STEIN MORPHISMES AND RIEMANN DOMAINS OVER STEIN SPACES

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In 1953 Serre [12] posed the conjecture that a holomorphic bundle with a Stein base and a Stein fiber is Stein. Through the years, various special cases of this conjecture have been settled (see [4] [10] [13] [14] [15] [18] [19]). But this conjecture is not true. A counterexample for Serre Problem was given by Skoda [19] in 1977. Demailly [3] showed that the first cohomology group of the bundle space of the structure sheaf of this counterexample is not Hausdorff in the canonical topology. This can be explained by the first main result of this paper which gives a criterion for the validity of Serre's conjecture:

THEOREM 1. *Let* $\widetilde{X}$ *be a complex space having a Stein morphism. Then* $\widetilde{X}$ *is Stein if and only if* $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ *is Hausdorff.*

The following are immediate consequences of Theorem 1.

COROLLARY 1. *Let* $\widetilde{X}$ *be a holomorphic bundle over a Stein space with Stein fiber. Then* $\widetilde{X}$ *is Stein if and only if* $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ *is Hausdorff.*

COROLLARY 2. *Let* Ω *be a locally Stein open set in a Stein space. Then* Ω *is Stein if and only if* $H^1(\Omega, \mathcal{O}_\Omega)$ *is Hausdorff.*

The second main result of this paper is also closely related with Serre's conjecture. It generalizes the well known fact that an open subset Ω of $\mathbf{C}^n$ is Stein if and only if $H^q(\Omega, \mathcal{O}_\Omega) = 0$ for $q = 1,2,\ldots, n-1$.

THEOREM 2. *Let $\widetilde{X}$ be a Riemann domain over a Stein space. Then $\widetilde{X}$ is Stein if and only if $H^q(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for $q = 1,2,\ldots, \dim \widetilde{X} - 1$.*

When $\widetilde{X}$ is a Riemann domain over a Stein manifold, this result has been established by Siu [17].

COROLLARY 3. *Let Ω be a relatively compact open set in a Stein space X. Then Ω is Stein if and only if $H^q(\Omega, \mathcal{O}_\Omega)$ is finite dimensional for every $q = 1,2,\ldots, \dim X - 1$.*

Proof. By a Theorem of Siu [16], $\dim H^q(\Omega, \mathcal{O}_\Omega) < \infty$ for every $q \geqslant 1$. Since X is Stein and $\overline{\Omega}$ is compact, by Cartan theorem A there exists an exact sequence

$$o \to \mathcal{O}_X^{p_l} \to \ldots \to \mathcal{O}_X^{p_0} \to N \to o$$

on Ω, where N is nilradical of $\mathcal{O}_X$. Hence, by the hypothesis it follows that $\dim H^q(\Omega, N) < \infty$ for all $q \geqslant 1$.

Take a holomorphic function f on X which is not constant on every irreducible branch of X. Since $\dim H^q(\Omega_{red}, \mathcal{O}_{\Omega_{red}}) < \infty$ for every $q = 1,2,\ldots, \dim X - 1$ and $H^q(\Omega_{red}, \mathcal{O}_{\Omega_{red}}) = 0$ for every $q \geqslant \dim X$, there exists $g = \sum_{j=1}^{m} \alpha_j f^j \neq 0$ such that $gH^q(\Omega_{red}, \mathcal{O}_{\Omega_{red}}) = 0$ for every $q \geqslant 1$. Observe that g is not constant on every irreducible branch of Ω_{red}. Therefore, the sequence:

$$0 \to \mathcal{O}_{\Omega_{red}} \xrightarrow{g} \mathcal{O}_{\Omega_{red}} \to \mathcal{O}_{\Omega_{red}} / g\mathcal{O}_{\Omega_{red}} \to 0$$

is exact. Thus by induction on $\dim X$ we have $H^q(\Omega_{red}, \mathcal{O}_{\Omega_{red}}) = 0$ for every $q \geqslant 1$. By Theorem 2, this implies that Ω is Stein.

COROLLARY 4. *Let $\widetilde{X}$ be a Riemann domain over a perfect Stein space X. Then $\widetilde{X}$ is Stein if and only if $\dim H^q(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ for every $q = 1, 2, \ldots, \dim \widetilde{X} - 1$.*

Proof. Since $\mathcal{O}_{\widetilde{X}} / g\mathcal{O}_{\widetilde{X}}$ is a Cohen-Macauly ring for every holomorphic function g on $\widetilde{X}$ which is not constant on every irreducible branch, as in Corollary 3 we get $H^q(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for every $q \geqslant 1$. Hence by Theorem 2, $\widetilde{X}$ is Stein.

COROLLARY 5. *Let $\tilde{X}$ be a Riemann domain over a Stein space X of dimension $\leqslant 4$. Then $\tilde{X}$ is Stein if and only if* $\dim H^q(\tilde{X}, O_{\tilde{X}}) < \infty$ *for every* $q=1,2,\ldots, \dim \tilde{X}-1$.

