', \varepsilon) \| \leqslant \mu e^{-\frac{\beta t}{\varepsilon}} \| x - x' \|,$$

$$\| X(t, x, \int_0^\infty \psi(\sigma, t - \varepsilon\sigma, x, \varepsilon)\, d\sigma) - X(t, x, \int_0^\infty \psi(\sigma, t, x, \varepsilon)\, d\sigma) \| \leqslant \Pi(\varepsilon),$$

for all $t \geqslant 0$, $s \in (-\infty, \infty)$, $x, x' \in G$, $y, y' \in G_1$, $\varepsilon \in (0, \mathscr{E}]$, $i = 1, 2, \ldots$;

$$\| \varphi(t, \varepsilon) \| \leqslant \rho \text{ for } t \leqslant 0, \quad \lim_{\varepsilon \to 0} \frac{\nu(\varepsilon)}{\varepsilon} = const > 0, \quad \sup_{\varepsilon \in (0, \mathscr{E}]} \frac{\nu(\varepsilon)}{\varepsilon} = C_1,$$

$$\lim_{\varepsilon \to 0} \frac{\Pi(\varepsilon)}{\varepsilon} = const > 0, \qquad \sup_{\varepsilon \in (0, \mathscr{E}]} \frac{\Pi(\varepsilon)}{\varepsilon} = C_2.$$

(iii) *The limits (4) and the limit*

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t < t_i < t + T} 1 = d, \quad d = const > 0$$

exist uniformly in $t \geqslant 0$ *and* $x \in G$.

(iv) *For each* $\varepsilon \in (0, \mathscr{E}]$ *the system of integro-differential equations (2) with impulses has a continuous and bounded solution* $x(t)$ *for* $t \geqslant 0$, $t \neq \tau_i$, $i = 1, 2, \ldots$ ($\| x(t) \| \leqslant \rho$ *for* $t \geqslant 0$) *which satisfies the matching conditions* $x(0 + 0) = \varphi(0, \varepsilon) = x_0$.

(v) *For each* $\varepsilon \in (0, \mathscr{E}]$ *the averaged initial value problem (5), (6) has a solution* $\bar{x}(t)$ *belonging to the domain* G *for* $t \geqslant 0$ *together with its neighbourhood of radius* $\rho' = const > 0$ *and satisfying the inequalities*

$$\frac{\partial t_i(\bar{x}(t))}{\partial x} I_i(\bar{x}(t)) \leqslant \nu < 0, \qquad \nu = const,$$

$$t \in (t'_i, t''_i), \quad t'_i = \inf_{x \in G} t_i(x), \quad t''_i = \sup_{x \in G} t_i(x), \quad i = 1, 2, \ldots,$$

or $\dfrac{\partial t_i(x)}{\partial x} \equiv 0$, *when* σ_i *is a hyperplane*.

Then for each $\eta > 0$ *and* $L > 0$ *there exists an* $\varepsilon_0 \in (0, \mathscr{E}]$ $(\varepsilon_0 = \varepsilon_0(\eta, L))$ *such that for* $\varepsilon \leqslant \varepsilon_0$ *the inequality*

$$\| x(t) - \bar{x}(t) \| < \eta$$

holds for $0 \leqslant t \leqslant L\varepsilon^{-1}$.

For the proof of Theorem 1 we shall use the following lemma:

LEMMA 1. *Let the conditions of Theorem 1 be fulfilled. Let* $T > 0$ *be a sufficiently large and fixed number. Then for each positive integer* $p \geqslant 1$ *the following inequality holds*

$$\| x(pT) - \bar{x}(pT) \| \leqslant$$
$$\leqslant \sum_{i=0}^{p-1} \{1 + [\nu(1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\lambda T\}^i [\varepsilon(1 + C_1)\alpha(T)T + \varepsilon^2 \overline{M}], \tag{7}$$

where $\overline{M} = (C_1 + d)[C_1 + \mathscr{E}\mu\beta^{-1}) + d]\lambda M T^2 + \max_{i=\overline{1,p}} M_i$ *and* $M_i = M_i(T, d_i)$ *are constants depending on* T *and on the constants* $d_j > 0$, $j = \overline{1, i}$.

Proof. The condition (iii) guarantees the existence of a function $\alpha(t)$ monotonuously decreasing to zero for t tending to infinity, such that for each $t \geqslant 0$ and $x \in G$ the following inequalities hold:

$$\| \int_t^{t+T} [X(\theta, x, \int_0^\infty \psi(\sigma, \theta, x, \varepsilon)d\sigma) - X_0(x, \varepsilon)] d\theta \| \leqslant \alpha(T)T, \tag{8}$$

$$\| \sum_{t < t_i < t+T} I_i(x) - I_0(x)T \| \leqslant \alpha(T)T.$$

We now prove the inequality (7) by induction on p. First, for $p = 1$ consider the system of integro-differential equations (2) with impulses in the interval $[0, T]$. Let

$$t_1(x_0) = t_1^{(0)}, \ldots, t_{d_1}(x_0) = t_{d_1}^{(0)}$$

be points lying in $(0, T)$, and $t_i^{(0)} < t_{i+1}^{(0)}$ for $i = \overline{1, (d_1 - 1)}$. Denote by $x_1^{(0)}(t, 0, x_0)$ the solution of the system

$$x_1^{(0)}(t, 0, x_0) = \begin{cases} x_0 + \nu \int_0^t X(\theta, x_1^{(0)}(\theta, 0, x_0), \frac{1}{\varepsilon} \int_{-\infty}^\theta \psi\left(\frac{t-s}{\varepsilon}, s,\right. \\ \left. x_1^{(0)}(s, 0, x_0), \varepsilon\right)ds)d\theta, \qquad t > 0, \\ \varphi(t, \varepsilon), \qquad t \leqslant 0. \end{cases} \tag{9}$$

The solution of (9) coincides with the solution of (2) till the moment τ_1 at which the trajectory $(t, x(t))$ meets the hypersurface σ_1, i.e. $x(t) = x_1^{(0)}(t, 0, x_0)$ for $t \leqslant \tau_1$.

Consider the function

$$\widetilde{x}_1^{(0)}(t, 0, x_0) = x_0 + \nu \int_0^t X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma) d\theta.$$

We shall obtain an estimate for the norm of the difference

$$R_1^{(0)}(t, 0, x_0, \varepsilon) = x_1^{(0)}(t, 0, x_0) - \widetilde{x}_1^{(0)}(t, 0, x_0).$$

For $0 < t \leqslant T$ we have

$$\| R_1^{(0)}(t, 0, x_0, \varepsilon) \| = \| x_1^{(0)}(t, 0, x_0) - \widetilde{x}_1^{(0)}(t, 0, x_0) \| \leqslant$$

$$\leqslant \nu \int_0^t \| X(\theta, x_1^{(0)}(\theta, 0, x_0), \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_1^{(0)}(s, 0, x_0), \varepsilon)\, ds) -$$

$$- X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma) \| \, d\theta \leqslant$$

$$\leqslant \nu \int_0^t \| X(\theta, x_1^{(0)}(\theta, 0, x_0), \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_1^{(0)}(s, 0, x_0), \varepsilon)\, ds) -$$

$$- X(\theta, x_0, \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_0, \varepsilon)\, ds) \| \, d\theta +$$

$$+ \nu \int_0^t \| X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta - \varepsilon\sigma, x_0, \varepsilon)\, d\sigma) -$$

$$- X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma) \| \, d\theta \leqslant \nu\lambda \int_0^t [\| x_1^{(0)}(\theta, 0, x_0) - x_0 \| +$$

$$+ \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \| \psi(\frac{\theta - s}{\varepsilon}, s, x_1^{(0)}(s, 0, x_0), \varepsilon) - \psi(\frac{\theta - s}{\varepsilon}, s, x_0, \varepsilon) \| ds]\, d\theta +$$

$$+ \nu \Pi(\varepsilon) T \leqslant \nu^2 \lambda M \int_0^t d\theta \int_0^\theta dl +$$

$$+ \frac{\nu\lambda\mu}{\varepsilon} \int_0^t d\theta \int_{-\infty}^{\theta} e^{-\beta\frac{\theta - s}{\varepsilon^2}} \| x_1^{(0)}(s, 0, x_0) - x_0 \| \, ds + \nu \Pi(\varepsilon) T \leqslant$$

$$\leqslant \frac{1}{2}\nu^2\lambda MT^2 + \frac{2\nu\lambda\mu\rho}{\varepsilon}\int_0^t d\theta \int_{-\infty}^{\theta} e^{-\beta\frac{\theta - s}{\varepsilon^2}} ds + \nu\Pi(\varepsilon)T \leqslant$$

$$\leqslant \nu^2\lambda MT^2/2 + 2\varepsilon\nu\lambda\mu\rho\beta^{-1}T + \nu\Pi(\varepsilon)T \equiv \omega_1(\varepsilon^2, T).$$

The obtained estimate shows that the function $\tilde{x}_1^{(0)}(t, 0, x_0)$ approximates the solution $x_1^{(0)}(t, 0, x_0)$ of the system (9) in the interval $(0, T]$ with accuracy of order ε^2.

