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A FINITE METHOD FOR GLOBALLY MINIMIZING CONCAVE FUNCTIONS OVER UNBOUNDED POLYHEDRAL CONVEX SETS AND ITS APPLICATIONS

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I. INTRODUCTION

We shall be concerned with the following concave programming problem:

$$\text{Globally minimize } f(x), \text{ subject to } x \in D, \qquad (1)$$

where $f: R^n \longrightarrow R$ is a real-value function, concave and finite on R^n, D is a polyhedral, not necessarily bounded, convex set in R^n. Usually. D is given explicitly by a finite system of linear equalities and (or) inequalities of the form

$$(a^i, x) + b_i = 0, \quad i \in I^o$$

$$(a^i, x) + b_i \leqslant 0, \quad i \in I^-$$

with a^i being n—dimensional vectors, b_i real numbers and I^o, I^- finite sets of indices.

For the case where the constraint set D is bounded, i.e. D is a polytope, this problem was first studied by *H.* Tuy in [8] and subsequently by a number of researchers. Recently, some authors have also been interested in the case where D may be unbounded (see [1], [9], [14]) and, as far as we know, the most general problem of globally minimizing a concave function over an arbitrary closed, convex, set was investigated for the first time in [13] (see also [10], [11]). The algorithm presented in [9], [14] are of the branch and bound type and proceed according to the cone splitting scheme and the cone bisection scheme worked out in Tuy [8] and in Thoai and Tuy [6] respectively. However, both these algorithms are in general infinite (though surely convergent) and require solving a linear subproblem at each step. The recent algorithm of V.T.Ban [1] which is a further development of the basic ideas proposed in [6], [8] — solves the problem in finitely many steps by exploiting the linear structure of the constraint set. However, in the applications we often have to treat a sequence of linearly constrained concave minimization problems each of which differs from the previous one only by one additional constraint. This occurs for instance in the context of outer approximation procedures or decomposition schemes. It is therefore of interest to have an algorithm which could take advantage of this property. One such algorithm was given in [4] for the case of bounded polyhedral convex constraint sets.

The purpose of the present paper is to extend the results in [4] to the general case of arbitrary polyhedral convex constraint sets.

The paper consists of 6 sections. After the Introduction, we shall present in Section 2 a practical and relatively simple technique for determining the vertices and the extreme directions of a polyhedral convex set that is obtained by adding a new linear constraint to a polyhedral convex set with known vertices and extreme directions. This technique can be regarded as a further development of the technique given in [4] for the case where the polyhedral convex set is bounded. Then, in Section *3* we shall develop a finite algorithm for globally minimizing a concave function over an arbitrary polyhedral convex set. This algorithm proceeds according to the same scheme as that presented in [4], namely: starting from a relaxed problem whose constraint set is a simple polyhedral convex set (e. g. the non negative orthant), we gradually add the constraints one by one at each iteration until an optimal solution of the current relaxed problem becomes feasible (and hence, solves the original problem). Our algorithm is quite different from that of V. T. Ban [1] and, as will be shown in the sequel, it offers the advantage of being particularly suitable for solving a sequence of problems each of which differs from the previous one just by one additional constraint. In Section *4* we shall discuss several particular features of applications of the present algorithm to the bilinear programming, the linear complementarity problems and concave minimization problems with special structure. In Section *5* a two—dimensional example is presented to illustrate how the algorithm works in practice. Finally, in Section *6* some preliminary computational experience is reported.

2. AN AUXILIARY PROBLEM

We begin with the following auxiliary problem: given a polyhedral convex set whose vertices and extreme directions are known, how to determine the vertices and extreme directions of a polyhedral convex set obtained from the previous one by adjoining just a new constraint? The main results, which will play a basic role in the solution of problem (1), may also have an independent interest.