Proof. Take a holomorphic function f on X which is not constant on every irreducible branch of X. Let $g = \sum_{j=1}^{m} \alpha_j f^j \neq 0$ such that $\tilde{g} H^q(\tilde{X}, O_{\tilde{X}}) = 0$, $\tilde{g} = g\pi$, for every $q \geqslant 1$. Observe that $\tilde{g}$ is not constant on every irreducible branch of $\tilde{X}$. Consider the exact sequence

$$0 \to \operatorname{Ker} \varphi \to O_{\tilde{X}} \xrightarrow{\varphi} \tilde{g} O_{\tilde{X}} \to 0$$

where φ is defined by multiplication by $\tilde{g}$. Since $\operatorname{supp} \operatorname{Ker} \varphi \subset \tilde{g}^{-1}(0)$ it follows that $\dim \operatorname{supp} \operatorname{Ker} \varphi \leq \dim \tilde{X} - 1 \leq 3$. Hence by a theorem of Siu [16], $H^q(\tilde{X}, \operatorname{Ker} \varphi) = 0$ for every $q \geqslant 3$. Then by the exactness of cohomology sequence it follows that the map $\widehat{\varphi}_q : H^q(\tilde{X}, O_{\tilde{X}}) \to H^q(\tilde{X}, \tilde{g} O_{\tilde{X}})$ induced by φ is surjective for every $q \geqslant 2$. This implies that $\dim H^q(\tilde{X}, \tilde{g} O_{\tilde{X}}) < \infty$ for every $q \geqslant 2$. By the hypothesis, considering the exact sequence

$$0 \to \tilde{g} O_{\tilde{X}} \xrightarrow{i} O_{\tilde{X}} \to O_{\tilde{X}} / \tilde{g} O_{\tilde{X}} \to 0$$

we infer that $\dim H^q(\tilde{X}, O_{\tilde{X}} / \tilde{g} O_{\tilde{X}}) < \infty$ for every $q \geqslant 1$. Applying the induction hypothesis to the Riemann domain $(\tilde{g}^{-1}(0), O_{\tilde{X}} / \tilde{g} O_{\tilde{X}})$ over the Stein space $(g^{-1}(0), O_X / g O_X)$ it follows that $H^q(\tilde{X}, O_{\tilde{X}} / \tilde{g} O_{\tilde{X}}) = 0$ for every $q \geqslant 1$. This implies that the map $\widehat{i}_q : H^q(\tilde{X}, \tilde{g} O_{\tilde{X}}) \to H^q(\tilde{X}, O_{\tilde{X}})$ induced by i is surjective for every $q \geqslant 1$. Then by the relation $\operatorname{Im} \widehat{i}_q = \tilde{g} H^q(\tilde{X}, O_{\tilde{X}})$ we have $H^q(\tilde{X}, O_{\tilde{X}}) = 0$ for every $q \geqslant 1$. By Theorem 2, $\tilde{X}$ is Stein.

The proofs of Theorems 1 and 2 will exploit some techniques of Fornaess and Narasimhan [5].

Let $\widetilde{X}$ be a (possibly non-reduced) complex space. We say that $\widetilde{X}$ has a Stein morphism if there exists a holomorphic map π of $\widetilde{X}$ into a Stein space X such that $\pi^{-1}(U)$ is Stein for every open subset U of X belonging to a Stein open covering of X. We also say that $\widetilde{X}$ is a Riemann domain over X if π is a local (open) isomorphism.

A sheaf $\mathcal{S}$ on X is called a Frechet sheaf if for every open subset U of X $H_0(U, \mathcal{S})$ has a Frechet space structure such that the restriction map $H^0(U, \mathcal{S}) \to H^0(V, \mathcal{S})$ is continuous for all open subsets V of U. For such a sheaf $\mathcal{S}$, the cohomology group $H^q(X, \mathcal{S})$ can be equipped with the induction topology

$$H^q(X, \mathcal{S}) = \lim_{\overrightarrow{\mathcal{U}}} H^q(\mathcal{V}, \mathcal{S})$$

where $\mathcal{U}$ is an open covering of X,

$$H^q(\mathcal{U}, \mathcal{S}) \underset{=}{\text{top}} \operatorname{Ker} \delta^q / \operatorname{Im} \delta^{q-1}$$

$\delta^q : C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$ are coboundary maps and $C^q(\mathcal{U}, \mathcal{S})$ are equipped with the product topology.

§ 2. MAIN LEMMA

In this section we establish the following Main lemma which is basic for the proof of Proposition 3.1.

Main lemma. Let Z be a reduced complex space, K a compact set in Z and f a holomorphic function on Z which is not constant on every irreducible branch of Z. Then there exist a compact set $L \subset Z$ and a number $\varepsilon > 0$ such that the following holds.

For every $|t| < \varepsilon$ there exists a constant $M_t > 0$ such that for, every constant r > 0 every holomorphic function g on Z_t is the restriction of a holomorphic function G to Z such that

$$(2.1) \qquad \| G \|_K \leqslant M_t \| g \|_{L \cap Z_t} + \tau.$$

The proof is based on the folowing.

2.1 LEMMA [5]. *Let Z be a Stein space, K a compact set in z. Let U be a relatively compact Runge domain in z, $K \subset$, $L = \overline{U}$. Let $\mathcal{S}$ and $\mathcal{S}''$ be coherent analytic shaves on z and $\alpha : \mathcal{S} \to ''$ a surjective morphism. Then there exists $C > 0$ such that for every $\sigma \in H^0(z, \mathcal{S}'')$ and constant $\tau > 0$ there exists a section $\beta \in H^0(Z, \mathcal{S})$ such that*

$$(2.2) \qquad x(\beta) = \sigma \text{ and } \| \beta \|_K = C \| \sigma \|_L + \tau.$$

2.2. LEMMA. *Let Z be a complex space, K a compact set in Z. Let f be a holomorphic function on Z which is not constant on every irreducible branch of Z. Let*

$$Z_t = V(f - t) = \{z \in Z : f(z) = t\},\ t \in \mathbf{C}$$

$$K_t = \widehat{(Z_t \cap K)}_{Z_t} = \{ z \in Z_t : | g(z) | \leqslant \| g \|_{K \cap Z_t}\ \forall g \in \mathcal{O}(Z_t) \}$$

Let Z and f satisfy the following conditions

(a) $H^1(Z, \mathcal{O}_z) = 0$

(b) Z_t is Stein for $t \in \mathbf{C}$.