The moment τ_1 at which the trajectory $(t, x(t))$ meets the hypersurface σ_1 appears to be a solution of the equation

$$t = t_1(x_1^{(0)}(t, 0, x_0)). \tag{10}$$

Since

$$t_1(x_1^{(0)}(t, 0, x_0)) = t_1(\tilde{x}_1^{(0)}(t, 0, x_0) + R_1^{(0)}(t, 0, x_0, \varepsilon)) =$$

$$= t_1(x_0 + \nu\int_0^t X(\theta, x_0, \int_0^{\infty}\psi(\sigma, \theta, x_0, \varepsilon)\,d\sigma)\,d\theta + O(\varepsilon^2)) =$$

$$= t_1(x_0) + \nu\frac{\partial t_1(x_0)}{\partial x}\int_0^{t^0} X(\theta, x_0, \int_0^{\infty}\psi(\sigma, \theta, x_0, \varepsilon)\,d\sigma)\,d\theta + O(\varepsilon^2) =$$

$$= t_1^{(0)} + \nu\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^{(0)}} X(\theta, x_0, \int_0^{\infty}\psi(\sigma, \theta, x_0, \varepsilon)\,d\sigma)\,d\theta +$$

$$+ \nu\frac{\partial t_1(x_0)}{\partial x}\int_{t_1^{(0)}}^{t} X(\theta, x_0, \int_0^{\infty}\psi(\sigma, \theta, x_0, \varepsilon)\,d\sigma)\,d\theta + O(\varepsilon^2) =$$

$$= t_1^{(0)} + \nu\frac{\partial t_1(x_0)}{\partial x}\int_0^{t_1^{(0)}} X(\theta, x_0, \int_0^{\infty}\psi(\sigma, \theta, x_0, \varepsilon)\,d\sigma)\,d\theta +$$

$$+ \nu\frac{\partial t_1(x_0)}{\partial x}(t - t_1^{(0)})X(\tilde{t}, x_0, \int_0^{\infty}\psi(\sigma, \tilde{t}, x_0, \varepsilon)\,d\sigma) + O(\varepsilon^2),$$

$$\tilde{t} = t_1^{(0)} + \delta(t - t_1^{(0)}), \qquad 0 \leqslant \delta \leqslant 1,$$

it follows from (10) that $\tau_1 = t_1^{(0)} + \varepsilon\Theta_1^{(0)} + O(\varepsilon^2)$, where

$$\Theta_1^{(0)} = \frac{v}{\varepsilon} \cdot \frac{\partial t_1(x_0)}{\partial x} \int_0^{t_1^{(0)}} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma)\, d\theta.$$

We note that the values of the constants δ for the various components of the vector $X(\tilde{t}, x_0, \int_0^{\infty} \psi(\sigma, \tilde{t}, x_0, \varepsilon)\, d\sigma)$ are different, in general.

From the inequality $t_1^{(0)} > 0$ it follows that $\tau_1 > \tau_0$, under the condition that ε is sufficiently small. Thus

$$x(t) = x_1^{(0)}(t, 0, x_0) = \tilde{x}_1^{(0)}(t, 0, x_0) + R_1^{(0)}(t, 0, x_0, \varepsilon)$$

for $\tau^0 < t \leqslant \tau_1 = t_1^{(0)} + \varepsilon\Theta_1^{(0)} + O(\varepsilon^2)$.

Further, let

$$x_1^+ = x_1^{(0)}(\tau_1, 0, x_0) + \varepsilon I_1(x_1^{(0)}(\tau_1, 0, x_0))$$

i.e.

$$x_1^+ = x_1^{(0)}(\tau_1, 0, x_0) + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) =$$

$$= x_0 + v\int_0^{\tau_1} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma)\, d\theta + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon),$$

where $I_1^{(0)} \equiv I_1(x_1^{(0)}(\tau_1, 0, x_0))$.

Denote by $x_2^{(0)}(t, \tau_1, x_1^+)$ the solution of the system

$$x_2^{(0)}(t, \tau_1, x_1^+) = \begin{cases} x_1^+ + v\displaystyle\int_{\tau_1}^{t} X\Big(\theta, x_2^{(0)}(\theta, \tau_1, x_1^+), \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi\big(\frac{\theta - s}{\varepsilon}, s, \\ \quad x_2^{(0)}(s, \tau_1, x_1^+), \varepsilon\big)\, ds\Big)\, d\theta, \qquad t > \tau_1, \\ x_1^{(0)}(t, 0, x_0), \qquad t \leqslant \tau_1. \end{cases} \tag{11}$$

The solution of (11) coincides with the solution of the system of integro-differential equations (2) with impulses till the moment τ_2 at which the trajectory $(t, x(t))$ meets the hypersurface σ_2, i.e. $x(t) = x_2^{(0)}(t, \tau_1, x_1^+)$ for $t \leqslant \tau_2$. To see this, consider the function

$$\tilde{x}_2^{(0)}(t, \tau_1, x_1^+) = x_1^+ + v\int_{\tau_1}^{t} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)d\sigma)\, d\theta.$$

Let us estimate the difference

$$\overset{(o)}{R_2}(t, \tau_1, \overset{+}{x_1}, \varepsilon) = \overset{(o)}{x_2}(t, \tau_1, \overset{+}{x_1}) - \overset{\sim(o)}{x_2}(t, \tau_1, \overset{+}{x_1}).$$

For $0 < \tau_1 < t \leqslant T$ we get

$$\| R_2^{(o)}(t, \tau_1, \overset{+}{x_1}, \varepsilon) \| = \| x_2^{(o)}(t, \tau_1, \overset{+}{x_1}) - \tilde{x}_2^{(o)}(t, \tau_1, \overset{+}{x_1}) \| \leqslant$$

$$\leqslant \nu \int_{\tau_1}^{t} \| x(\theta, x_2^{(o)}(\theta, \tau_1, \overset{+}{x_1}), \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_2^{(o)}(s, \tau_1, \overset{+}{x_1}), \varepsilon)\, ds) -$$

$$- X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon) d\sigma) \| d\theta \leqslant$$

$$\leqslant \nu \int_{\tau_1}^{t} \| X(\theta, x_2^{(o)}(\theta, \tau_1, \overset{+}{x_1}), \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_2^{(o)}(s, \tau_1, \overset{+}{x_1}), \varepsilon)\, ds) -$$

$$- X(\theta, x_0, \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, x_0, \varepsilon)\, ds) \| d\theta +$$

$$+ \nu \int_{\tau_1}^{t} \| X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta - \varepsilon\sigma, x_0, \varepsilon)\, d\sigma) -$$

$$- X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma) \| d\theta \leqslant$$

$$\leqslant \nu\lambda \int_{\tau_1}^{t} [\, \| x_2^{(o)}(\theta, \tau_1, \overset{+}{x_1}) - x_0 \| + \frac{1}{\varepsilon}\int_{-\infty}^{\theta} \| \psi(\frac{\theta - s}{\varepsilon}, s, x_2^{(o)}(s, \tau_1, \overset{+}{x_1}), \varepsilon) -$$

$$- \psi(\frac{\theta - s}{\varepsilon}, s, x_0, \varepsilon) \| ds\,]\, d\theta + \nu\Pi(\varepsilon) T \leqslant$$

$$\leqslant \nu^2 \lambda M \int_{\tau_1}^{t} d\theta \int_0^{\theta} dl + \varepsilon\nu\lambda M T + \nu\lambda\omega_1(\varepsilon^2, T) T + 2\varepsilon\nu\lambda\mu\rho\beta^{-1} T + \nu\Pi(\varepsilon) T \leqslant$$

$$\nu^2\lambda M T^2/2 + 2\varepsilon\nu\lambda\mu\rho\beta^{-1} T + \nu\Pi(\varepsilon) T + \varepsilon\nu\lambda M T + \nu\lambda\omega_1(\varepsilon^2, T) T \equiv \omega_2(\varepsilon^2, T).$$

Thus the function $\tilde{x}_2^{(o)}(t, \tau_1, \overset{+}{x_1})$ approximates the solution $x_2^{(o)}(t, \tau_1, \overset{+}{x_1})$ of the system (11) in the interval $(\tau_1, t] \subset (0, T]$ with accuracy of order ε^2.

