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ON AN EFFECTIVE METHOD OF PURSUIT IN LINEAR DISCRETE GAMES WITH DIFFERENT TYPES OF CONSTRAINTS ON CONTROLS

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1. INTRODUCTION

In the pursuit process there are two control objects: the pursuer $x = (x_1, x_2)$ and evader $y = (y_1, y_2)$ whose dynamics can be described by the following difference equations

$$x(k+1) = f(x(k), u(k)), \tag{1}$$

$$y(k+1) = g(y(k), v(k)), \tag{2}$$

where u and v are controls. The initial values $x(0) = x_o$ and $y(0) = y_o$ are given. The components x_1, y_1 (respectively x_2, y_2) of phase vectors are referred to as geometrical positions (respectively velocities) of the objects. The pursuit process is completed when the objects coincide geometrically i.e.

$$x_1 = y_1. \tag{3}$$

To simplify the notations we transform the pursuit process into the discrete game. Namely, we couple the phase vectors x and y into a single vector $z = (x, y)$ which belongs to the phase vector space R of the game being the direct sum of the phase vector spaces of both objects. Thus we can rewrite equations (1) and (2) as a single difference equation

$$z(k+1) = F(z(k), u(k), v(k)), z(o) = z_o. \tag{4}$$

Condition (3) defines in the space R a certain subset M on which the game is completed. Consider the game independently of the pursuit process. The game is given if its phase vector space R, the equation (4) and the subset M are given. In the sequel we shall restrict ourself to the following two problems:

1°. The pursuit game: Find the value $u(k)$ at each step k in order to complete the game.

2°. The evasion game: Find the value $v(k)$ at each step k in order to prevent the end of the game.

In recent years the discrete games of pursuit have received much attention from researchers (see for example [2—7]). In these papers some sufficient conditions for pursuit have been presented. Note that the controls u and v in [2—7] satisfy either

$$u(k) \in P_k,\ v(k) \in Q_k,\ k = 0, 1, \ldots$$

or

$$\sum_{k=0}^{\infty} \|u(k)\|^2 \leqslant \rho^2; \quad \sum_{k=0}^{\infty} \|v(k)\|^2 \leqslant \varrho^2.$$

In this paper, we consider the discrete game of pursuit with different types of constraints on controls.. Moreover, under general hypotheses of usage informations we obtain some new effective sufficient conditions which guarantee the possibility of completing the pursuit from a point of the phase vector space. The present paper is a continuation of the earlier works [4-8].

2. THE MAIN RESULTS

Denote the phase vector by $z \in R^n$ and assume that its motion is described by the following difference equation:

$$z(k+1) = Az(k) - Bu(k) + Cv(k);\ z(0) = z_0, \tag{5}$$

where $u \in R^p$, $v \in R^q$ are pursuit and evasion controls respectively; $k = 0,1,\ldots$ are the index of step in the game, and A, B, C are the matrixs of appropriate degrees. The controls $u(k)$ and $v(k)$ are assumed to satisfy the condition

$$\sum_{k=0} \| u(k) \|^2 \leqslant \rho^2;\ v(k) \in Q_k, \tag{6}$$

where $\rho > 0$ and Q_k are subsets of R^q. Let M be a subset of R^n such that $M = M_1 + M_2$, where M_1 is a subspace in R^n and M_2 a subset of the orthogonal complement to M_1 in R^n. Let π denote orthogonal projection from R^n onto L (with respect to a given basics of R^n).

Let there be given sets $N(s)$, $s = 0,1,\ldots$ satisfying

$$N(s) \subset \{0,1,2,\ldots,s\}. \tag{7}$$

We shall say that the pursuit process in the discrete game (5)—(6) is complete after k_1 steps, if for any controls $v(o)$, $v(1)$, ..., $v(k_1-1)$; $v(i) \in Q_i$, $i = 0, 1, \ldots, k_1 - 1$, there exist controls $u(o)$, $u(1)$, ..., $u(k_1-1)$ such that

$$\sum_{i=o}^{k_1-1} \| u(i) \|^2 \leqslant \rho^2,$$

and the solution $z = z(k)$, $0 \leqslant k \leqslant k_1 - 1$ of the equation

$$z(k+1) = Az(k) - Bu(k) + Cv(k)\,;\; z(o) = z_o$$

satisfy the equality

$$z(k_1) = A^{k_1} z_0 - \sum_{i=0}^{k_1-1} A^{k_1-1-i} Bu(i) + \sum_{i=0}^{k_1-1} A^{k_1-1-i} Cv(i) \in M.$$