Then

(i) *the restriction map $\mathcal{O}(Z) \to \mathcal{O}(Z_t)$ is surjective for $t \in C$*

(ii) *there exists a compact set L in Z such that $K_t \subset L$ for sufficiently small t,*

Proof Let $f_t = f - t$ and let $\widehat{f_t} : \mathcal{O}_Z \to \mathcal{O}_Z$ denote the morphism defined by multiplication by f_t. Using the exact sequences.

$$0 \to \operatorname{Ker} \widehat{f_t} \to \mathcal{O}_Z \xrightarrow{\widehat{f_t}} f_t\, \mathcal{O}_Z \to 0$$

$$0 \to f_t \mathcal{O}_Z \to \mathcal{O}_Z \to \mathcal{O}_Z / f_t\, \mathcal{O}_Z \to 0$$

$$\mathcal{O}_Z / f_t\, \mathcal{O}_Z \to \mathcal{O}_{Z_t} \to 0$$

and the relation supp Ker $\widehat{f_t} \subset Z_t$ we derive (i) from the Steiness of Z_t.

To prove (ii) observe that by (i) there exists a holomorphic map $G : Z \to C_N$ such that $g = G \mid_{Z_o} : Z_o \to \mathbf{C}^N$ is proper. Let $r = \| g \|_{K \cap Z_o} + 1$ and let $W_t\ (r)$ denote the union of components of $G^{-1}(\Delta(o,r)) \cap Z_t$ meeting K, where $\Delta(o,r) = \{ z = (z_1, \ldots, z_N) \in \mathbf{C}^N : \max | Z_j | < r \}$

Choose a relatively compact neighbourhood V of $K \cup \overline{W_o(2r)}$ in Z. We claim that

$$(2.2) \qquad K \cap Z_t \subset W_t(3r/2) \subset V$$

for sufficiently small t.

Indeed, since $K \cap Z_o \subset W_o(r)$, it follows that $K \cap Z_t \subset W_t(3r/2)$ for sufficiently small t. Assume now that there exists a sequence $\{t_j\}$ converging to zero such that $W_{t_j}(3r/2) \setminus V \neq Q$. Then we find a sequence $\{z_j\}$, $z_j \in W_{t_j}(3r/2) \cap \partial V$. Let z be a limit point of $\{z_j\}$. Then it is easy to see that $z \in \overline{W_o(2r)} \setminus V$. This is impossible, since $W(2r) \subset V$. From (2.2) and the fact that $W_t(3r/2)$ is a Runge domain in Z_t we get

$$K_t = (K \widehat{\cap} Z_t)_{Z_t} = (K \widehat{\cap} Z_t)_{Z_t} \cap W_t(3r/2) \subset V$$

for sufficiently small t.

2.3 Proof of Main Lemma. Let $\mathcal{U} = \{U_j\}$ be a Stein open covering of Z. Let $K_j \subset U_j$, $j = 1,2,\ldots, N$ be compact sets such that $K \subset \bigcup_{j=1}^{N} K_j$. By hypothesis $H^1(\mathcal{U}, \mathcal{O}_z) = H^1(Z, \mathcal{O}_z) = 0$, the coboundary map : $\sigma : C^o(\mathcal{U}, \mathcal{O}_z) \to Z^1(\mathcal{U}, \mathcal{O}_z)$ is surjective.

Put

$$W = \{(g_i) \in C^0(\mathcal{U}, \mathcal{O}_z) : \max_{1 \leqslant i \leqslant N} \|g_i\|_{K_i} < 1\}$$

Then, by the open mapping theorem $\delta W \supset \widetilde{W} \cap Z^1(\mathcal{U}, \mathcal{O}_Z)$ for some neighbourhood $\widetilde{W}$ of zero in $\pi_{i,j}\, \mathcal{O}(U_i \cap U_j)$. Take compact sets $K_{ij} \subset U_i \cap U_j$, $i, j = 1,2,\ldots, N_1$ $N_1 \geqslant N$ and an $\varepsilon > 0$ such that

$$\widetilde{W} \geqslant \{(g_{ij}) \in \pi\, \mathcal{O}(U_i \cap U_j) : \|g_{ij}\|_{K_i \cap K_j} < \varepsilon, \qquad i,j = 1,2,\ldots, N_1\}.$$

Since Z_0 is nowhere dense, the restriction maps $\mathcal{O}(U_i \cap U_j) \to \mathcal{O}(U_i \cap U_j \setminus Z_0)$ are embeddings. Thus we may assume that $K_{ij} \cap Z_0 = \emptyset$ for $i,j = 1,2,\ldots,N_1$. Put

$$K'_i = \bigcup_{j=1}^{N} K_{ij} \cup K_{ji}) \quad i = 1,2,\ldots, N_1 .$$

Then

$$K'_i \subset U_1 \setminus Z_0 \text{ and } K_{ij} \subset K'_i \cap K'_j$$

and

$$\delta W = Z^i(\mathcal{U}, \mathcal{O}_z) \cap \{(g_{ij}) \in \pi\, \mathcal{O}(U_i \cap U_j) : \|g_{ij}\|_{K'_i \cap K'_j} < \varepsilon$$
$$i,j = 1, 2, \ldots, N_1$$

Hence, there exists a constant $C > 0$ such that for every $(g_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_Z)$ we can find $(g_i) \in C^0(\mathcal{U}, \mathcal{O}_z)$ such that $\delta(g_i) = (g_{ij})$ and

$$\max_{1 \leqslant i \leqslant N_1} \|g_i\| = C \max_{1 \leqslant i,j \leqslant N_1} \|g_{ij}\|_{K'_i \cap K'_j}$$