To state this auxiliary problem precisely, let there be given a polyhedral convex set M defined by a system of linear inequality constraints

$$g_i(x) = (a^i, x) + b_i \leqslant 0, \quad i = 1, \dots, m, \tag{2}$$

where a^i are n—dimensional vectors, b_i are real numbers, $m \geqslant n$. Suppose we already know the set U of vertices and the set V of extreme directions of M, so that

$$M = \text{co } U + \text{cone } V.$$

Suppose that $U \neq \varnothing$ (M has at least one vertex), while V may be empty (M is a polytope). Consider an affine function

$$h(x) = (c, x) + d$$

with c being an n — dimensional non-zero vector, d a real number. Define the polyhedral convex set

$$M' = M \cap \{ x \in R^n : \; h(x) \leqslant 0 \}. \tag{3}$$

We wish to compute the set U' of vertices and the set V' of extreme directions of M'.

Let us denote :

$$U^- = \{ u \in U : h(u) < 0 \}, \; U^+ = \{ u \in U : h(u) > 0 \} \tag{4}$$

$$V^- = \{ v \in V : (c,v) < 0 \}, \; V^+ = \{ v \in V : (c,v) > 0 \} \tag{5}$$

$$H = \{ x \in R^n : h(x) = 0 \}.$$

PROPOSITION 1. *If $U^+ = V^+ = \phi$ then $M' = M$, i.e. $U' = U$ and $V' = V$.*

Proof. From the hypotheses $h(u) \leqslant 0$ for all $u \in U$ and $(c,v) \leqslant 0$ for all $v \in V$. Every $x \in M$ can be expressed in the form

$$x = \sum_{u \in U} \alpha_u u + \sum_{v \in V} \beta_v v$$

with $\alpha_u \geqslant 0$, $\beta_v \geqslant 0$ and $\Sigma \alpha_u = 1$, hence $h(x) = \Sigma \alpha_u h(u) + \Sigma \beta_v (c,v) \leqslant 0$. This means that $x \in M'$ and therefore, $M \subset M'$. The converse inclusion is obvious. Thus $M' = M$, as was to be proved.

PROPOSITION 2. *Suppose $U^- = V^- = \phi$.*

a) If $U^+ = U$ then $M' = \phi$, i.e. $U' = V' = \phi$.

b) Otherwise, $U' = U \setminus U^+$ and $V' = V \setminus V^+$.

Proof. a) $U^+ = U$ and $V^- = \phi$ mean that $h(u) > 0$ for all $u \in U$ and $(c,v) \geqslant 0$ for all $v \in V$. Therefore, we have for every $x \in M$

$$h(x) = \sum_{u \in U} \alpha_u h(u) + \sum_{v \in V} \beta_v (c,v) > 0$$

(note that there exists at least one $\alpha_u > 0$). So, $M' = \phi$.

b) $U^- = V^- = \phi$ mean that $h(u) \geqslant 0$ for all $u \in U$ and $(c,v) \geqslant 0$ for all $v \in V$. Therefore, $h(x) \geqslant 0$ for all $x \in M$ and hence, $M \subset \{x : h(x) \geqslant 0\}$. So we have

$$M' = M \cap \{x : h(x) = 0\} = M \cap H \neq \phi$$

(since $U \setminus (U^+ \cup U^-) = U \setminus U^+ \neq \phi$). This shows that M' is a face of M. Therefore, each vertex (extreme direction) of M' is also a vertex (an extreme direction) of M. Thus $U' \subset U \setminus U^+$ and $V' \subset V \setminus V^+$. The converse inclusions being obvious, the Proposition is proved.

PROPOSITION 3. *Suppose* $\dot{U}^+ \cup \dot{V}^+ \neq \phi$ *and* $U^- \cup \dot{V}^- \neq \phi$.

a) $U' \cap U = U \setminus U^+$.

b) any vertex $w \in U' \setminus U$ *must be the intersection of the hyperplane H with an edge of M connecting a vertex* $u^- \in U^-$ *with a vertex* $u^+ \in U^+$, *or emanating from a vertex* $u \in U^-$ *(or* U^+*) in a direction* $v \in V^+$ *(or* V^-*, resp.).*

Proof. a) For any $u \in U^+$ we have $u \notin M'$ and hence, $u \notin U'$ (since $h(u) > 0$). On the other hand, eervy $u \in U \setminus U^+$ belongs to M' (since $h(u) \leqslant 0$), hence is a vertex of M'. So $U \setminus U^+ = U' \cap U$ (this relation holds even if $U \setminus U^+ = \phi$).

