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APPLICATION OF THE HORMANDER – METHOD TO PROVE AN EXTENSION – THEOREM FOR VECTOR – FIELDS

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Let G be a domain in R^3 and K be a compact subset of G such that $G \setminus K$ is connected. Consider the system of partial differential equations:

$$(1) \begin{cases} \operatorname{div} \vec{U} = 0 \\ \operatorname{rot} \vec{U} = 0 \end{cases}$$

where $\vec{U} = \vec{U}(x) = \{u_1(x), u_2(x), u_3(x)\}$ is a vector — function of $x = (x_1, x_2, x_3) \in R^3$. This system is called the Riesz — system in R^3 and has applications in mathematical physics.

In the sequel we shall be concerned with the following extension problems for the system (1):

Let $\vec{u} = (u_1, u_2, u_3)$ be a given continuous solution of (1) in $G \setminus K$. Under which conditions can $\vec{u}$ be extended to a solution over G?

For the case where $G = G_1 \times G_2 \times G_3$ is a polycylindrical domain in R^3. This problem has been solved with the help of the integral formula — method (see [1]). The aim of this paper is to show that in the general case it be solved by means of the Hormander method (see [5])

Let $\varphi \in \mu_0^\infty(G)$ be a function such that $\varphi = 1$ in a neighbourhood of K (the existence of such a function has been proved in [6, 7]. Denote

$$(2)\ \vec{u}^0 = \begin{cases} (1-\varphi)\vec{u} & \text{in } G \setminus K \\ 0 & \text{in } K. \end{cases}$$

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We wish to find a vector — function $\vec{v} \in C_0^2(R^3)$ such that the vector — function

$$(3) \qquad \vec{\tilde{u}} = \vec{u} - \vec{v}$$

Solves (1) and is an extension of $\vec{u}$ to the whole of G. The vector — function $\vec{\hat{u}}$ is a solution of (1) if

$$(4) \quad \operatorname{div} \vec{\tilde{u}} = \operatorname{div} \vec{u^o} - \operatorname{div} \vec{v} = 0$$

$$\operatorname{rot} \vec{\tilde{u}} = \operatorname{rot} \vec{u^o} - \operatorname{rot} \vec{v} = 0$$

or

$$(5) \quad \operatorname{div} \vec{v} = f$$

$$\operatorname{rot} \vec{v} = \vec{F}$$

Where

$$(6) \quad f = \begin{cases} \operatorname{div} \vec{u^o} & \text{in } G \\ 0 & \text{otherwise,} \end{cases}$$

$$(7) \quad \vec{F} = \begin{cases} \operatorname{rot} \vec{u^o} & \text{in } G \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 1 : *Assume that the system (5) has a solution v in the whole of R^3, such that $\vec{v} \in C^2(R^3)$ and*

(8) $\vec{v} \equiv 0$ for all sufficiently large $|x_3|$.

Then any given solution $\vec{u}$ of (1) in $G \setminus K$ can be extended to a solution over G.

Proof: By assumption the vector — function $\vec{v}$ is a solution of the Riesz — system (see [2]) in C (Supp φ) $= R^3 \setminus$ Supp φ. From (8) and the uniqueness theorem for the Riesz-system we get

(9) $\vec{v} \equiv 0$ in the unbounded connected components of the complement of Supp φ.

Since the boundary of this set belongs to $G \setminus K$, there exists an open set $\sigma \neq \phi$ such that $\sigma \subset G \setminus K$ and

$$(10) \quad \vec{v} = 0, \; \vec{u} = \vec{u^o} \text{ in } \sigma$$

Clearly, the vector — function u defined by (3) is a solution of (1). In view of (10) we get

$$(11) \qquad \vec{\tilde{u}} = \vec{u} \text{ in } \sigma .$$

On the other hand, since $G \setminus K$ is connected, it follows from (11) and the uniqueness theorem for the solutions of (1) that

$$\vec{\tilde{u}} = \vec{u} \text{ in } G \setminus K.$$

This means that $\vec{\tilde{u}}$ is the extension of $\vec{u}$ to the whole of G.

q. e. d.