We shall be interested in computing the value $u(k)$ of the pursuit control at each step k when the values $v(s)$ of the evasion control are know for all $s \in N(k)$. In other words, we shall be interested in finding the function

$$u(k) = u(v(s) : s \in N(k)).$$

Let us denote

$\Delta_1(K) = \{ 0 \leqslant k \leqslant K - 1 : N(k) \neq \varnothing \}$; $\Delta_2(K) = \{ 0, 1, \ldots, K-1\} \setminus \Delta_1(K)$;

$\Delta_3(K) = \bigcup_{k \in \Delta_1(K)} N(k)$; $\Delta_4(K) = \{0, 1, \ldots, K-1\} \setminus \Delta_3(K)$.

It is evident that

$$\Delta_1(K) = \{s_1, s_2, \ldots, s_{|\Delta_1(K)|}\},$$

$$\Delta_3(K) = \{r_1, r_2, \ldots, r_{|\Delta_3(K)|}\},$$

where $s_1 < s_2 < \ldots < s_{|\Delta_1(K)|}$; $r_1 < r_2 < \ldots r <_{|\Delta_3(K)|}$ and $|\Delta_i(K)|$, $i = 1,3$ means the number of elements of sets $\Delta_i(K)$. Furthermore we denote

$$H(K) = \sum_{i \in \Delta_4(K)} \pi A^{K-1-i} CQ_i.$$

THEOREM 1. *Assume that K_1 is positive integral number satisfying*

1/ $M_2 \overset{*}{-} H(K_1) \neq \varnothing$, w' re is the geometrical difference in the sence of L. S. Pontryagin [1].

2/ ..., There exist controls $u^*(i)$, $i \in \Delta_2(K_1)$, a matrix $\varphi(K_1) = (\gamma K_1(s_i, r_j))$, $i = 1, 2, \ldots, |\Delta_1(K_1)|$; $j = 1, 2, \ldots, |\Delta_3(K_1)|$ of degree $|\Delta_3(K_1)| \times |\Delta_1(K_1)|$ and matrixs $F_{\varphi(K_1)}(s_i, r_j)$, $i = 1, 2, \ldots, |\Delta_1(K_1)|$; $j = 1, 2, \ldots, |\Delta_3(K_1)|$ of degree $q \times p$, such that

2a) $$\sum_{i\in\Delta_2(K_1)} \| u^*(i) \|^2 \leqslant \rho^2,$$

2b) $$\gamma_{K_1}(s_i, r_j) = 0 \quad \text{if } r_j \notin N(s_i),$$

2c) $$\sum_{i=1}^{|\Delta_1(K_1)|} \gamma_{K_1}(s_i, r_j) = 1 \text{ for all } j = 1,2\ldots, |\Delta_3(K_1)|,$$

2d) $$F_{\varphi(K_1)}(s_i, r_j) = \tilde{O} \text{ if } r_j \notin N(s_i),$$

2e) $$\pi A^{K_1-1-i} BF_{\varphi(K_1)}(s_i, r_j) = \gamma_{K_1}(s_i, r_j)\, \pi A^{K_1-1-r_j} C(r_j)$$

for all $i = 1,2,\ldots, |\Delta_1(K_1)|$; $r_j \in N(s_i)$, *where* $\tilde{O}$ *denotes the null matrix of degree* $q \times p$.