It can easily be seen that after the moment τ_1, the trajectory $(t, x(t))$ never meets again the hypersurface σ_1. Indeed, if

$$\bar{t}_1 = \tau_1 + \varepsilon \frac{\partial t_1(x_0)}{\partial x} I_1^{(o)} + O(\varepsilon^2)$$

is the root of the equation

$$t = t_1 (x_2^{(0)}(t, \tau_1, x_1^+))$$

then, from assumption (v) and the continuity of the vector-function $I_1(x)$, it follows that $\bar{t}_1 < \tau_1$ when ε is sufficiently small.

The moment at which the trajectory meets the hypersurface σ_2 is

$$\tau_2 = t_2^{(0)} + \varepsilon\, \Theta_2^{(0)} + O(\varepsilon^2),$$

where

$$\Theta_2^{(0)} = \frac{\partial t_2(x_0)}{\partial x} \left[\frac{\nu}{\varepsilon} \int_0^{t_2^{(0)}} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma)\, d\theta + I_1^{(0)}\right].$$

From $t_2^{(0)} > t_1^{(0)}$ it follows that $\tau_2 > \tau_1$ when ε is sufficiently small. Then, for $\tau_1 < t \leqslant \tau_2$ we have

$$x(t) = x_2^{(0)}(t, \tau_1, x_1^+) = \tilde{x}_2^{(0)}(t, \tau_1, x_1^+) + R_2^{(0)}(t, \tau_1, x_1^+, \varepsilon) =$$

$$= \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon I_1^{(0)} + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) + R_2^{(0)}(t, \tau_1, x_1^+, \varepsilon)$$

and

$$x_2^+ = x_2^{(0)}(\tau_2, \tau_1, x_1^+) + \varepsilon I_2 (x_2^{(0)}(\tau_2, \tau_1, x_1^+)) =$$

$$= \tilde{x}_1^{(0)}(\tau_2, 0, x_0) + \varepsilon (I_1^{(0)} + I_2^{(0)}) + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) +$$

$$+ R_2^{(0)}(\tau_2, \tau_1, x_1^+, \varepsilon) = x_0 + \nu \int_0^{\tau_2} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon)\, d\sigma)\, d\theta +$$

$$+ \varepsilon (I_1^{(0)} + I_2^{(0)}) + R_1^{(0)}(\tau_1, 0, x_0, \varepsilon) + R_2^{(0)}(\tau_2, \tau_1, x_1^+, \varepsilon),$$

where $I_2^{(0)} = I_2 (x_2^{(0)}(\tau_2, \tau_1, x_1^+))$.

In the general case $j = \overline{(2, (d_1 + 1))}$, denote by $x_j^{(0)}(t, \tau_{j-1}, x_{j-1}^+)$ the solution of the system

$$x_j^{(0)}(t, \tau_{j-1}, x_{j-1}^+) = \begin{cases} x_{j-1}^+ + \nu \int_{\tau_{j-1}}^{t} X(\theta, x_j^{(0)}(\theta, \tau_{j-1}, x_{j-1}^+), \frac{1}{\varepsilon} \int_{-\infty}^{\theta} \psi(\frac{\theta - s}{\varepsilon}, s, \\ \quad x_j^{(0)}(s, \tau_{j-1}, x_{j-1}^+), \varepsilon)\, ds)\, d\theta, \quad t > \tau_{j-1}; \\ x_{j-1}^{(0)}(t, \tau_{j-2}, x_{j-2}^+), \quad t \leqslant \tau_{j-1}, \end{cases} \tag{12}$$

where

$$x_{j-1}^{+}=x_{j-1}^{(0)}(\tau_{j-1},\tau_{j-2},x_{j-2}^{+})+\varepsilon I_{j-1}(x_{j-1}^{(0)}(\tau_{j-1},\tau_{j-2},x_{j-2}^{+}))=$$

$$=x_0+\nu\int_0^{\tau_{j-1}}X(\theta,x_0,\int_0^{\infty}\psi(\sigma,\theta,x_0,\varepsilon)\,d\sigma)\,d\theta+\varepsilon\sum_{i=1}^{j-1}I_i^{(0)}+\sum_{i=1}^{j-1}R_i^{(0)}(\tau_i,\tau_{i-1},x_{i-1}^{+},\varepsilon),$$

$$I_{j-1}^{(0)}\equiv I_{j-1}(x_{j-1}^{(0)}(\tau_{j-1},\tau_{j-2},x_{j-2}^{+})),\qquad x_0^{+}=x_0.$$

The solution of (12) coincides with the solution of the system (2) till the moment τ_j at which the trajectory $(t, x(t))$ meets the hypersurface σ_j, i.e.

$$x(t)=x_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})\text{ for }t\leqslant\tau_j.$$

Let

$$\tilde{x}_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})=x_{j-1}^{+}+\nu\int_{\tau_{j-1}}^{t}X(\theta,x_0,\int_0^{\infty}\psi(\sigma,\theta,x_0,\varepsilon)\,d\sigma)\,d\theta.$$

One can show, as in the cases $j=1$ and $j=2$, that the difference

$$R_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+},\varepsilon)=x_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})-\tilde{x}_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})$$

in the interval $0<\tau_{j-1}<t\leqslant T$ satisfies the inequality

$$\|R_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+},\varepsilon)\|\leqslant\nu^2\lambda MT^2/2+2\varepsilon\nu\lambda\mu\rho\beta^{-1}T+$$

$$+\nu\Pi(\varepsilon)T+\varepsilon\nu(j-1)\lambda MT+\nu\lambda T\sum_{i=1}^{j-1}\omega_i(\varepsilon^2,T)\equiv\omega_j(\varepsilon^2,T).$$

Hence the function $\tilde{x}_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})$ approximates the solution $x_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})$ of the system (12) in the interval $(\tau_{j-1},t]\subset(0,T]$ with accuracy of order ε^2.

Since for $j=\overline{2,(d_1+1)}$ we have

$$\tilde{x}_j^{(0)}(t,\tau_{j-1},x_{j-1}^{+})=x_{j-1}^{+}+\nu\int_{\tau_{j-1}}^{t}X(\theta,x_0,\int_0^{\infty}\psi(\sigma,\theta,x_0,\varepsilon)\,d\sigma)\,d\theta=$$

$$=x_0+\nu\int_0^{\tau_{j-1}}X(\theta,x_0,\int_0^{\infty}\psi(\sigma,\theta,x_0,\varepsilon)\,d\sigma)\,d\theta+\varepsilon\sum_{i=1}^{j-1}I_i^{(0)}+$$

$$+\sum_{i=1}^{j-1}R_i^{(0)}(\tau_i,\tau_{i-1},x_{i-1}^{+},\varepsilon)+\nu\int_{\tau_{j-1}}^{t}X(\theta,x_0,\int_0^{\infty}\psi(\sigma,\theta,x_0,\varepsilon)\,d\sigma)\,d\theta=$$

$$= \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon \sum_{i=1}^{j-1} I_i^{(0)} + \sum_{i=1}^{j-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon),$$

we can write

$$x(t) = x_j^{(0)}(t, \tau_{j-1}, x_{j-1}^+) = \tilde{x}_1^{(0)}(t, 0, x_0) + \varepsilon \sum_{i=0}^{j-1} I_i^{(0)} + \sum_{i=1}^{j-1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_j^{(0)}(t, \tau_{j-1}, x_{j-1}^+, \varepsilon). \tag{13}$$

for

$$t_{j-1}^{(0)} + \varepsilon \Theta_{j-1}^{(0)} + \gamma_{j-1} O(\varepsilon^2) = \tau_{j-1} < t \leqslant \tau_j = t_j^{(0)} + \varepsilon \Theta_j^{(0)} + O(\varepsilon^2),$$

where

$$\Theta_j^{(0)} = \frac{\partial t_j(x_0)}{\partial x} \left[\frac{\nu}{\varepsilon} \int_0^{t_j^{(0)}} X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon) d\sigma) d\theta + \sum_{i=0}^{j-1} I_i^{(0)} \right],$$

$$t_0^{(0)} = \Theta_0^{(0)} = \gamma_0 = 0, \; I_0^{(0)} = R_0^{(0)}(\tau_0, \tau_{-1}, x_{-1}^+, \varepsilon) = 0, \; \gamma_j = 1, \; j = \overline{1, d_1},$$

as well as for

$$t_{d_1}^{(0)} + \varepsilon \Theta_{d_1}^{(0)} + O(\varepsilon^2) = \tau_{d_1} < t \leqslant T, \; j = d_1 + 1.$$