Take $\varepsilon_0 > 0$ such that

$$|f(z) > 2_{\varepsilon_0} \text{ for } z \in \bigcup_{i=1}^{N} K'_i$$

Put $K_i = \emptyset$ for every $N < i \leqslant N_1$. Applying (2.1) to the canonical map α: $\mathcal{O}_{U_i} \to \mathcal{O}_{U_i} / f_t\, \mathcal{O}_{U_i}$, with $K = K_i \cup K'_i$, $U = \operatorname{Int} L'_i$, where L'_i is some compact set containing $(K_i \cup K'_i)_{U_i}$ we get a constant $C^i_{1t} > 0$ satisfying (2.1). Let $L' = \bigcup_{i=1}^{N_1} L'_i$, $C_{1t} = \max_{1 \leqslant i \leqslant N_1} C^i_{1t}$ and let L be a compact set in Z such that

$$L'_t \subset \operatorname{Int} L \qquad \text{for } |t| < \varepsilon_1 < \varepsilon_0\,,\ \varepsilon_1 > 0.$$

Let $g \in \mathcal{O}(z_t)$. Applying Lemma 2.1 to the canonical map $\alpha = \mathcal{O}_z / f_t\, \mathcal{O}_z \to \mathcal{O}_{zt}$, with $K = L'_t \subset \operatorname{Int} L \cap Z_t$ we see that there exists $\tilde{g} \in H^0(Z_t, \mathcal{O}_Z / f_t\, \mathcal{O}_Z)$ such that

$$(2.4) \qquad \alpha(\tilde{g}) = g \qquad \text{and } \|\tilde{g}\|_{L'_t} = C_{2t}\, \|g\|_{L \cap Z_t} + \tau.$$

where C_{2t} is independent of g.

For each i take $h_i \in \mathcal{O}(U_i)$ such that $h_i \mid_{U_i \cap Z_t} = \tilde{g} \mid U_i \cap Z_t$ and

$$(2.5) \qquad \| h_i \|_{K_i \cup K'_i} = C_{1t}\, \| \tilde{g} \|_{L' \cap Z_t} + \tau$$

$i = 1, 2, \ldots, N_1$.

From (2. 3) and (2. 4) we obtain

(2.6) $$\max_{1\le i\le N_1} \| h_i \|_{K_i \cup K_i'} \le C_{1t} C_{2t} \| g \|_{L\cap Z_t} + \tau C_{1t} + \tau$$

Now $h_i - h_j = (f - t)\, g_{ij}$, $g_{ij} \in \mathcal{O}(U_i \cap U_j)$. Since Z_t is nowhere dense. $(g_{ij}) \in Z^1(U, \mathcal{O})$ and for $|t| < \varepsilon_{1/2}$ we have

$$\| g_{ij} \|_{K_i' \cap K_j'} \le 1/\varepsilon_0 \, \| h_i - h_j \|_{K_i' \cap K_j'} \le 2/\varepsilon_0 \max_{1\le i\le N_1} \| h_i \|_{K_i'}$$

$$\le 2C_{1t} C_{2t} / \varepsilon_0 \| g \|_{L\cap Z_t} + (2\tau C_{1t} + \tau) / \varepsilon_0$$

Hence we can find $g_i \in \mathcal{O}(U_i)$ such that $\delta(g_i) = (g_{ij})$ and

$$\max_{1\le i\le N} \| g_i \|_{K_i} = C \max_{1\le i,j\le N} \| g_{ij} \|_{K_i' \cap K_j'}$$

$$= 2CC_{1t} C_{2t} / \varepsilon_0 \| g \|_{L\cap Z_t} + (2\tau CC_{1t} + \tau C) / \varepsilon_0$$

If we define $G = h_i - (f - t)\, g_i$ on U_i we get $G \in \mathcal{O}(Z)$, $G|_{Z_t} = g$ and

$$\| G \|_K \le \max_{1\le i\le N} \left(\| h_i \|_{K_i} + \| f - t \|_K \| g_i \|_{K_i} \right)$$

$$= C_{1t} C_{2t} \| g \|_{L\cap Z_t} + \tau C_{1t} + \tau + 2 \| f - t \|_K CC_{1t} C_{2t} / \varepsilon_0 \| g \|_{L\cap Z_t}$$
$$+ \| f - t \|_K (2\tau CC_{1t} + C\tau) / \varepsilon_0$$
$$= \| f - t \|_K (2\tau CC_{1t} + C\tau) / \varepsilon_0 + \tau C_{1t} + \tau$$
$$+ (C_{1t} + C_{2t} + 2 \| f - t \|_K CC_{1t} C_{2t} / \varepsilon_0) \| g \|_{L\cap Z_t}$$

Remark. When Z is a relatively compact locally Stein open set in a Stein space the above proposition has been proved by Fornaess and Narasimhan in [5]. Our proof in only an adaptation of their proof.

§ 3. MAIN PROPOSITION

Using our Main Lemma we now prove a proposition which is basic for the proofs of Theorems 1 and 2.

3. 1. PROPOSITION. *Let $\widetilde{X}$ be a complex space having a holomorphic map π of $\widetilde{X}$ into a Stein space X such that there exists a holomorphic function f on X which is not constant on every irreducible branch of $\widetilde{X}$. Then $\widetilde{X}$ is Stein if and only if $\pi^{-1}(V(f - t))$ is Stein for every $t \in \mathbf{C}$ and $H^1(\widetilde{X}_{red}, \mathcal{O}_{\widetilde{X}_{red}}) = 0$.*

Proof. By a theorem of Grauert it suffices to show that $\widetilde{X}_{red}$ is Stein. Since $V(\widetilde{f}-t)$, where $\widetilde{f}=f\pi$ is Stein for every $t\in\mathbf{C}$, from Lemma 2.2 we infer that $\mathcal{O}(\widetilde{X}_{red})$ separates points. Hence it remains to prove that $\widetilde{X}_{red}$ is holomorphically convex.