3) $$\chi(K_1) \leqslant \tilde{\rho}^2 \text{ , where}$$

$$\tilde{\rho}^2 = \rho^2 - \sum_{i\in\Delta_2(K_1)} \| u^*(i) \|^2 \text{ and}$$

$$\chi^2(K_1) = \sup_{\substack{v(i)\in Q_i \\ i\in\Delta_3(K_1)}} \sum_{i=1}^{|\Delta_1(K_1)|} \left\| \sum_{r_j\in N(s_i)} F_{\varphi(K_1)}(s_i, r_j)\, v(r_j) \right\|^2$$

4) $$\pi A^{K_1} z_o \in G(K_1) + (M_2 \,\bar{*}\, H(K_1)) + \sum_{i\in\Delta_2(K_1)} \pi A^{K_1-1-i} Bu^*(i), \quad (8)$$

where

$$G(K_1) = \left\{ \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s_i} B\omega(s_i) : \sum_{i=1}^{|\Delta_1(K_1)|} \| \omega(s_i) \|^2 \leqslant (\tilde{\rho} - \chi(K_1))^2 \right.$$

Then the pursuit process in the discrete game (5) — (6) is completed after K_1 *steps.*

Proof. From (8), it follows that there exist vectors $m \in M_2 \,\bar{*}\, H(K_1)$, $\tilde{\omega}(s_i)$, $i = 1,\ldots, |\Delta_1(K_1)|$ such that

$$\sum_{i=1}^{|\Delta_1(K_1)|} \| \tilde{\omega}(s_i) \|^2 \leqslant (\tilde{\rho} - \chi(K_1))^2, \quad (9)$$

$$\pi A^{K_1} z_o - \sum_{i\in\Delta_2(K_1)} \pi A^{K_1-1-i} Bu^*(i) = m + \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s_i} B\tilde{\omega}(s_i). \quad (10)$$

Assume now that $v(i)$, $i = 0,1,\dots, K_1 - 1, \overline{v}(i) \in Q_i$ is an arbitrary evasion control. Then the pursuit control $\overline{u}(i)$, $i = 0,1,\dots, K_1 - 1$ is defined by the formula

$$\overline{u}(i) = \begin{cases} u^*(i) \text{ if } i \in \Delta_2(K_1) \\ \sum_{r_j \in N(s_k)} F_{\varphi(K_1)}(s_k, r_j)\, \overline{v}(r_j) + \tilde{\omega}(s_k) \text{ if } i = s_k,\ k = 1,\dots, |\Delta_1(K_1)|. \end{cases}$$

By virtue of the Minkowski's inequality we have

$$\left(\sum_{i=1}^{|\Delta_1(K_1)|} \left\| \tilde{\omega}(s_i) + \sum_{r_j \in N(s_i)} F_{\varphi(K_1)}(s_i, r_j)\overline{v}(r_j) \right\|^2 \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \left| \sum_{i=1}^{|\Delta_1(K_1)|} \| \tilde{\omega}(s_i) \|^2 \right|^{\frac{1}{2}} +$$

$$\left| \sum_{i=1}^{|\Delta_1(K_1)|} \left\| \sum_{r_j \in N(s_i)} F_{\varphi(K_1)}(s_i, r_j)\overline{v}(r_j) \right\|^2 \right|^{\frac{1}{2}} \leqslant$$

$$\leqslant \tilde{\rho} - \chi(K_1) + \chi(K_1) = \tilde{\rho}.$$

Consequently,

$$\sum_{i=0}^{K_1-1} \| \overline{u}(i) \|^2 = \sum_{i \in \Delta_2(K_1)} \| u^*(i) \|^2 + \sum_{i=1}^{|\Delta_1(K_1)|} \left\| \tilde{\omega}(s_i) + \sum_{r_j \in N(s_i)} F_{\varphi(K_1)}(s_i, \tau_j)\overline{v}(\tau_j) \right\|^2 \leqslant$$

$$\leqslant \sum_{i \in \Delta_2(K_1)} \| u^*(i) \|^2 + \tilde{\rho}^2 = \rho^2,$$

i.e. $\overline{u}(i)$, $i = 0,1,\dots, K_1 - 1$ is a capable pursuit control. Moreover, for every vector $m \in M_2 \overset{*}{-} H(K_1)$ there exists a vector $m_2 \in M_2$ such that

$$m = m_2 - \sum_{i \in \Delta_4(K_1)} \pi A^{K_1 - 1 - i} C\overline{v}(i). \tag{11}$$