Hence

$$x(T) = x_{d_1+1}^{(0)}(T, \tau_{d_1}, x_{d_1}^{(+)}) = x_0 + \nu \int_0^T X(\theta, x_0, \int_0^{\infty} \psi(\sigma, \theta, x_0, \varepsilon) d\sigma) d\theta + \varepsilon \sum_{i=0}^{d_1} I_i^{(0)} + \sum_{i=0}^{d_1} R_i^{(0)}(\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_{d_1+1}^{(0)}(T, \tau_{d_1}, x_{d_1}^+, \varepsilon).$$

Let $\overline{x}(t)$ be the solution of the averaged system (5) with initial condition (6). Then for $t \geqslant 0$

$$\overline{x}(t) = x_0 + \varepsilon \int_0^t \left[\frac{\nu}{\varepsilon} X_0(\overline{x}(\theta), \varepsilon) + I_0(\overline{x}(\theta)) \right] d\theta$$

and

$$\overline{x}(T) = x_0 + \varepsilon \int_0^T \left[\frac{\nu}{\varepsilon} X_0(\overline{x}(\theta), \varepsilon) + I_0(\overline{x}(\theta)) \right] d\theta.$$

In order to estimate the difference $x(T) - \overline{x}(T)$, taking into account (8), we rewrite $x(T)$ in the form

$$x(T) = x_0 + \varepsilon \left[\frac{\nu}{\varepsilon} X_0(x_0, \varepsilon) + I_0(x_0) \right] T +$$

$$+ \nu \int_0^T [X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon)d\sigma) - X_0(x_0, \varepsilon)]d\theta + \tag{14}$$

$$+ \varepsilon \left[\sum_{i=0}^{d_1} I_i^{(o)} - I_0(x_0) T \right] + \sum_{i=0}^{d_1} R_i^{(o)} (\tau_i, \tau_{i-1}, x_{i-1}^+, \varepsilon) + R_{d_1+1}^{(o)} (T, \tau_{d_1}, x_{d_1}^+, \varepsilon).$$

Define the operator A_0 as follows

$$A_0 x = x + \varepsilon \left[\frac{\nu}{\varepsilon} X_0(x, \varepsilon) + I_0(x) \right] T, \quad x \in G.$$

From (14) according to (8) and the conditions of Theorem 1 we get

$$\| x(T) - A_0 x_0 \| \leqslant \nu \| \int_0^T [X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon) \, d\sigma) - X_0(x_0, \varepsilon)] \, d\theta \| +$$

$$+ \varepsilon \| \sum_{i=0}^{d_1} I_i^{(o)} - I_0(x_0) T \| + \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) \leqslant \nu \alpha (T) T +$$

$$+ \varepsilon \| \sum_{i=1}^{d_1} I_i (x_0) - I_0(x_0) T \| + \varepsilon \| \sum_{i=1}^{d_1} (I_i^{(o)} - I_i (x_0)) \| + \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) \leqslant$$

$$\leqslant \nu \alpha (T) T + \varepsilon \alpha (T) T + \varepsilon \sum_{i=1}^{d_1} \| I_i (x_i^{(o)}(\tau_i, \tau_{i-1}, x_{i-1}^+) - I_i (x_0) \| +$$

$$+ \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) \leqslant (\nu + \varepsilon) \alpha (T) T + \varepsilon \lambda \sum_{i=1}^{d_1} \| x_i^{(o)} (\tau_i, \tau_{i-1}, x_{i-1}^+) - x_0 \| +$$

$$+ \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) = (\nu + \varepsilon) \alpha (T) T + \varepsilon \lambda \sum_{i=1}^{d_1} \| x_0 +$$

$$+ \nu \int_0^{\tau_i} X (\theta, x_0, \int_0^\infty \psi (\sigma, \theta, x_0, \varepsilon) \, d\sigma) \, d\theta + \varepsilon \sum_{l=0}^{i-1} I_l^{(o)} + \tag{15}$$

$$+ \sum_{l=0}^{i} R_l^{(o)} (\tau_l, \tau_{l-1}, x_{l-1}^+, \varepsilon) - x_0 \| + \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) \leqslant$$

$$\leqslant (\nu + \varepsilon) \alpha (T) T + \varepsilon \nu \lambda M T d_1 + \varepsilon^2 \lambda \sum_{i=1}^{d_1} \sum_{l=0}^{i-1} \| I_l^{(o)} \| +$$

$$+ \varepsilon \lambda \sum_{i=1}^{d_1} \sum_{l=0}^{i} \| R_l^{(o)} (\tau_l, \tau_{l-1}, x_{l-1}^+, \varepsilon) \| + \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) \leqslant$$

$$\leqslant (\nu + \varepsilon) \alpha (T) T + \varepsilon \lambda M d_1 [2 \nu T + \varepsilon (d_1 - 1)]/2 +$$

$$+ \sum_{i=0}^{d_1+1} \omega_i (\varepsilon^2, T) + \varepsilon \lambda \sum_{i=1}^{d_1} \sum_{l=0}^{i} \omega_l (\varepsilon^2, T) \leqslant$$

$$\leqslant \varepsilon (1 + C_1) \alpha (T) T + \varepsilon^2 M_1,$$

where $\omega_0 (\varepsilon^2, T) = 0$ and $M_1 = M_1(T, d_1)$ is a constant.

For $t \geqslant 0$, $\tau \in [0, T)$ and $x \in G$ we have

$$\| X_0(x, \varepsilon) \| \leqslant \lim_{T\to\infty} \frac{1}{T} \left\| \int_t^{t+T} X\left(\theta, x, \int_0^\infty \psi(\sigma, \theta, x, \varepsilon)\, d\sigma\right) d\theta \right\| \leqslant M,$$

$$\| I_0(x) \| \leqslant \lim_{T\to\infty} \frac{1}{T} \sum_{t < t_i < t+T} \| I_i(x) \| \leqslant Md,$$

$$\| X_0(\overline{x}(\tau), \varepsilon) - X_0(x_0, \varepsilon) \| \leqslant \lim_{T\to\infty} \frac{1}{T} \left\| \int_t^{t+T} \left[X(\theta, \overline{x}(\tau), \int_0^\infty \psi(\sigma, \theta, \overline{x}(\tau), \varepsilon) d\sigma) - \right.\right.$$

$$\left.\left. - X(\theta, x_0, \int_0^\infty \psi(\sigma, \theta, x_0, \varepsilon) d\sigma) \right] d\theta \right\| \leqslant (\nu + \varepsilon d)(1 + \varepsilon\mu\beta^{-1})\lambda MT,$$

$$\| I_0(\overline{x}(\tau)) - I_0(x_0) \| \leqslant \lim_{T\to\infty} \frac{1}{T} \left\| \sum_{t < t_i < t+T} I_i(\overline{x}(\tau)) - I_i(x_0) \right\| \leqslant$$

$$\leqslant (\nu + \varepsilon d)\, \lambda d MT.$$

Using these estimates we obtain

$$\| \overline{x}(T) - A_0 x_0 \| = \left\| x_0 + \varepsilon \int_0^T \left[\frac{\nu}{\varepsilon} X_0(\overline{x}(\theta), \varepsilon) + I_0(\overline{x}(\theta)) \right] d\theta - \right.$$

$$\left. - x_0 - \nu X_0(x_0, \varepsilon) T - \varepsilon I_0(x_0) T \right\| \leqslant \varepsilon \int_0^T \left\{ \frac{\nu}{\varepsilon} \| X_0(\overline{x}(\theta), \varepsilon) - X_0(x_0, \varepsilon) \| + \right. \tag{16}$$

$$\left. + \| I_0(\overline{x}(\theta)) - I_0(x_0) \| \right\} d\theta \leqslant (\nu + \varepsilon d)[\nu(1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\, \lambda MT^2.$$