Let K be a given compact set in $\widetilde{X}_{red}$. We claim that $\widehat{K}$ is compact. Indeed, suppose the contrary. Then, since $\pi\widehat{K}\subset\widehat{\pi K}$ and $\widehat{\pi K}$ is compact there exists a infinite discrete sequence $\{z_n\}\subset\widehat{K}$ such that $\pi z_n\to p\in X$. We may assume that $f(p)=0$. Obviously $\eta\widetilde{f}$ is also non constant on every irreducible branch of $\widetilde{X}_{red}$ and $V(\eta\widetilde{f}-t)$ is Stein for all $t\in\mathbf{C}$, where $\eta:\mathcal{O}_{\widetilde{X}}\to\mathcal{O}_{\widetilde{X}_{red}}$ denotes the canonical map. Let $L\subset\widetilde{X}_{red}$ be a compact set and let $\varepsilon>0$ such that (2.1) holds. Take a compact set $\widetilde{L}\subset\widetilde{X}_{red}$ such that

$$L_t\subset\widetilde{L}\ \text{for}\ |t|<\varepsilon_1<\varepsilon,\qquad \varepsilon_1>0.$$

For $|t|<\varepsilon_1$ we have

$$\widehat{K}\cap V(\eta\widetilde{f}-t)\subset L_t\subset\widetilde{L}. \tag{3.1}$$

Indeed, if $q\in V(\eta\widetilde{f}-t)\setminus L_t$, then there exists $g\in\mathcal{O}(V(\eta\widetilde{f}-t))$ such that $|g(q)|>1$ and $\|g\|_{L_t}<1/2M_t$. Let $\widetilde{g}$ be a holomorphic extension of g on $\widetilde{X}_{red}$ such that

$$\|\widetilde{g}\|_K=M_t\ \|g\|_{L_t}+1/3.$$

Then $\|\widetilde{g}\|_K<1$, $|\widetilde{g}(q)|>1$. Hence $q\,\bar{\in}\,\widehat{K}$.

From (3.1) it follows that $z_n\in\widehat{L}$ for sufficiently large n. This contradicts the choice of $\{z_n\}$.

The proposition is proved.

4. Proof of Theorem 1.

Let $\pi:\widetilde{X}\to X$ be a Stein morphism. By a covering lemma of Stehle [20] we find a Stein open covering $\mathcal{U}=\{V_j\}$ of X such that

(i) $\Omega_j = \bigcup_{1 \leqslant i \leqslant j} V_i$ is Stein for every $j \geqslant 1$

(ii) $\Omega_j \cap V_{j+1}$ is Runge in V_{j+1} for every $j \geqslant 1$.

(iii) $\widetilde{\mathcal{U}} = \{X'_j\}$, where $X'_j = \pi^{-1}(V_j)$ is a Stein open covering of $\widetilde{X}$. Since $\pi(\partial\pi^{-1}V) \subset \partial V$ for every open set $V \subset X$ it is easy to see that $\pi^{-1}(V)$ is Stein for every Stein open set $V \subset U \in \mathcal{U}$. Thus by (ii) we have $X_j \cap X'_{j+1}$ is Runge in X_{j+1}, where $X_j = \pi^{-1}(\Omega_j)$.

Let $\widetilde{\mathcal{U}}_1$ be a Stein open covering of $\widetilde{X}$ such that

$\widetilde{\mathcal{U}}_1 < \widetilde{\mathcal{U}}$, $\mathcal{U}_1$ forms a basis of open sets of $\widetilde{X}$ and

$$\widetilde{\mathcal{U}}_1 |_{X_{j+1}} = \widetilde{\mathcal{U}}_1 |_{X_j} \cup \widetilde{\mathcal{U}}_1 |_{X'_{j+1}} \quad \text{for every } j \geqslant 1$$

where $\widetilde{\mathcal{U}}_1 |_G = \{ U \in \widetilde{\mathcal{U}}_1 : U \subset G \}$.

Adapting an argument of Jeannane [8] we get the following

4.1. PROPOSITION. *Let $\widetilde{X}$ be as above and $\mathcal{S}$ a coherent analytic sheaf on X. Then*

(i) $H^q(\widetilde{X}, \mathcal{S}) = 0$ for every $q \geqslant 2$.

(ii) the boundary map $\delta : C^0(\widetilde{\mathcal{U}}_1, \mathcal{S}) \to Z^1(\widetilde{\mathcal{U}}_1, \mathcal{S})$ has dense image.

Proof of Theorem. a) From 5.1 and the hypothesis we get:

$$H^q(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0 \quad \text{for every } q \geqslant 1$$

and

$$H^q(\widetilde{X}, \widetilde{N}) = 0 \qquad \text{for every } q \geqslant 2.$$

Whence,

$$H^q(\widetilde{X}_{red}, \mathcal{O}_{\widetilde{X}_{red}}) = 0 \text{ for every } q \geqslant 1.$$

b) By a) we may assume that X is reduced. Assume that the theorem has been established for every subspace of $\widetilde{X}$ of dimension $<$ dim $\widetilde{X}$. To prove the Steiness of X, it suffices, by a theorem of Narasimhan [11], to show that every irreducible branch Z of $\widetilde{X}$ is Stein.