From the formula for solution of equation (5) we have

$$z(K_1) = A^{K_1} z_0 - \sum_{i=0}^{K_1-1} A^{K_1 - 1 - i} B\overline{u}(i) + \sum_{i=0}^{K_1-1} A^{K_1 - 1 - i} C\overline{v}(i).$$

Therefore,

$$\pi z(K_1) = \pi A^{K_1} z_o - \sum_{i \in \Delta_2(K_1)} \pi A^{K_1-1-i} Bu^*(i) -$$

$$- \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s} B\overline{u}(s_i) + \sum_{i \in \Delta_3(K_1)} \pi A^{K_1-1-i} C\overline{v}(i) +$$

$$+ \sum_{i \in \Delta_1(K_1)} \pi A^{K_1-1-i} C\overline{v}(i).$$

From (10) and (11) it follows that

$$\pi z(K_1) = m_2 - \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s_i} Bu(s_i) + \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{k_1-1-s_i} B\widetilde{\omega}S_i) +$$

$$+ \sum_{j \in \Delta_3(K_1)} \pi A^{K_1-1-j} C\overline{v}(j), \tag{12}$$

where for $i = 1,2,\dots, |\Delta_1(K_1)|$, $\overline{u}(s_i)$ is defined by

$$\overline{u}(s_i) = \widetilde{\omega}(s_i) + \sum_{r_j \in N(s_i)} F_{\varphi(K_1)}(s_i, r_j)\ \overline{v}(r_j).$$

Hence

$$\sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s} B\overline{u}(s_i) = \sum_{i=1}^{|\Delta_1(K_1)|} \pi A^{K_1-1-s_i} B(s_i) +$$

$$+ \sum_{i=1}^{|\Delta_1(K_1)|} \left(\sum_{r_j \in N(s_i)} \pi A^{K_1-1-s_i} BF_{\varphi(K_1)}(s_i, r_j)\,\overline{v}(r_j) \right). \tag{13}$$

From (12) and (13) we get

$$\pi z(K_1) = m_2 + \sum_{j \in \Delta_3(K_1)} \pi A^{K_1-1-j} C\overline{v}(j) -$$

$$- \sum_{i=1}^{|\Delta_1(K_1)|} \left(\sum_{r_j \in N(s_i)} \pi A^{K_1-1-s_i} BF_{\varphi(K_1)}(s_i, r_j)\,\overline{v}(r_j)) \right). \tag{14}$$

From property 2e/we obtain

$$\sum_{i=1}^{|\Delta_1(K_1)|} \left(\sum_{r_j \in N(s_i)} \pi A^{K_1-1-s_i} BF_{\varphi(K_1)}(s_i, r_j)\bar{v}(r_j) \right) =$$

$$= \sum_{i=1}^{|\Delta_1(K_1)|} \left| \sum_{j=1}^{|\Delta_3(K_1)|} \gamma_{K_1}(s_i, r_j)\pi A^{K_1-1-r_j} C\bar{v}(r_j) \right| =$$

$$= \sum_{j=1}^{|\Delta_3(K_1)|} \left| \sum_{i=1}^{|\Delta_1(K_1)|} \gamma_{K_1}(s_i, r_j) \right| \pi A^{K_1-1-r_j} C\bar{v}(r_j).$$

It follows readily from 2c, that

$$\sum_{i=1}^{|\Delta_1(K_1)|} \left| \sum_{r_j \in N(s_i)} \pi A^{K_1-1-s_i} BF_{\varphi(K_1)}(s_i, r_j)\bar{v}(r_j) \right| =$$

$$= \sum_{j=1}^{|\Delta_3(K_1)|} \pi A^{K_1-1-r_j} C\bar{v}(r_j). \tag{15}$$

From (14) (15) we have $\pi z(K_1) = m_2$, i. e. $z(K_1) \in M$. Thus, the proof is complete.

We now consider the case where.

$$N(k) = \{k\} \qquad k \geqslant 0.$$

Then

$$\Delta_1(K) = \Delta_3(K) = \{0,1,\dots, K-1\};\ \Delta_2(K) = \Delta_4(K) = \phi \text{ for any } K > 1.$$

As an immediate consequence of Theorem 1 we obtain.