b) Let $w \in U' \setminus U$. Denote by $F(w)$ the smallest face of M containing w. Since $w \notin U$ we must have $F(w) \neq \{w\}$ and hence, dim $F(w) \geqslant 1$. If dim $F(w) > 1$ then $F(w)$ would have in common with H a line segment containing w in its relative interior. This would conflict with w being a vertex of M'. Therefore, dim $F(w) = 1$ and $F(w)$ is an edge (bounded or unbounded) of M. Let us distinguish two cases:

CASE 1: $F(w)$ is a bounded edge of M. For example, $F(w) = [u,v]$ with $u.\ v \in U$ Then $w = tu + (1-t)v$ for some $t : 0 < t < 1$. This implies $h(u) \neq 0$, $h(v) \neq 0$. From the relation

$$h(w) = t.h(u) + (1-t).h(v) = 0$$

it follows that $h(u).h(v) < 0$.

CASE 2: $F(w)$ is an unbounded edge of M. For example, $F(w) = \{u + \theta v : \theta \geqslant 0\}$ with $u \in U$, $v \in V$. Then $w = u + tv$ for some $t > 0$. This implies $h(u) \neq 0$, $(c, v) \neq 0$. Since

$$h(w) = h(u) + t.(c, v) = 0$$

we must have $h(u).(c, v) < 0$, completing the proof.

PROPOSITION 4. *Under the same hypotheses as in Proposition 3:*

a) $V' \cap V = V \setminus V^+$,

b) any extreme direction $v \in V' \setminus V$ *satisfies* $(c, v) = 0$ *and is of the form* $v = \lambda p + \mu q$ *with* $\lambda, \mu > 0$, $(p, q) \in V^- \times V^+$ *defining a two-dimensional face of the recession cone of M.*

Proof. a) Let K and K' denote the recession cone of M and M' respectively. It follows from (2), (3) that

$$K = \text{cone } V = \{x \in R^n :\ (a^i, x) \leqslant 0,\ i = 1,\dots, m\},$$

$$K' = \text{cone } V' = K \cap \{x \in R^n : (c, x) \leqslant 0\}.$$

This shows that $V \setminus V^+ = V' \cap V$.

b) Let now $v \in V' \setminus V$. Since $v \in V'$ there are among the constraints defining K' $(n\text{-}1)$ linearly independent constraints binding for v. Further, since $v \notin V$, one of these n-1 binding constraints must be $(c, v) = 0$. Let J denote the index set of the remaining n-2 binding constraints: $J \subset \{1,..., m\}$, $|J| = n$-2. Then

$$Z(v) = \{x \in K : (a^j, x) = 0, \ j \in J\}$$

is the smallest face of K containing the ray $\{tv : t \geqslant 0\}$. Certainly, $Z(v) \neq \{tv : t \geqslant 0\}$, for otherwise v would be an extreme direction of M, i. e. $v \in V$. Therefore, dim $Z(v) = 2$ and hence, $Z(v)$ is a two-dimensional cone defined, for instance, by two extreme directions p and q belonging to V. We thus have

$$v = \lambda p + \mu q \text{ with some } \lambda, \mu > 0.$$

This implies $(c, p) \neq 0$, $(c, q) \neq 0$. From the relation

$$(c, v) = \lambda(c, p) + \mu(c, q) = 0$$

it follows that $(c, p) \cdot (c, q) < 0$, completing the proof.

On the basic of Propositions 3, 4 one can determine the new vertices of M' (i.e. the members of $U' \setminus U$) and the new extreme directions of M' (i.e. the members of $V' \setminus V$) in the case $U^+ \cup V^+ \neq \phi$ and $U^- \cup V^- \neq \phi$, as follows:

RULE A (for finding the new vertices of M'):

a) For any pair $(u^-, u^+) \in U^- \times U^+$ determine the point

$$w = tu^- + (1 - t)u^+, \text{ where } t = h(u^+)/(h(u^+) - h(u^-)).$$

b) For any pair $(u, v) \in \{U^- \times V^+\} \cup \{U^+ \times V^-\}$ determine the point

$$w = u + tv, \text{ where } t = -h(u)/(c, v).$$

For each w defined by a) or b) denote by $I(w)$ the index set of the constraints of form (2) that define M and are binding for w:

$$I(w) = \{i : g_i(w) = 0, \ i = 1,..., m\}.$$

It can be seen that in the case a)

$$I(w) = \{i : g_i(u^-) = g_i(u^+) = 0, \ i = 1,..., m\}$$

and in the case b)

$$I(w) = \{i : g_i(u) = (a^i, v) = 0, \ i = 1,..., m\}.$$

Then, as can easily be verified

$$F(w) = \{x \in M : g_i(x) = 0, \ i \in I(w)\}$$

is the smallest face of M containing w. Therefore, if $|I(w)| < n - 1$ or if there exists a vertex $z \in U \setminus \{u^-, u^+\}$ (in the case a)) or $z \in U / \{u\}$ (in the case b)), such that $g_i(z) = 0$ for all $i \in I(w)$ (i.e. $z \in F(w) \cap U$), then dim $F(w) > 1$ and hence, by Proposition 3, w cannot be a vertex of M': $w \notin U'$. Otherwise, dim $F(w) = 1$ and w is a vertex of M': $w \in U'$.

RULE B (for finding the new extreme directions of M'):

For any pair $(p, q) \in V^- \times V^+$ determine the point $v = (c, q)p - (c, p)q$. It is easily seen that $v \in K$ and $(c, v) = 0$. Let $J(v) = \{j : (a^j, v) = 0,$

$j = 1, \ldots, m\}$. Clearly $J(v) = \{j: (a^j, p) = (a^j, q) = 0, j = 1, \ldots, m\}$. Then, as can easily be verified,

$$Z(v) = \{x \in K: (a^j, x) = 0, j \in J(v)\}$$

is the smallest face of K containing the ray $\{tv: t \geqslant 0\}$. Therefore, if $|J(v)| < n-2$ or if there is at least one $z \in V \setminus \{p, q\}$ such that $(a^j, z) = 0$ for all $j \in J(v)$ (i.e. $z \in Z(v) \cap K$), then dim $Z(v) > 2$ and hence, by Proposition 4, v cannot be an extreme direction of M': $v \notin V'$. Otherwise, dim $Z(v) = 2$ and v is an extreme direction of M': $v \in V'$.

We have thus established the rules for determining the vertices and the extreme directions of a polyhedral convex set M' of the form (3), once the vertices and the extreme directions of the polyhedral convex set M are known.

The above results can also be applied to the case where M' is defined by

$$M' = M \cap \{x \in R^n: h(x) = (c, x) + d = 0\}, \qquad (6)$$

i.e. M' is obtained from M by adding a linear equality constraint. Actually, since one equality constraint is equivalent to a system of two opposite inequality constraints, we can draw from Propositions 1 — 4 the following.

COROLLARY 1. *Let there be given a polyhedral convex set M with vertex set U and extreme direction set V. Let M' be defined by (6), and let U' and V' be the vertex set and the extreme direction set of M' respectively. Let U^-, U^+, V^-, V^+ be as before (see (4), (5)).*

a) Suppose $U^+ = V^+ = \phi$. If $U^- = U$ then $M' = \phi$, i.e. $U' = V' = \phi$. Otherwise, $U' = U \setminus U^-$, $V' = V \setminus V^-$.

b) Suppose $U^- = V^- = \phi$. If $U^+ = U$ then $M' = \phi$, i. e. $U' = V' = \phi$. Otherwise, $U' = U \setminus U^+$, $V' = V \setminus V^+$.

c) Suppose $U^+ \cup V^+ \neq \phi$ and $U^- \cup V^- \neq \phi$. Then $U' \cap U = U \setminus \{U^+ \cup U^-\}$, $V' \cap V = V \setminus \{V^+ \cup V^-\}$. Furthermore, any vertex $w \in U' \setminus U$ must be the intersection of the hyperplane $h(x) = 0$ with a bounded edge $[u, u']$ of M, such that $h(u).h(u') < 0$, or an unbounded edge $\{u + \theta v: \theta \geqslant 0\}$ of M such that $h(u).(c, v) < 0$. Any extreme direction $v \in V' \setminus V$ satisfies $(c, v) = 0$ and is of the form $v = \lambda p + \mu q$ with $\lambda, \mu > 0$, $(p, q) \in V^- \times V^+$ defining a two-dimensional face of the recession cone of M.