c) Let Z be a given irreducible branch of $\widetilde{X}$. From the exact sequence

$$0 \to J_Z \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_Z \to 0$$

it follows that

$$H^q(Z, \mathcal{O}_Z) = 0 \text{ for every } q \geqslant 1.$$

If $\pi(Z) = \{\star\}$, then Z is Stein, since π is a Stein morphism. Assume now that $\pi(Z) \neq \{\star\}$. Take $f \in \mathcal{O}(X)$ such that $f \mid \pi(Z) \neq$ const. Since Z is irreducible the sequence

$$0 \to \mathcal{O}_Z \xrightarrow{\widetilde{f}_t} \mathcal{O}_Z \to \mathcal{O}_{Z/(\widetilde{f}_t)} \to 0$$

where $\widetilde{f}_t = f\pi - t$, is exact for all $t \in C$. Whence, we have

$$H^q(Z \cap V(\widetilde{f}_t), \mathcal{O}_{Z/(\widetilde{f}_t)}) = 0 \text{ for every } q \geqslant 1.$$

Since $\pi \mid_{Z \cap V(\widetilde{f}_t)} : Z \cap V(\widetilde{f}_t) \to V(f_t)$ is a Stein morphism, from Proposition 4.1 we get

$$H^q(Z \cap V(\widetilde{f}_t), \mathcal{O}_{V(\widetilde{f}_t)}) = 0 \text{ for every } q \geqslant 1.$$

Thus by induction hypothesis, $Z \cap V(\widetilde{f}_t)$ is Stein for every $t \in C$. By Proposition 3.1 we then derive the Steiness of $\widetilde{X}$. The theorem is proved.

5. THE SHEAF $F \otimes_{\mathcal{O}(X)} \mathcal{O}$

First we recall some facts about the tensor product of Frechet modules. All algebras are assumed to be commutative with unit element. Let B be a Frechet algebra. By $\mathcal{C}(B)$ we denote the category of Frechet modules and continuous B-linear maps. A complex

$$\ldots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \ldots$$

in $\mathcal{C}(B)$ is called direct if for each n there exists a continuous C-linear map $h_n : M_n \to M_{n+1}$ such that $d_{n+1} h_n + h_{n-1} d_n = \text{id}$.

A Frechet B-module P is called free if it is isomorphic to $B \widehat{\otimes}_\pi A$, where $B \widehat{\otimes}_\pi A$ denotes the projective tensor product of B with A, a Frechet space, and the B-module structure on $B \widehat{\otimes}_\pi A$ is given by the formula.

$$b(u \otimes v) = bu \otimes v \text{ for } b \in B \text{ and } u \otimes v \in B \widehat{\underset{\pi}{\otimes}} A.$$

We say that P is D-projective if for every direct sequence $M \xrightarrow{g} M \to 0$ in $\mathcal{C}(B)$ the map $\widehat{g} : H_0M_B(P,M) \to H_0M_B(P,M'')$ induced by g is surjective.

Let $E \in \mathcal{C}(B)$. A complex

$$P(E) \; :... \to P_n \xrightarrow{d_n} P_{n-1} \to ... \to P_1 \xrightarrow{d_1} P_0 \to 0$$

in $\mathcal{C}(B)$ is called a D-projective resolution of E if.

(i) D_j is D-projective for every $j \geqslant 0$.

(ii) there exists a continuous B-linear map $\varepsilon : P_0 \to E$ such that $P(E) \xrightarrow{\varepsilon} E \to 0$ is exact. Moreover, if $(P)E \xrightarrow{\varepsilon} E \to 0$ is direct, then $P(E)$ is called a direct D-projective resolution of E.

Setting

$$P_n = \underbrace{B \widehat{\underset{\pi}{\otimes}} ... \widehat{\underset{\pi}{\otimes}} B}_{n+1} \widehat{\underset{\pi}{\otimes}} E$$

$$d_n(a_0 \otimes ... \otimes a_n \otimes u) = \sum_{j=1}^{n} (-1)^{j-1} a_0 \otimes ... \otimes a_{j-1} a_j \otimes \otimes$$

$$a_n \otimes u + (-1)^n a_0 \otimes ... \otimes a_n u$$

$$h_n(u) + 1 \otimes u, \; \varepsilon(a \otimes u) = au$$

we get a direct free resolution of F.

$$F(E) : ... \to B \widehat{\underset{\pi}{\otimes}} B \widehat{\underset{\pi}{\otimes}} E \to B \widehat{\underset{\pi}{\otimes}} E \xrightarrow{\varepsilon} E \to 0.$$

Hence for the covariant functor

$$E \widehat{\underset{B}{\otimes}} : \mathcal{C}(B) \to \mu(B) : M \to E \widehat{\underset{B}{\otimes}} M \overset{def}{=} E \widehat{\underset{\pi}{\otimes}} M / \mathrm{Imd}$$

where $\mu(B)$ denotes the category of B-modules and

$$d: E \widehat{\underset{\pi}{\otimes}} B \widehat{\underset{\pi}{\otimes}} M \to E \widehat{\underset{\pi}{\otimes}} M \; d(u \otimes a \otimes m), = au \otimes m - u \otimes am$$

we can construct the left derived functor

$$\widehat{\mathrm{Tor}}_q^B(E,.) : \mathcal{C}(B) \to \mu(B)$$

of $E \widehat{\otimes}_B$ by putting

$$\widehat{\mathrm{Tor}}_q^B(E, M) = H_q\ (F(E) \widehat{\otimes}_B M).$$

It is known [7] that

(i) $\widehat{\mathrm{Tor}}_q^B(E,.) = E \widehat{\otimes}_B$

(ii) $\widehat{\mathrm{Tor}}_q^B(E, F) = \mathrm{Tor}_q^B\ (B\ F,\ E)$

(iii) For every direct sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathcal{C}\,(B)$ there exists an exact homology sequence

$$\ldots - \widehat{\mathrm{Tor}}_1^B\,(\mathrm{E},\ M'') - E \widehat{\otimes}_B M' \to E \widehat{\otimes}_B M \to E \widehat{\otimes}_B M'' \to 0.$$