COROLLARY 1. *Let K_1 be a positive integral number such that*

a) *There exist matrixs $F(k)$, $k = 0,1,\dots, K_1 - 1$ of degree $q \times p$ satisfying*

$$\pi A^{K_1-1-k} BF(k) = \pi A^{K_1-1-k} C(k).$$

b/ $\chi^2(K_1) \leqslant \rho^2$, *where*

$$\chi^2(K_1) = \sup_{v(i) \in Q_i} \sum_{i=0}^{K_1-1} \| F(i)\, v(i) \|^2.$$

$$i = 0,1,\dots, K_1-1$$

c/ $\pi A^{K_1} z_0 \in G(K_1)$, where

$$G(K_1) = \left\{ \sum_{i=0}^{K_1 - 1} \pi A^{K_1 - 1 - i} B\omega(i) : \sum_{i=0}^{K_1 - 1} \| \omega(i) \|^2 \leqslant (\rho - \chi(K_1))^2 \right.$$

Then the pursuit process in the discrete game (5) — (6) is completed after K_1 steps.

Note that an analogous result for discrete games with integral constraints on controls has been obtained earlier by N.Yu. Satimov [3] and Phan Huy Khai, A.Ya. Azimov [5], [7].

3. APPLICATION

In this section, for making the purpose and usefulness of the result in §2. more apparent, illustrative examples will be given.

I. Assume that the motions of vectors $z_1 \in R^n$, $z_2 \in R^n$ are described by the difference equations:

$$z_1(k+1) = \alpha z_1(k) + \frac{1 - (-1)^k}{2} u(k);\ z_1(0) = z_1^0 \tag{16}$$

$$z_2(k+1) = \beta z_2(k) + v(k) \quad ;\ z_2(0) = z_2^0 \tag{17}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $z_1^0 \neq z_2^0$. The controls u, v are vectors of R^n satisfying:

$$\sum_{k=0}^{\infty} \| u(k) \|^2 \leqslant \rho^2 ;\ \| v(k) \| \leqslant \delta. \tag{18}$$

The pursuit process in the discrete game (16) — (18) is said to be completed after k_1 steps if $z_1(k_1) = z_2(k_1)$. Let us calculate the value $u(k)$ of the control parameter u at each step k, assuming that the values $v(s)$ of the evasion control are known for all $s \in N(k)$, where

$$N(k) = \begin{cases} (k-1, k) & \text{if } k \text{ is odd number} \\ \phi & \text{if } k \text{ is even number} \end{cases}$$

Let us denote $z = (z_1, z_2)^T \in R^{2n}$,

$$A = \begin{vmatrix} \alpha E & O \\ \tilde{O} & \beta E \end{vmatrix}; \quad B(K) = -\begin{vmatrix} \frac{1-(-1)^k}{2} E \\ O \end{vmatrix}; \quad C(k) = \begin{vmatrix} \breve{O} \\ E \end{vmatrix},$$

where F and $\breve{O}$ is the unit and null matrix of degree n, respectively and $(z_1, z_2)^T$ is the transpose of the vector (z_1, z_2).

We now can rewrite equation (16), (17) as a single difference equation

$$z(k+1) = Az(k) - B(k)u(k) + C(k)v(k)\,; \; z(o) = (z_1^0, z_2^0)^T.$$

In this case we have

$$M = \left\{ z = (z_1, z_2)^T : z_1 = z_2 \right\}; \; N = \left\{ z = (z_1, z_2)^T : z_1 = -z_2 \right\}.$$

Denote orthogonal projection of the space R^{2n} onto its subspace L by π. Let $\Pi = (E, -E)$. Then for a suitable system of coordinate of L, the matrix of π can be written as Π.