From (15) and (16) the following inequality holds

$$\| x(T) - \overline{x}(T) \| \leqslant \| x(T) - A_0 x_0 \| + \| \overline{x}(T) - A_0 x_0 \| \leqslant$$

$$\leqslant \varepsilon (1 + C_1)\, \alpha(T)\, T + \varepsilon^2 \overline{M}, \tag{17}$$

where $\overline{M} = (C_1 + d)\, [C_1(1 + \varepsilon\mu\beta^{-1}) + d]\lambda MT^2 + M_1.$

Hence we get an estimate for $\| x(T) - \overline{x}(T) \|$ which shows the closeness of the points $x(T)$ and $\overline{x}(T)$. Since $\overline{x}(T)$ belongs to the domain G with its neighbourhood of radius ρ, it follows from (16) and (17) that the points $A_0 x_0$ and $x(T)$ belong to G too. This completes the proof of the inequality (7) for $p = 1$.

For the next step of the induction proof, we introduce the notations

$$\tau_i^{(r-1)} = \tau_{d_0 + d_1 + \dots + d_{r-1} + i},$$

$$x_i^{(r-1)+} = x^+_{d_0 + d_1 + \dots + d_{r-1} + i}, \; d_0 = 0, \; i = \overline{1, d_r},$$

$$\tau_0^{(r-1)} = (r-1)T,\ \tau_{d_r+1}^{(r-1)} = rT,$$

$$x_0^{(r-1)+} = X((r-1)T),\ x_{d_r+1}^{(r-1)+} = x(rT),\ r = 1, 2, \ldots$$

Then we have $\tau_0^{(r-1)} = \tau_{d_{r-1}+1}^{(r-2)}$ and

$$x_0^{(r-1)+} = x_{d_{r-1}+1}^{(r-2)+},\ r = 2, 3, \ldots$$

Let us assume that for $p = r (r \geqslant 2)$ the inequality (7) as well as of the type (13), (15) — (17) hold, that is

$$x(t) = x_j^{(r-1)}(t, \tau_{j-1}^{(r-1)}, x_{j-1}^{(r-1)+}) = \tilde{x}_1^{(r-1)}(t, (r-1)T, x((r-1)T)) +$$

$$+ \varepsilon \sum_{i=0}^{j-1} I_i^{(r-1)} + \sum_{i=0}^{j-1} R_i^{(r-1)}(\tau_i^{(r-1)}, \tau_{i-1}^{(r-1)}, x_{i-1}^{(r-1)}, \varepsilon) +$$

$$+ R_j^{(r-1)}(t, \tau_{j-1}^{(r-1)}, x_{j-1}^{(r-1)+}, \varepsilon)$$

for

$$t_{j-1}^{(r-1)} + \varepsilon\Theta_{j-1}^{(r-1)} + \gamma_{j-1} O(\varepsilon^2) = \tau_{j-1}^{(r-1)} < t \leqslant \tau_j^{(r-1)} = t_j^{(r-1)} + \varepsilon\Theta_j^{(r-1)} +$$
$$+ O(\varepsilon^2),$$

where

$$\Theta_j^{(r-1)} = \frac{\partial t_{d_1 + \ldots + d_{r-1} + j}(x(r-1)T)}{\partial x}\Big[\int_{(r-1)T}^{t_j^{(r-1)}} X(\theta, x((r-1)T),$$

$$\int_0^\infty \psi(\sigma, \theta, x((r-1)T), \varepsilon)\, d\sigma) d\theta + \sum_{i=0}^{j-1} I_i^{(r-1)}\Big],$$

$$t_0^{(r-1)} = rT,\quad \Theta_0^{(r-1)} = \gamma_0 = 0,\ I_0^{(r-1)} = 0,$$

$$R_0^{(r-1)}(\tau_0^{(r-1)}, \tau_{-1}^{(r-1)}, x_{-1}^{(r-1)+}, \varepsilon) = 0,\ \gamma_j = 1,\ j = \overline{1, d_r},$$

as well as for

$$t_{d_r}^{(r-1)} + \varepsilon\Theta_{d_r}^{(r-1)} + O(\varepsilon^2) = \tau_{d_r}^{(r-1)} < t \leqslant \tau_{d_{r+1}}^{(r-1)} = rT,\ j = d_r + 1;$$

$$\| x(rT) - A_0 x((r-1)T) \| \leqslant (\nu + \varepsilon)\alpha(T)T + \varepsilon\lambda M d_r[2\nu T + \varepsilon(d_r - 1)]/2 +$$

$$+ \sum_{i=0}^{d_r+1} \omega_i(\varepsilon^2 T) + \varepsilon\lambda \sum_{i=1}^{d_r} \sum_{l=0}^{i} \omega_l(\varepsilon^2, T) \leqslant \varepsilon(1 + C_1)\alpha(T)T + \varepsilon^2 M_{r-1},$$

where

$\omega_0^{(r-1)}(\varepsilon^2, T) = 0$ and $M_r = M_r(T, d_r)$ is a constant;

$$\| A_0 x((r-1)T) - A_0 \overline{x}((r-1)T) \| \leqslant \{1 + [\nu(1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\lambda T\}.$$

$$\sum_{i=0}^{r-2} \{1+[\nu(1+\varepsilon\mu\beta^{-1})+\varepsilon d]\lambda T\}^i \cdot [\varepsilon(1+C_1)\alpha(T)T+\varepsilon^2\overline{M}],$$

where

$$\overline{M}=(C_1+d)[C_1(1+\sigma\mu\beta^{-1})+d]\lambda MT^2+\max_{i=\overline{1,(r-1)}} M_i;$$

$$\| A_0\overline{x}((r-1)T)-\overline{x}(rT)\| \leqslant \varepsilon^2(C_1+d)[C_1(1+\sigma\mu\beta^{-1})+d]\lambda MT^2;$$

$$\| x(rT)-\overline{x}(rT)\| \leqslant \| x(rT)-A_0x((r-1)T)\| +$$

$$+\| A_0x((r-1)T)-A_0\overline{x}((r-1)T)\| + \| A_0\overline{x}((r-1)T)-\overline{x}(rT)\| \leqslant$$

$$\leqslant \sum_{i=0}^{r-1} \{1+[\nu(1+\varepsilon\mu\beta^{-1})+\varepsilon d]\lambda T\}^i [\varepsilon(1+C_1)\alpha(T)T+\varepsilon^2\overline{M}],$$

where

$$\overline{M}=(C_1+d)[C_1(1+\sigma\mu\beta^{-1})+d]\lambda MT^2+\max_{i=\overline{1,(r-1)}} M_i .$$

Consider d_{r+1} points

$$t_{d_1+\dots d_r+1}(\overline{x}(rT)),\ \dots,\ t_{d_1+\dots+d_r+d_{r+1}}(\overline{x}(rT))$$

in the interval $(rT, (r+1)T)$ so that

$$t_{d_1+\dots+d_r+i}(\overline{x}(rT)) < t_{d_1+\dots+d_r+i+1}(\overline{x}(rT)),$$

$$i=\overline{1,(d_{r+1}-1)}.$$

Then from (7) for $p=r$, and from the continuity of the functions $t_i(x)$, $i=1, 2, \dots$ it follows that for ε sufficiently small these $d_{\tau+1}$ points

$$t_{d_1+\dots+d_r+1}(x(rT)) = t_1^{(r)},\ \dots,\ t_{d_1+\dots+d_r+d_{r+1}}(x(rT)) = \\ = t^{(r)}_{d_r+1} \tag{18}$$

lie in the interval $(rT, (r+1)T)$, so that $t_i^{(r)} < t_{i+1}^{(r)}$, $i=\overline{1,(d_{r+1}-1)}$.

From the conditions of Lemma 1 and from (7) for $p=r$ it follows that for ε sufficiently small there exists a constant $\beta_r \in [-\beta, 0)$ such that for $i=\overline{1, d}_{r+1}$

$$\frac{\partial t_{d_1+\dots+d_{r+i}}(x(rT))}{\partial x} I_{d_1+\dots+d_r+i}(x(rT)) \leqslant \beta_r < 0 . \tag{19}$$

We shall prove (7) for $p=r+1$.