(iv) If *$P(E)$ is a nuclear D–projective resolution of E, i. e. P_j is nuclear for every $j \geqslant 0$, then* $\widehat{\mathrm{Tor}}_q^B\,(E,\ M) = H_q\ (P\,(E)\ \widehat{\otimes}_B \mathrm{M})$. Now let X be a complex space having a countable topology and F a Frechet B-module. We denote by $F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}$ the sheaf on X given by the formula:

$$U \to F \widehat{\otimes}_{\mathcal{O}\,(X)} (U) \quad \text{for every open subset } U \text{ of } X.$$

5.1. PROPOSITION. *Let $\mathcal{S}$ be an analytic Frechet sheaf on X, Suppose that for every relatively compact Stein open set $U \subset X$ there exists a Frechet $\mathcal{O}\,(U)$-module $F\,(U)$ such that*

$$(5.1)\ \mathcal{S}\,|_U \cong F\,(U) \widehat{\otimes}_{\mathcal{O}\,(U)} \mathcal{O} \ \text{and}\ \widehat{\mathrm{Tor}}_1^{\mathcal{O}(U)}\,(F\,(U),\ \mathcal{O}(\mathrm{V})) = 0$$

for all Stein open subsets $V \subset U$.
Then for every Stein open subset $G \subset X$ there exists a Frechet $\mathcal{O}\,(G)$. module $E(G)$ such that

$$(5.2)\ \mathcal{S}\,|_G = F\,(G \otimes_{\mathcal{O}(G)} \mathcal{O} \ \text{and}\ \widehat{\mathrm{Tor}}^{\mathcal{O}(G)}\ F(G),\ \mathcal{O}\ (V)) = 0$$

for all relatively compact Stein open subsets $V \subset G$.

We need the following

5.1. LEMMA. (Mittag-Leffer [6]. *Let*

$$0 \to \{\,E'_j\,\} \to \{\,E_j\,\} \to \{\,E''_j\,\} \to 0$$

be an exact sequence of projective systems of Frechet space such that the canonical map $E'_{j+1} \to E'_j$ has dense image. Then the sequence

$$0 \longrightarrow \varprojlim E'_j \longrightarrow \varprojlim E_j \longrightarrow \varprojlim E''_j \longrightarrow 0$$

is exact.

From the definition of the projective tensor product of Frechet spaces we get the following

5.2. LEMMA. *Let $\{E_j\}$ and $\{F_j\}$ be projective systems of Frechet spaces such that the maps $E_{j+1} \to E_j$ and $F_{j+1} \to F_j$ have dense images. Then*

$$\varprojlim (E_j \widehat{\otimes}_{\pi} F_j) \cong (\varprojlim E_j) \widehat{\otimes}_{\pi} (\varprojlim F_j).$$

Proof of Proposition 5.1. a) We first show that for every relatively compact Stein set $U \subset X$ and for every Stein open set $V \subset U$ the following holds

$$\text{(5.3)} \qquad H^0(V, \mathcal{S}) \cong H^0(U, \mathcal{S}) \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}(V).$$

Obviously (5.3) holds in the case where $F(U)$ is free. In the general case, consider the direct sequence

$$\text{(5.4)} \quad 0 \to F_i \to P_{i-1} \to F_{i-1} \to 0$$

$i = 1, 2$, where $F_0 = F(U)$, and P_0 and P_1 are free Frechet $\mathcal{O}(U)$ — modules. From (5.1) and the exactness of the homology sequence associated with (5.4) we get the exact sequence

$$\text{(5.5)} \qquad 0 \to F_i \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O} \to P_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O} \to F_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O} \to 0$$

Since $H^1(U, F_i \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}) = 0$ (Proposition 5.2) we obtain the commutative and exact diagram:

$$\begin{array}{ccccccc}
F_i \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}(V) & \to & P_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}(V) & \to & F_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}(V) & \to & 0 \\
\downarrow & & \| \wr & & \downarrow & & \\
H^0(V, F_i \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}) & \to & H^0(V, P_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}) & \to & H^0(V, F_{i-1} \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}) & \to & 0
\end{array}$$

Therefore $H^0(V, \mathcal{S}) \cong F_0 \widehat{\otimes}_{\mathcal{O}(U)} \mathcal{O}(V)$.

b) We write $G = \bigcup_{n=1}^{\infty} G_n$, where $\{G_n\}$ is an increasing sequence of relatively compact holomorphically convex domain in G with $V = G_1$. For each n consider the **canonical** free resolution of $\mathcal{O}(G_n)$— and $\mathcal{O}(G)$— modules $\mathcal{O}(V)$.

$$\ldots \xrightarrow{d_1} \mathcal{O}(G) \widehat{\otimes}_\pi \mathcal{O}(V) \to \mathcal{O}(V) \to 0$$

$$\ldots \xrightarrow{d_1^n} \mathcal{O}(G_n) \widehat{\otimes}_\pi \mathcal{O}(V) \to \mathcal{O}(V) \to 0.$$

By a) $\widehat{\mathrm{Tor}}_1^{\mathcal{O}(G_n)} (H^0(G_n, \mathcal{S}), \mathcal{O}(V)) = 0$ and the restriction maps $H^0(G_{n+1}, \mathcal{S}) \to$ $\to H^0(G_n, \mathcal{S})$ have dense images. Hence, applying Lemmas 5.1 and 5.2 to the exact sequence of projective systems

$$\{ \mathcal{O}(G_n) \widehat{\otimes}_\pi \mathcal{O}(G_n) \widehat{\otimes}_\pi \mathcal{O}\, G_n) \widehat{\otimes}_\pi H^0(G_n, \mathcal{S}) \} \to \{ \mathcal{O}(G_n) \widehat{\otimes}_\pi \mathcal{O}(G_n) \widehat{\otimes}_\pi H^0(G_n, \mathcal{S}) \}$$

$$\to \{ \mathcal{O}(G_n) \widehat{\otimes}_\pi H^0(G_n, \mathcal{S}) \} \to H^0(V, \mathcal{S}) \to 0$$

we obtain (5.1).