By a direct computation it follows that

$$\Pi A^{K-1-k} B(k) = \begin{cases} \tilde{O}, \text{ if } k \text{ is even number} \\ \alpha^{K-1-k} E, \text{ if } k \text{ is odd number,} \end{cases}$$

$$\Pi A^{K-1-k} C(k) = \beta^{K-1-k} E.$$

For any positive even number K^* we define a square matrix of degree K^* by

$$\varphi(K^*) = \begin{pmatrix} 0\;1 & 00 \ldots\ldots 00 \\ 0\;1 & 00 \ldots\ldots 00 \\ 0\;0 & 01 \ldots\ldots 00 \\ 0\;0 & 01 \ldots\ldots 00 \\ \cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0\;0 & 00 \ldots\ldots 01 \\ 0\;0 & 00 \ldots\ldots 01 \end{pmatrix}$$

Next, we denote

$$F_{\varphi(K^*)}(i, j) = \begin{cases} 0, \text{ if } i \text{ is even number} \\ 0, \text{ if i is odd number and } j > i, \text{ or } j < i-1 \\ \dfrac{\beta^{K^*-1-i}}{\alpha^{K^*-1-i}} E, \text{ if } i \text{ is odd number and } j = i \\ \dfrac{\beta^{K^*-i}}{\alpha^{K^*-1-1}} E, \text{ if } i \text{ is odd number and } j = i-1. \end{cases}$$

It should be noted that the assumptions 2b, 2c, 2d, 2e in Theorem 1 are fulfilled. Moreover, we have

$$\sum_{i=o}^{K^*-1} \left\| \sum_{j=o}^{i} F_{\varphi(K^*)}(i,j)\, v(j) \right\|^2 = \left\| \frac{\beta^{K^*-1}}{\alpha^{K^*-2}} (o) + \frac{\beta^{K^*-2}}{\alpha^{K^*-2}} v(1) \right\|^2 +$$

$$+ \left\| \frac{\beta^{K^*-3}}{\alpha^{K^*-4}} v(2) + \frac{\beta^{K^*-4}}{\alpha^{K^*-4}} v(3) \right\|^2 + \dots + \left\| \beta v(K^*-2) + v(K^*-1) \right\|^2 \leqslant$$

$$\leqslant (\beta^2+1) \left(\frac{\beta^{K^*-2}}{\alpha^{K^*-2}} \right)^2 \left(\| v(o) \|^2 + \| v(1) \|^2 \right) +$$

$$+ (\beta^2+1) \left(\frac{\beta^{K^*-4}}{\alpha^{K^*-4}} \right)^2 \left(\| v(2) \|^2 + \| v(3) \|^2 \right) +$$

$$+ \dots + (\beta^2+1) \left(\| v(K^*-2) \|^2 + \| v(K^*-1) \|^2 \right) \leqslant$$

$$\leqslant 2\delta^2 (\beta^2+1) \left(1 + \left(\frac{\beta^2}{\alpha^2} \right)^2 + \dots + \left(\frac{\beta^{K^*-4}}{\alpha^{K^*-4}} \right)^2 + \left(\frac{\beta^{K^*-2}}{\alpha^{K^*-2}} \right)^2 \right).$$

Then, for $\beta < \alpha$, we get

$$\sum_{i=0}^{K^*-1} \left\| \sum_{j=o}^{i} F_{\varphi(K^*)}(i,j)\, v(j) \right\|^2 < 2\delta^2 (\beta^2+1) \frac{1}{1 - \frac{\beta^4}{\alpha^4}}$$

Consequently, the assumption 3 in Theorem 1 will be fulfilled if

$$2\delta^2 (\beta^2+1)\, \alpha^4 \leqslant \rho^2 (\alpha^4 - \beta^4).$$

Hence, if $0 < \alpha < 1$ and $0 < \beta < 1$, we obtain

$$\lim_{K \to +\infty} \left\| \Pi A^K z_o \right\| = \lim_{K \to +\infty} \left\| \alpha^K z_1^0 - \beta^K z_2^0 \right\| = 0 \text{ for any } z_o = (z_1^0, z_2^0)^T \in R^{2n}.$$

Since $\lim_{K \to +\infty} \left\| \Pi A^K z_o \right\| = 0$ the assumption 4 in Theorem 1 is fulfilled.