In the case c) the method for determining the new vertices of M' (i.e. the members of $U' \setminus U$) and the new extreme directions of M' (i.e. the members of $V' \setminus V$) is exactly the same as in the case where M' is of the form (3).

Furthermore, if M is a polytope (i.e. $V = \phi$) then, of course, M' is a polytope too and the determination of the vertex set U' of M' is a relatively easy task. Namely, we get

COROLLARY 2 (*Inequality constraint case*). *Let M be a polytope with vertex set U and let M' be defined by (3). Let U' be the vertex set of M' and let U^-, U^+ be defined by (4):*

a) If $U^+ = \phi$ then $M' = M$, i.e. $U' = U$.

b) If $U^- = \phi$ then $M' = \phi$, i.e. $U' = \phi$, when $U^+ = U$ and $U' = U \setminus U^+$ when $U^+ \neq U$.

c) If $U^+ \neq \phi$, $U^- \neq \phi$ then $U' \cap U = U \setminus U^+$ and any vertex $w \in U' \setminus U$ must be the intersection of the hyperplane $h(x) = 0$ with some edge of M connecting a vertex $u \in U^-$ with a vertex $v \in U^+$.

We thus recover in this special case the results of [4].

COROLLARY 3. *(Equality constraint case). Let M be a polytope with vertex set U and let M' be defined by (6). Let U' be the vertex set of M' and let U^-, U^+ be defined by (4):*

a) Suppose $U^+ = \phi$. If $U^- = U$ then $M' = U' = \phi$. Otherwise, $U' = U \setminus U^-$.

b) Suppose $U^- = \phi$. If $U^+ = U$ then $M' = U' = \phi$. Otherwise, $U' = U \setminus U^+$.

c) If $U^+ \neq \phi$ and $U^- \neq \phi$ then $U' \cap U = U \setminus \{U^+ \cup U^-\}$ and any vertex $w \in U' \setminus U$ must be the intersection of the hyperplane $h(x) = 0$ with some edge of M connecting a vertex $u \in U^-$ with a vertex $v \in U^+$.

In the case c) of Corollaries 2,3 to determine the new vertices of M' (i.e. the members of $U' \setminus U$) one can apply Rule A as before (but $w = tu + (1-t)v$ with $u \in U^-$, $v \in U^+$ *should be examined*).

Remark 1. It can easily be verified that the above results still hold even if M is given by a finite mixed system of linear equality and inequality constraints.

Remark 2. When M is a simplex the intersection of the hyperplane $h(x) = 0$ with any edge of M connecting a vertex $u \in U^-$ with a vertex $v \in U^+$ is exactly a vertex of M', so the determination of the new vertices of M' in this case is quite easy.

Remark 3. By repeated application of the above results, one can compute all the vertices and all the extreme directions of a polyhedral convex set D of the form

$D = \{x \in R^n_+ : (a^i, x) + b_i R_i \, 0,\ i = 1,\dots, m\}$ with R_i being one of the relations $=, \leqslant, \geqslant$. Indeed, one can start from $S_0 = R^n_+$ which obviously has only one vertex 0 — the origin of coordinates, and n extreme directions e^j — the j-th unit vector in R^n $(j = 1,\dots, n)$.

3. FINITE METHOD FOR CONCAVE MINIMIZATION UNDER LINEAR CONSTRAINTS

Let us turn now to the main problem we are concerned with in this paper, namely:

Minimize f(x) subject to (7)

$$(a^i, x) + b_i \leqslant 0, \quad i = 1, \dots, m, \tag{8}$$

$$x_j \geqslant 0, \quad j = 1, \dots, n, \tag{9}$$

where $f : R^n \to R$ is a concave function, defined throughout R^n (hence continuous), a^i are n-dimensional vectors and b_i are real numbers. As previously, denote by D the set of all points x satisfying (8), (9).

We first observe the following properties.