5.2. PROPOSITION. *Let $\mathcal{S}$ be a coherent analytic sheaf on a Stein space X. Then:*

(5.6) $\mathcal{S} \cong H^0(X, \mathcal{S}) \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}$ *and* $\widehat{\mathrm{Tor}}_1^{(X)}(H^0(X, \mathcal{S}), \mathcal{O}(V)) = 0$

for every relatively compact Stein open set $V \subset X$.

Proof. By Proposition 5.1 it suffices to show that (5.1) holds for every relatively compact Stein open set $U \subset X$ and for all Stein open subset $V \subset U$. By Cartan theorem A there exists an exact sequence

(5.7) $\quad 0 \longrightarrow \mathcal{O}^{p_l} \longrightarrow \ldots \longrightarrow \mathcal{O}^{p_0} \longrightarrow \mathcal{S} \longrightarrow 0$

on U.

The by Cartan theorem B and by induction on l we obtain (5.1).

5. 3. PROPOSITION. *Let $\widetilde{X}$ be a Riemann domain over a complex space X such that $H^q(\widehat{X}, \mathcal{O}) = 0$ for every $q \geqslant 1$. Let F be a Frechet $\mathcal{O}(X)$—module such that $\widehat{Tor}_1^{(X)}(F, \mathcal{O}(V)) = 0$ for every sufficiently small Stein open set $V \subset X$. Then*

$$H^q(\tilde{X}, F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\tilde{X}}) = 0 \;\; \textit{for every } q \geqslant 1 .$$

Proof. a) We first assume that $F \cong \mathcal{O}(X) \widehat{\otimes}_\pi A$, where A is a Frechet space.

Then $F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\hat{X}} \cong \mathcal{O}_X \widehat{\otimes}_\varepsilon A$. Let U be a Stein open covering of X. Then U is a

Leray covering for $\mathcal{O}_{\widetilde{X}}$ and $\mathcal{O}_{\widetilde{X}} \widehat{\otimes}_{\varepsilon} A$ [2]. Hence, by hypothesis the sequence

$$0 \longrightarrow \mathcal{O}(\widetilde{X}) \longrightarrow C^0(U, \mathcal{O}_{\widetilde{X}}) \longrightarrow \dots$$

is exact. Therefore, the sequence

$$0 \longrightarrow \mathcal{O}(\widetilde{X}) \widehat{\otimes}_{\varepsilon} A \longrightarrow C^0(U, \mathcal{O}_{\widetilde{X}} \widehat{\otimes}_{\varepsilon} A) \longrightarrow \dots$$

is also exact. Hence $H^q(\widetilde{X}, F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\widetilde{X}}) = 0$ for all $q \geqslant 1$.

b) In the general case, consider the direct sequences (5.4), where $i=1,2,\dots$. By hypothesis the sequences (5.5) are exact. Hence, by a) we have:

$$H^q(\widetilde{X}, F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}) \cong \dots \cong H^{q+p}(\widetilde{X}, F \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}) = 0$$

for every $q \geqslant 1$ and for p sufficiently large.

6. Proof of Theorem 2.

a) Let $\widetilde{N}$ and N denote the nilradicals of $\mathcal{O}_{\widetilde{X}}$ and $\mathcal{O}_X$ respectively. By Proposition 5.4 we have: $N \cong N(X) \widehat{\otimes}_{\mathcal{O}(X)} N$ and $\widehat{\mathrm{Tor}}_1^{(X)}(N(X), \mathcal{O}(V)) = 0$ for every relatively compact Stein open set $V \subset X$. Since $\pi : \widetilde{X} \longrightarrow X$ is locally isomorphic it follows that

$$\widetilde{N}_x \cong N_{\pi x} \cong (N(X) \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_X)_{\pi x} \cong (N(X) \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\widetilde{X}})_x .$$

for every $x \in \widehat{X}$.

Hence $\widetilde{N} \cong N(X) \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\widetilde{X}}$. By Proposition 5.5 we have

$$H^q(\widetilde{X}, \widetilde{N}) = 0 \text{ for every } q \geqslant 1 .$$

This yields $H^q(\widetilde{X}_{red}, \mathcal{O}_{\widetilde{X}_{red}}) = 0$ for every $q \geqslant 1$.

b) Take a holomorphic function f on X which is not constant on every irreducible branch of X. Then $\widehat{f} = f\pi$ is also nonconstant on every irreducible branch of X. Consider the Riemann domain $\pi : V(\widetilde{f-t}) \to V(f-t)$. As in a) we have

$$\mathcal{O}_{V(\widetilde{f-t})} \cong \mathcal{O}_{V(f-t)} \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}_{\widetilde{X}_{red}}$$

Hence $H^q(V(\widetilde{f}-t), \mathcal{O}_{V(\widetilde{f}-t)}) = 0$ for every $q \geqslant 1$. Thus by induction on dim $\widetilde{X}$ it follows that $V(\widetilde{f}-t)$ is Stein for every $t \in \mathbf{C}$. From Proposition 3.1 we derive the Steiness of X.

The proof is complete.

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