As an immediate consequence of Theorem 1 we have

PROPOSITION 1. *Assume that $0 < \alpha < 1$, $0 < \beta < 1$, $\beta \leqslant \alpha$, $2\delta^2\alpha^4(\beta^2+1) \leqslant \rho^2(\alpha^4 - \beta^4)$. Then the pursuit process in the discrete game (16) — (18) is completed after a finite number of steps for any position $z^0 = (z_1^0, z_2^0)^T \in R^{2n}$.*

II. Assume now that the motions of vectors $z_i \in R^n, i=1, 2, 3, 4$ are described by the difference equations:

$$z_1(k+1) = z_1(k) + \mu z_2(k); \; z_1(0) = z_1^0, \quad (19)$$

$$z_2(k+1) = -\alpha_2 \mu z_1(k) + (1-\mu\alpha_1) z_2(k) + \mu u(k); \; z_2(0) = z_2^0, \quad (20)$$

$$z_3(k+1) = z_3(k) + \mu z_4(k); \; z_3(0) = z_3^0, \quad (21)$$

$$z_4(k+1) = -\beta_2 \mu z_3(k) + (1-\mu\beta_1) z_4(k) + \mu\sigma v(k); \; z_4(0) = z_4^0, \quad (22)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_3$ are real number, μ is a small positive parameter; $z_1^0 \neq z_3^0$. The controls u, v are vectors of R^n satisfying:

$$\sum_{k=0}^{\infty} \| u(k) \|^2 \leqslant \rho^2 , \; \| v(k) \| \leqslant 1. \quad (23)$$

The pursuit process in the discrete game (19) – (23) is said to be completed after k_1 steps if $z_1(k_1) = z_3(k_1)$. Let us calculate the value $u(k)$ of the control parameter u at each step k, assuming that the value $v(k)$ of the evasion control is known. That is, let us compute

$$u(k) = u(v(k)).$$

Let us denote $z = (z_1, z_2, z_3, z_4)^T \in R^{4n}$,

$$A = \begin{pmatrix} E & \mu E & \tilde{0} & \tilde{0} \\ -\alpha_2 \mu E & (1-\mu\alpha_1)E & \tilde{0} & \tilde{0} \\ \tilde{0} & \tilde{0} & E & \mu E \\ \tilde{0} & \tilde{0} & -\beta_2 \mu E & (1-\mu\beta_1)E \end{pmatrix}; \quad B = \begin{pmatrix} \tilde{0} \\ -\mu E \\ \tilde{0} \\ \tilde{0} \end{pmatrix}; \quad C = \begin{pmatrix} \tilde{0} \\ \tilde{0} \\ \tilde{0} \\ \mu\sigma E \end{pmatrix}$$

We now can rewrite equations (19) – (22) as a single difference equation $z(k+1) = Az(k) - Bu(k) + Cv(k)$; $z(0) = (z_1^0, z_2^0, z_3^0, z_4^0)^T$.

we have then

$$M = \left\{ z = (z_1, z_2, z_3, z_4)^T \quad R^{4n} : z_1 = z_3 \right\},$$

$$L = \left\{ z = (z_1, z_2, z_3, z_4)^T \quad R^{4n} : z_1 = -z_3, z_2 = z_4 = 0 \right\}.$$

Note that M is subspace of R^{4n} and L is its orthogonal complement in R^{4n}. The orthogonal projection of the space R^{4n} onto L is denoted by π. Let $\Pi = (E, 0, -E, 0)$. Then for a suitable systems of coordinate of L, the matrix of π can be written as Π.

Let λ_1 and λ_2 be the solutions of the quadratic equation

$$\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0, \tag{24}$$

and γ_1 γ_2 be the solutions of the quadratic equation

$$\gamma^2 + \beta_1 \gamma + \beta_2 = 0. \tag{25}$$

By a direct computation it follows that (see for example [7])

$$A^k = \begin{pmatrix} x_k E & y_k E & \tilde{O} & \tilde{O} \\ p_k E & q_k E & \tilde{O} & \tilde{O} \\ \tilde{O} & \tilde{O} & s_k E & r_k E \\ \tilde{O} & \tilde{O} & t_k E & h_k E \end{pmatrix}, \; k = 1, 2,$$

where $x_1 = 1$, $y_1 = \mu$; $s_1 = 1$, $r_1 = \mu$ and for $k = 2, 3, ..$ we have

$$x_k = \begin{cases} \dfrac{\lambda_2 (1+\mu \lambda_1)^k - \lambda_1 (1+\mu\lambda_2)^k}{\lambda_2 - \lambda_1} & \text{if the equation (24) has real roots } \lambda_1, \lambda_2; \\ (1+\mu\lambda^*)[1-(k-1)\mu\lambda^*] & \text{if the equation (24) has a double root } \lambda^*, \end{cases}$$