The solution of the system of integro - differential equations with impulses, which assume to have been construc.ed in the intervals $((p-1)T,\ pT)$ $p=\overline{1,r}$ will be extended to the next interval $(rT, (r+1)T]$. For the sake of simplicity we shall write x_{pT} instead of $x(pT)$.

Let $x_1^{(r)}(t, rT, x_{rT})$ be the solution of the system

$$x_1^{(r)}(t, rT, x_{rT}) = \begin{cases} x_{rT} + \nu \int\limits_{rT}^{t} X\left(\theta, x_1^{(r)}(t, rT, x_{rT}), \frac{1}{\varepsilon} \int\limits_{-\infty}^{\theta} \psi\left(\frac{\theta - s}{\varepsilon}, s, \right.\right. \\ \left.\left. x_1^{(r)}(s, rT, x_{rT}), \varepsilon\right) ds\right) d\theta, \qquad t > rT, \\ x_{d_r+1}^{(r-1)}(t, \tau_{d_r}^{(r-1)}, x_{d_r}^{(r-1)+}), \qquad t \leqslant rT. \end{cases} \tag{20}$$

The solution of (20) coincides with the solution of the system of integro-differential equations (2) with impulses till the moment $\tau_1^{(r)}$ at which the trajectory $(t, x(t))$ meets the hypersurface $\sigma_{d_1 + \ldots + d_r + 1}$, that is for $t \leqslant \tau_1^{(r)}$ we have $x(t) = x_1^{(r)}(t, rT, x_{rT})$.

Consider the function

$$\tilde{x}_1^{(r)}(t, rT, x_{rT}) = x_{rT} + \nu \int\limits_{rT}^{t} X\left(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma\right) d\theta.$$

For $rT < t \leqslant (r+1)T$ we have

$$\| R_1^{(r)}(t, rT, x_{rT}, \varepsilon) \| = \| x_1^{(r)}(t, rT, x_{rT}) - \tilde{x}_1^{(r)}(t, rT, x_{rT}) \| \leqslant$$

$$\leqslant \nu \int\limits_{rT}^{t} \left\| X\left(\theta, x_1^{(r)}(t, rT, x_{rT}), \frac{1}{\varepsilon} \int\limits_{-\infty}^{\theta} \psi\left(\frac{\theta - s}{\varepsilon}, s, x_1^{(r)}(s, rT, x_{rT}), \varepsilon\right) ds\right) - \right.$$

$$\left. - X\left(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma\right) \right\| d\theta \leqslant$$

$$\leqslant \nu \int\limits_{rT}^{t} \left\| X\left(\theta, x_1^{(r)}(t, rT, x_{rT}), \frac{1}{\varepsilon} \int\limits_{-\infty}^{\theta} \psi\left(\frac{\theta - s}{\varepsilon}, s, x_1^{(r)}(s, rT, x_{rT}), \varepsilon\right) ds\right) - \right.$$

$$\left. - X\left(\theta, x_{rT}, \frac{1}{\varepsilon} \int\limits_{-\infty}^{\theta} \psi\left(\frac{\theta - s}{\varepsilon}, s, x_{rT}, \varepsilon\right) ds\right) \right\| d\theta +$$

$$+ \nu \int\limits_{rT}^{t} \left\| X\left(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta - \varepsilon\sigma, x_{rT}, \varepsilon)\, d\sigma\right) - \right.$$

$$\left. - X\left(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma\right) \right\| \leqslant$$

$$\leqslant \nu\lambda \int_{rT}^{t} [\| x_1^{(r)}(t, rT, x_{rT}) - x_{rT} \| + \frac{1}{\varepsilon} \int_{-\infty}^{\theta} \| \psi\left(\frac{\theta - s}{\varepsilon}, s, x_1^{(r)}(s, rT, x_{rT}), \varepsilon\right) - \psi\left(\frac{\theta - s}{\varepsilon}, s, x_{rT}, \varepsilon\right) \| ds] \, d\theta + \nu\Pi(\varepsilon) T \leqslant$$

$$\leqslant \nu^2 \lambda M \int_{rT}^{t} d\theta \int_{rT}^{\theta} dl + 2\varepsilon\nu\lambda\mu\rho\beta^{-1} T + \nu\Pi(\varepsilon) T \leqslant$$

$$\leqslant \nu^2 \lambda M T^2/2 + 2\varepsilon\nu\lambda\mu\rho\beta^{-1} T + \nu\Pi(\varepsilon) T \equiv \omega_1(\varepsilon^2, T).$$

Hence the function $\tilde{x}_1^{(r)}(t, rT, x_{rT})$ approximates the solution of (20) in the interval $(rT, (r+1)T]$ with accuracy of order ε^2.

For the root of the equation

$$t = t_{d_1 + \ldots + d_r + 1}(x_1^{(r)}(t, rT, x_{rT}))$$

we find

$$\tau_1^{(r)} = t_1^{(r)} + \varepsilon\Theta_1^{(r)} + O(\varepsilon^2), \tag{21}$$

where

$$\Theta_1^{(r)} = \frac{\nu}{\varepsilon} \frac{\partial t_{d_1 + \ldots + d_r + 1}(x_{rT})}{\partial x} \int_{rT}^{t_1^{(r)}} X(\theta, x_{rT}, \int_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon) \, d\sigma) \, d\theta.$$

From (18) and (21) it follows that for ε sufficiently small the inequality $\tau_1^{(r)} > rT$ holds. Thus, for $rT < t \leqslant \tau_1^{(r)}$,

$$x(t) = x_1^{(r)}(t, rT, x_{rT}) = \tilde{x}_1^{(r)}(t, rT, x_{rT}) + R_1^{(r)}(t, rT, x_{rT}, \varepsilon).$$

Further,

$$x_1^{(r)+} = x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}) + \varepsilon I_{d_1 + \ldots + d_r + 1}(x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT})) =$$

$$= \tilde{x}_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}) + \varepsilon I_1^{(r)} + R_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}, \varepsilon) =$$

$$= x_{rT} + \nu \int_{rT}^{\tau_1^{(r)}} X(\theta, x_{rT}, \int_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon) d\sigma) \, d\theta + \varepsilon I_1^{(r)} + R_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}, \varepsilon),$$

where $I_1^{(r)} \equiv I_{d_1 + \ldots + d_r + 1}(x_1^{(r)}(\tau_1^{(r)}, rT, x_{rT}))$.

In the general case $j = \overline{2, (d_{r+1} + 1)}$ we denote by $x_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+})$ the solution of the system

$$x_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) = \begin{cases} x_{j-1}^{(r)+} + \nu \int\limits_{\tau_{j-1}^{(r)}}^{t} X(\theta, x_j^{(r)}(\theta, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}), \\ \frac{1}{\varepsilon} \int\limits_{-\infty}^{\theta} \psi\left(\frac{\theta - s}{\varepsilon}, s, x_j^{(r)}(s, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}), \varepsilon) \, ds\right) d\theta, \quad t > \tau_{j-1}^{(r)}, \\ x_{j-1}^{(r)}(t, \tau_{j-2}^{(r)}, x_{j-2}^{(r)+}), \quad t \leqslant \tau_{j-1}^{(r)}, \end{cases} \tag{22}$$

where

$$x_{j-1}^{(r)+} = x_{j-1}^{(r)}(\tau_{j-1}^{(r)}, \tau_{j-2}^{(r)}, x_{j-2}^{(r)+}) +$$

$$+ \varepsilon I_{d_1 + \ldots + d_r + j - 1}(x_{j-1}^{(r)}(\tau_{j-1}^{(r)}, \tau_{j-2}^{(r)}, x_{j-2}^{(r)+})) =$$

$$= x_{rT} + \nu \int\limits_{rT}^{\tau_{j-1}^{(r)}} X(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon) \, d\theta +$$

$$+ \varepsilon \sum_{i=1}^{j-1} I_i^{(r)} + \sum_{i=1}^{j-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon),$$

$$I_{j-1}^{(r)} \equiv I_{d_1 + \ldots + d_r + j - 1}(x_{j-1}^{(r)}(\tau_{j-1}^{(r)}, \tau_{j-2}^{(r)}, x_{j-2}^{(r)+})).$$

The solution of (22) coincides with the solution of (2) till the moment $\tau_j^{(r)}$ at which the trajectory $(t, x(t))$ meets the hypersurface $\sigma_{d_1 + \ldots + d_r + j}$.