The converse of Lemma is also true:

LEMMA 2: *Assume that $\vec{u}$ can be extended to a solution of (1) then the system (5) has a solution $\vec{v}$ such that condition (8) is fulfilled for all sufficiently large $|x_3|$.*

Proof: Let $\vec{\tilde{u}}$ be the extension of $\vec{u}$ to the whole of G,

Denote:

$$(12) \qquad \vec{v} = \begin{cases} \vec{u^0} - \vec{\tilde{u}} & \text{in } G \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\vec{v} \in C^2$, $v = 0$ for all sufficiently large $|x_3|$ and $\vec{v}$ is a solution of (5). Q.e.d.

Now consider the system (5). From a theorem of potential theory (see ([3]) p. 161) it follows that the system (5) is solvable if

$$(13) \qquad \operatorname{div} \vec{F} = 0$$

This condition is fulfilled if $\vec{F}$ is given by (7). Hence we get from [4] (chapters 3), and 7, the following lemma

LEMMA 3: *The system (5) is always solvable, and the vector-function*

$$(14) \qquad \vec{v_0} = -\operatorname{grad} \int_{R^3} \frac{f(\xi)}{4\pi r(\xi,x)}\, d\xi + \operatorname{rot} \int_{R^3} \frac{\vec{F}(\xi)}{4\pi r(\xi,x)}\, d\xi$$

is a particular solution of this system, where $r(\xi, x)$ is the distance from x to ξ in R^3.

Remark 1: Let $\vec{v_0}$ be a particular solution of (5) defined by (14) and $\vec{v'}$ be a solution of the Riesz — system (1) in the whole of R^3. Then

$$\vec{v} = \vec{v'} + \vec{v_0}$$

is also always a solution of (5).

If we assume that a solution $\vec{v'}$ of the Riesz — system (1) exists such that

(15) $\vec{v}_0 = -\vec{v'}$ for all sufficiently large $| x_3 |$ then this solution $\vec{v'}$ satisfies the condition (8) (the existence of such a solution $\vec{v'}$ is not obvious.

Consider now the following

EXTENSION — PROBLEM

1) Special : Let T be a domain in R^3 of the form

(16) $$T = \{ x = (x_1 , x_2 , x_3) \in R^3 / | x_3 | \leqslant c_0 \}$$

where $c_0 > 0$ is sufficiently large. Can every solution $\vec{v}_0$ of the Riesz— system (1) in $T := R^3 \setminus T$

be extended to a solution over R^3 ?

It turns out that

THEOREM: *If the special extension — problem is solvable, then the above stated extension — problem is solvable, i. e every solution $\vec{u}$ of (1) in $G \setminus K$ can be extended to a solution over G*

Proof: Given a solution $\vec{u}$ of (1) in $G \setminus K$, we can define f and $\vec{F}$ from the above mentioned function.

Further we choose $c_0 > 0$, such that

$$f = 0 \text{ and } \vec{F} = 0 \text{ in } {}^cT.$$

then the system (5) is the Riesz — system in ${}^cT$ and the vector — function $-\vec{v}_0$ is a solution of (1) in ${}^cT$, where $\vec{v}_0$ is the vector —function defined by (14). From the assumption it follows that $-\vec{v}_0$ can be extended to a solution of the Riesz – system (1) over T, such that

$$\vec{v'} = -\vec{v}_0 \text{ in } {}^cT \text{ or}$$

17) $$\vec{v}_0 = -\vec{v'} \text{ in } {}^cT$$

Now consider the vector — function

18) $$\vec{v} = \vec{v'} + \vec{v}^0 .$$

It follows immediately from (16) and (18) that $\vec{v}$ is a solution of (5) satisfying

$$\vec{v} \equiv 0 \text{ in } {}^cT$$

or $$\vec{v} \equiv 0 \text{ for sufficiently large } | x_3 | .$$

Therefore, by Lemma 1 $\vec{u}$ can be extended to a solution of (1) over G.Q.e.d.

Thus, the above extension — problem can be reduced to the special extension—problem 1. If this special extension — problem 1 has a solution then a solution $\vec{v'}$ of the Riesz — system exists, such that condition (15) is fulfilled.

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