$$y_k = \begin{cases} \dfrac{(1+\mu\lambda_2)^k (1+\mu\lambda_1)^k}{\lambda_2 - \lambda_1} & \text{if the equation (24) has real roots } \lambda_1, \lambda_2; \\ k\mu(1+\mu\lambda^*) & \text{if the equation (24) has a double root } \lambda^*, \end{cases}$$

$$s_k = \begin{cases} \dfrac{\gamma_2 (1+\mu\gamma_1)^k - \gamma_1 (1+\mu\gamma_2)^k}{\gamma_2 - \gamma_1} & \text{if the equation (25) has real roots } \gamma_1, \gamma_2, \\ (1+\mu\gamma^*)[1-(k-1)\mu\gamma^*] & \text{if the equation (25) has a double root } \gamma^*, \end{cases}$$

$$r_k = \begin{cases} \dfrac{(1+\mu\gamma_2)^k - (1+\mu\gamma_1)^k}{\gamma_2 - \gamma_1} & \text{if the equation (25) has real roots } \gamma_1, \gamma_2, \\ k\mu(1+\mu\gamma^*) & \text{if the equation (25) has a double root } \gamma^*. \end{cases}$$

A simple computation shows that

$\Pi A^k B = -\mu y_k E$; $\Pi A^k C = -\mu \sigma r_k E$ for any $k = 1, 2, ..$.

$\Pi A^o B = \Pi A^o C = \tilde{O}$.

Putting

$$F(k) = \begin{cases} \sigma \dfrac{r_k}{y_k} E, & \text{if } k = 1, 2, \ldots \\ \tilde{O}, & \text{if } k = 0; \end{cases}$$

we have

$$\chi^2(K) = \sup_{\substack{\| v(k) \| \leqslant 1 \\ k=0,\dots,K-1}} \sum_{k=0}^{K-1} \| F(k)\, v(k) \|^2 \leqslant \sigma^2 \sum_{k=0}^{K-1} \frac{r_k^2}{y_k^2} .$$

Hypothesis 1. The equations (24), (25) have double roots λ^* and γ^* satisfying $\gamma^* < \lambda^* \leqslant 0$ and $\sigma^2 < \rho^2 \delta_1^2$, where

$$\delta_1^2 = 1 - \left(\frac{1+\mu\gamma^*}{1+\mu\lambda^*}\right)^2.$$

Hypothesis 2. The equation (24) has a double root λ^* and the equation (25) has real roots $\gamma_1 < \gamma_2$ satisfying $\gamma_2 < \lambda^* \leqslant 0$

and $\sigma^2 < \rho^2 \delta_2^2$, where $\delta_2^2 = 1 - \left(\frac{1+\mu\gamma_2}{1+\mu\lambda^*}\right)^2.$

Hypothesis 3. The equation (24) has real roots $\lambda_1 < \lambda_2$ and the equation (25) has a double root γ^* satisfying $\gamma^* < \lambda_1 \leqslant 0$ and $\sigma^2 < \rho^2 \delta_3^2$, where $\delta_3^2 = 1 - \left(\frac{1+\mu\gamma^*}{1+\mu\lambda_1}\right)^2.$

Hypothesis 4. The equations (24), (25) have real roots λ_1, λ_2 and γ_1, γ_2 satisfying $\lambda_1 < \lambda_2$, $\gamma_1 < \gamma_2$, $\gamma_2 < \lambda_2 \leqslant 0$ and $\sigma^2 < \rho^2 \delta_4^2$, where $\delta_4^2 = 1 - \left(\frac{1+\mu\gamma_2}{1+\mu\lambda_1}\right)^2.$

As an immediate consequence of Corollary 1 we have

PROPOSITION 2. *Assume that one of the hypotheses 1,2,3,4 is fulfilled. Then the pursuit process in the discrete game (19) — (23) is completed after a finite number of steps for any position* $z^0 = (z_1^0, z_2^0, z_3^0, z_4^0)^T \in R^{4n}$.

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