Consider the function

$$\tilde{x}_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) = x_{j-1}^{(r)+} + \nu \int\limits_{\tau_{j-1}^{(r)}}^{t} X(\theta, x_{rT}, \int\limits_0^{\infty} \psi(\sigma, \theta, x_{rT}, \varepsilon) \, d\sigma) \, d\theta.$$

It is easy to check that the following estimation holds in the interval $rT < \tau_{j-1}^{(r)} < t \leqslant (r+1)T$

$$\| R_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}, \varepsilon) \| = \tag{23}$$

$$= \| x_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) - \tilde{x}_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) \| \leqslant \omega_j(\varepsilon^2, T).$$

Since

$$\tilde{x}_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) = x_{rT} + \nu \int_{rT}^{t} X(\theta, x_{rT}, \int_0^\infty \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma)\, d\theta +$$

$$+ \varepsilon \sum_{i=1}^{j-1} I_i^{(r)} + \sum_{i=1}^{j-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon) =$$

$$= \tilde{x}_1^{(r)}(t, rT, x_{rT}) + \varepsilon \sum_{i=1}^{j-1} I_i^{(r)} + \sum_{i=1}^{j-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon)$$

we get that

$$x(t) = x_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}) = \tilde{x}_1^{(r)}(t, rT, x_{rT}) + \varepsilon \sum_{i=0}^{j-1} I_i^{(r)} +$$

$$+ \sum_{i=0}^{j-1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon) + R_j^{(r)}(t, \tau_{j-1}^{(r)}, x_{j-1}^{(r)+}, \varepsilon) \tag{24}$$

for

$$t_{j-1}^{(r)} + \varepsilon\Theta_{j-1}^{(r)} + \gamma_{j-1} O(\varepsilon^2) = \tau_{j-1}^{(r)} < t \leqslant \tau_j^{(r)} + \varepsilon\Theta_j^{(r)} + O(\varepsilon^2),$$

where

$$\Theta_j^{(r)} = \frac{\nu}{\varepsilon} \frac{\partial t_{d_1 + \dots + d_{r+j}}(x_{rT})}{\partial x} \left[\int_{rT}^{t_j^{(r)}} X(\theta, x_{rT}, \int_0^\infty \psi(\sigma, s, x_{rT}, \varepsilon)\, d\sigma)\, d\theta + \right.$$

$$\left. + \sum_{i=0}^{j-1} I_i^{(r)} \right], \quad t_0^{(r)} \equiv rT, \quad \Theta_0^{(r)} = \gamma_0 = 0, \quad I_0^{(r)} = 0,$$

$$R_0^{(r)}(\tau_0^{(r)}, \tau_{-1}^{(r)}, x_{-1}^{(r)+}, \varepsilon) = 0, \quad \gamma_j = 1, \quad j = \overline{1, d}_{r+1},$$

as well as for

$$t_{d_{r+1}}^{(r)} + \varepsilon\Theta_{d_{r+1}}^{(r)} + O(\varepsilon^2) = \tau_{d_{r+1}}^{(r)} < t \leqslant \tau_{d_{r+1}+1}^{(r)} = (r+1)T, \; j = d_{r+1} + 1.$$

Now we compute $x((r+1)T)$ and $\overline{x}((r+1)T)$:

$$x((r+1)T) = x_{d_{r+1}+1}^{(r)}((r+1)T, \tau_{d_{r+1}}^{(r)}, x_{d_{r+1}}^{(r)+}) =$$

$$= x_{rT} + \nu \int_{rT}^{(r+1)T} X(\theta, x_{rT}, \int_0^\infty \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma)\, d\theta +$$

$$+ \varepsilon \sum_{i=0}^{d_{r+1}} I_i^{(r)} + \sum_{i=0}^{d_{r+1}+1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon) =$$

$$= x_{rT} + \varepsilon \left[\frac{\nu}{\varepsilon} X_0(x_{rT}, \varepsilon) + I_0(x_{rT}) \right] \cdot T +$$

$$+ \nu \int_{rT}^{(r+1)T} [Y(\theta, x_{rT}, \int_0^\infty \psi(\sigma, \theta, x_{rT}, \varepsilon)\, d\sigma) - X_0(x_{rT}, \varepsilon)]\, d\theta +$$

$$+ \varepsilon [\sum_{i=0}^{d_{r+1}} I_i^{(r)} - I_0(x_{rT})T] + \sum_{i=0}^{d_{r+1}+1} R_i^{(r)}(\tau_i^{(r)}, \tau_{i-1}^{(r)}, x_{i-1}^{(r)+}, \varepsilon)$$

$$\overline{x}((r+1)T) = x_0 + \varepsilon \int_0^{(r+1)T} \left[\frac{\nu}{\varepsilon} X_0(\overline{x}(\theta), \varepsilon) + I_0(\overline{x}(\theta))\right] d\theta =$$

$$= \overline{x}(rT) + \varepsilon \int_{rT}^{(r+1)T} \left[\frac{\nu}{\varepsilon} X_0(\overline{x}(\theta), \varepsilon) + I_0(\overline{x}(\theta))\right] d\theta.$$

Taking into account the definition of the operator A_0 we can write the difference $x((r+1)T) - \overline{x}((r+1)T)$ in the following form

$$x((r+1)T) - \overline{x}((r+1)T) = [x((r+1)T) - A_0 x_{rT}] +$$
$$+ [A_0 x_{rT} - A_0 \overline{x}(rT)] + [A_0 \overline{x}(rT) - \overline{x}((r+1)T)].$$

Hence

$$\| x((r+1)T) - \overline{x}((r+1)T) \| \leqslant \| x((r+1)T) - A_0 x_{rT} \| +$$
$$+ \| A_0 x_{rT} - A_0 \overline{x}(rT) \| + \| A_0 \overline{x}(rT) - \overline{x}((r+1)T) \|. \tag{25}$$

Proceeding further in the same manner as in obtaining the estimate (15) for the first term on the right hand side of (25) we obtain

$$\| x((r+1)T) - A_0 x_{rT} \| \leqslant (\nu + \varepsilon)\, \alpha(T)T + \varepsilon\lambda M d_{r+1} [2\nu T +$$
$$\varepsilon(d_{r+1} - 1)]/2 + \sum_{i=0}^{d_{r+1}} \omega_i(\varepsilon_r^2 T) + \varepsilon\lambda \sum_{i=1}^{d_{r+1}} \sum_{l=0}^{i} \omega_l(\varepsilon_r^2 T) \leqslant \tag{26}$$
$$\leqslant \varepsilon(1 + C_1)\alpha(T)T + \varepsilon^2 M_{r+1},$$

where $M_{r+1} = M_{r+1}(T, d_r)$ is a constant.

For the second term on the right hand side of (25) we have

$$\| A_0 x_{rT} - A_0 \overline{x}(rT) \| = \| x_{rT} + \nu X_0(x_{rT}, \varepsilon)T + \varepsilon I_0(x_{rT})T -$$
$$- \overline{x}(rT) - \nu X_0(\overline{x}(rT)\varepsilon)T - \varepsilon I_0(\overline{x}(rT))T \| \leqslant$$
$$\leqslant \| x_{rT} - \overline{x}(rT) \| + \nu T \| X_0(x_{rT}, \varepsilon) - X_0(\overline{x}(rT), \varepsilon) \| + \tag{27}$$
$$+ \varepsilon T \| I_0(x_r) - I_0(\overline{x}(rT)) \| \leqslant [1 + \lambda\nu T(1 + \varepsilon\mu\beta^{-1}) + \varepsilon\lambda T d].$$
$$\| x_{rT} - \overline{x}(rT) \| \leqslant \{1 + [\nu(1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\lambda T\}.$$

$$\sum_{i=0}^{r-1} \{1+[\nu(1+\varepsilon\mu\beta^{-1})+\varepsilon d]\lambda T\}^i . [\varepsilon(1+C_1)\alpha(T)T+\varepsilon^2 \overline{M}],$$

where

$$\overline{M} = (C_1 + d)[C_1(1+\varepsilon\mu\beta^{-1})+d]\lambda M T^2 + \max_{i=\overline{1,r}} M_i .$$

Since for $t \in (rT, (r+1)T]$,

$$\|\overline{x}(t) - \overline{x}(rT)\| \leqslant \varepsilon \int_{rT}^{t} \left[\frac{\nu}{\varepsilon}\|X_0(\overline{x}(\theta),\varepsilon)\| + \|I_0(\overline{x}(\theta))\|\right] d\theta \leqslant (\nu+\varepsilon d)MT$$

for the third term on the right hand side of (25) we get

$$\|A_0\overline{x}(rT) - \overline{x}((r+1)T)\| = \|\overline{x}(rT) + \nu X_0(\overline{x}(rT),\varepsilon)T + \varepsilon I_0(\overline{x}(rT))T -$$

$$- \overline{x}(rT) - \int_{rT}^{(r+1)T} [\nu X_0(\overline{x}(\theta),\varepsilon) + \varepsilon I_0(\overline{x}(\theta))]\, d\theta \| \leqslant$$

$$\leqslant \int_{rT}^{(r+1)T} [\nu\| X_0(\overline{x}(\theta),\varepsilon) - X_0(\overline{x}(rT),\varepsilon)\| + \tag{28}$$

$$+ \varepsilon\| I_0(\overline{x}(\theta)) - I_0(\overline{x}(rT))\|]\, d\theta \leqslant [\lambda\nu(1+\varepsilon\mu\beta^{-1}) + \varepsilon\lambda d] .$$

$$\cdot \int_{rT}^{(r+1)T} \|\overline{x}(\theta) - \overline{x}(rT)\|\, d\theta \leqslant (\nu+\varepsilon d)[\nu(1+\varepsilon\mu\beta^{-1})+\varepsilon d]\lambda M T^2 \leqslant$$

$$\leqslant \varepsilon^2 (C_1 + d)[C_1(1+\varepsilon\mu\beta^{-1}) + d]\lambda M T^2 .$$

From (25) — (28) it follows that

$$\| x((r+1)T) - \overline{x}((r+1)T)\| \leqslant$$

$$\leqslant \sum_{i=0}^{r} \{1+[\nu(1+\varepsilon\mu\beta^{-1})+\varepsilon d]\lambda T\}^i . [\varepsilon(1+C_1)\alpha(T)T + \varepsilon^2\overline{M}],$$

where

$$\overline{M} = (C_1 + d)[C_1(1+\varepsilon\mu\beta^{-1})+d]\lambda M T^2 + \max_{i=\overline{1,r}} M_i .$$

The last inequality shows that (7) also holds for $p = r+1$ and that $x((r+1)T$ belongs to the domain G.

The proof of Lemma 1 is complete.

Proof of Theorem 1. According to the condition (iii) of Theorem 1, there exists a constant $C(T) < \infty$ such that for each $i = 1, 2, \ldots$ the inequality $d_i < C(T)$ holds. Hence there exists a constant $M_0(T) < \infty$ such that

$$\overline{M}(C_1 + d)[C_1(1+\varepsilon\mu\beta^{-1}) + d]\lambda M T^2 + \max_{i=\overline{1,r}} M_i \leqslant M_0(T) . \tag{29}$$

Let q be the integer part of the number $L/\varepsilon T$. Then for each $p \in \overline{1, q}$ according to (29) and Lemma 1, we have

$$\| x(pT) - \bar{x}(pT) \| \leqslant$$

$$\leqslant \varepsilon \sum_{i=0}^{p-1} \{1 + [\nu(1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\, \lambda T\}^{i}\, [(1 + C_1)\alpha(T)T + \varepsilon M_0(T)] \leqslant$$

$$\leqslant [(1 + C_1)\, \alpha(T)T + \varepsilon M_0(T)]\, \{ 1 + [\nu (1 + \varepsilon\mu\beta^{-1}) + \varepsilon d]\, \lambda T \}^{p} \cdot$$

$$\cdot \{ [(C_1(1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda T \}^{-1} \leqslant [\exp \{ [C_1(1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda L \} +$$

$$+ O(\varepsilon)] \cdot [(1 + C_1)\, \alpha (T)T + \varepsilon M_0(T)]\, \{[C_1 (1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda T \}^{-1}.$$

Choose T so large that

$$\frac{(1 + C_1)\, \alpha (T) \exp \{ [C_1 (1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda L \}}{[C_1 (1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda} < \frac{\eta}{4},$$

and then choose ε so small that

$$\frac{O(\varepsilon) [(1 + C_1)\, \alpha (T)T + \varepsilon M_0(T)] + \varepsilon \exp \{ [C_1(1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda L \}\, M_0(T)}{[C_1(1 + \varepsilon\mu\beta^{-1}) + d]\, \lambda} < \frac{\eta}{4}.$$

Then for each $p \in \overline{1, q}$ the inequality

$$\| x(pT) - \bar{x}(pT) \| < \eta / 2 \tag{30}$$

holds. Further, estimating $\| \bar{x}(t) - \bar{x}((p-1)T) \|$ and $\| x(t) - x((p-1)T) \|$ in the interval $(p-1)T \leqslant t \leqslant pT$, we have

$$\| \bar{x}(t) - \bar{x}((p-1)T) \| \leqslant \| \int_{(p-1)T}^{t} [\nu X_0(\bar{x}(\theta), \varepsilon) + \varepsilon I_0(\bar{x}(\theta))]\, d\theta \| \leqslant$$

$$\leqslant \varepsilon (C_1 + d)\, MT,$$

$$\| x(t) - x((p-1)T) \| = \| x_j^{(p-1)}(t, \tau_{j-1}^{(p-1)}, x_{j-1}^{(p-1)+}) - x((p-1)T) \| =$$

$$= \| \tilde{x}_1^{(p-1)}(t, (p-1)T, x_{(p-1)T}) + \varepsilon \sum_{i=0}^{j-1} I_i^{(p-1)} + \tag{32}$$

$$+ \sum_{i=0}^{j-1} R_i^{(p-1)}(\tau_i^{(p-1)}, \tau_{i-1}^{(p-1)}, x_{i-1}^{(p-1)+}, \varepsilon) +$$

$$+ R_j^{(p-1)}(t, \tau_{j-1}^{(p-1)}, x_{j-1}^{(p-1)+}, \varepsilon) - x((p-1)T) \| \leqslant$$

$$\leqslant \nu \int_{(p-1)T}^{t} \| X(\theta, x_{(p-1)T}, \int_{0}^{\infty} \psi(\sigma, \theta, x_{(p-1)T}, \varepsilon)\, d\sigma \|\, d\theta +$$

$$+ \varepsilon \sum_{i=0}^{j-1} \| I_i^{(p-1)} \| + \sum_{i=0}^{j-1} \| R_i^{(p-1)} (\tau_i^{(p-1)}, \tau_{i-1}^{(p-1)}, x_{i-1}^{(p-1)+}, \varepsilon) \| +$$

$$+ \| R_j^{(p-1)} (t, \tau_{j-1}^{(p-1)}, x_{j-1}^{(p-1)+}, \varepsilon) \| \leqslant \varepsilon M [C_1 T + (j-1)] +$$

$$+ \sum_{i=0}^{j} \omega_i (\varepsilon^2, T) \leqslant \varepsilon M [C_1 T + C(T)] + \varepsilon^2 M_0(T) \equiv \Psi (\varepsilon, T).$$

Hence, for sufficiently small ε, the choice of T ensures that

$$\Psi (\varepsilon, T) < \eta/2. \tag{33}$$

From (30) — (33), under the above choice of T, for sufficiently small ε and for $p = \overline{1, q}$ we get the following inequality

$$\| x(t) - \bar{x}(t) \| \leqslant \| x(t) - x((p-1)T) \| +$$
$$+ \| x((p-1)T) - \bar{x}((p-1)T) \| + \| \bar{x}((p-1)T) - \bar{x}(t) \| < \eta$$

in the interval $(p-1)T \leqslant t \leqslant pT$.

Hence, for small ε ($0 < \varepsilon \leqslant \varepsilon_0 \leqslant \mathscr{E}$) in the whole interval $0 \leqslant t \leqslant L\varepsilon^{-1}$ the inequality

$$\| x(t) - \bar{x}(t) \| < \eta$$

holds. This completes the proof of Theorem 1.

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