LEMMA 1. *Let M be any convex set in R^n. If f is unbounded below on some ray contained in M with direction w, then f is also unbounded below on any ray contained in M with the same direction w.*

Proof. Let $f(x)$ be continuous and unbounded below over the ray $\Gamma_1 = \{u + \lambda w : \lambda \geqslant 0\}$ and let $\Gamma_2 = \{v + \lambda w : \lambda \geqslant 0\} \subset M$. Suppose that $f(x)$ is bounded below over Γ_2, i.e. $f(x) \geqslant \gamma$ for all $x \in \Gamma_2$. Define $\beta = \min \{\gamma, f(u)\}$. Since $f(x)$ is unbounded below over Γ_1 there is $\lambda_1 > 0$ such that $f(u + \lambda_1 w) < \beta$. By virtue of the continuity of f on Γ_1 there exists a ball W around $u + \lambda_1 w$ such that $f(x) < \beta$ for all $x \in M \cap W$. Consider any point

$$x = \alpha v + (1 - \alpha)u + \lambda_1 w = u + \lambda_1 w + \alpha(v - u),\ 0 < \alpha < 1.$$

For α small enough, $x \in M \cap W$ and hence $f(x) < \beta$. But we also have

$$\begin{aligned} f(x) &= f\left(\alpha \left(v + \frac{\lambda_1}{\alpha} w\right) + (1 - \alpha) u\right) \\ &\geqslant \alpha f\left(v + \frac{\lambda_1}{\alpha} w\right) + (1 - \alpha) f(u) \\ &\geqslant \alpha\gamma + (1 - \alpha) f(u) \geqslant \beta. \end{aligned}$$

This contradiction completes the proof.

LEMMA 2. *Let M be any convex set in R^n. It f is bounded below on every extreme ray of M then f is also bounded below on any ray contained in M.*

Proof. From the hypotheses it follows that f attains its minimum over M. If Γ is a ray contained in M then obviously $\inf \{f(x) : x \in \Gamma\} \geqslant \inf \{f(x) \cdot x \in M\} > -\infty$.

Denote now by S_o the orthant R^n_+. Let U_o be the vertex set and V_o be the extreme direction set of S_o. Obviously $U_o = \{0\}$, $V_o = \{e^1, \dots, e^n\}$, where e^j is the j-th unit vector in $R^n (j = 1, \dots, n)$. Let $I_o = \{m + 1, \dots, m + n\}$ be the index set of the constraints defining S_o (the index $m + j$ corresponds to the constraint $x_j \geqslant 0$).

ITERATION $k = 0,1, \dots, m$. At this iteration we already have a polyhedral convex set $S_k \supset D$ along with the set U_k of vertices, the set V_k of extreme directions of S_k (generally, U_k is non-empty, but V_k may be empty), and the index set I_k of the constraints defining S_k: $I_k \subset \{1, \dots, m+n\}$. Let $J_k = \{1, \dots, m\} \setminus I_k$.

STEP 1. It is known that a concave function which is bounded below over a ray attains its minimum at the origin of this ray (see e. g. [2]). Therefore, if there exist $v \in V_k$ and $\theta > 0$ such that $f(\theta . v) < f(0)$, then f is unbounded below over the ray $\{tv : t \geqslant 0\}$ and hence, by Lemma 1, f is unbounded below over any ray emanating from some point of S_k in the direction v. Compute

$$\alpha = \max_{i \in J_k} \{(a^i, v)\}. \tag{10}$$

a) If $\alpha \leqslant 0$, i. e. $(a^i, v) \leqslant 0$ for all $i = 1, \dots, m$, stop: either $D = \phi$ or the problem has no finite optimal solution and v is a direction of recession of D over which $f(x)$ is unbounded below.

b) Otherwise, select

$$i_k = \arg\max \{(a^i, v) : i \in J_k\} \tag{11}$$

and go to step 3.

STEP 2. If no extreme direction v as in Step 1 is discovered then the minimum of $f(x)$ over S_k is attained in at least one vertex of S_k. So we select

$$w^k = \arg\min \{f(u) : u \in U_k\}$$

(if there are several candidates, take any one of them). Compute

$$\beta = \max_{i \in J_k} \{(a^i, w^k) + b_i\}. \tag{12}$$

a) If $\beta \leqslant 0$, i. e. $(a^i, w^k) + b_i \leqslant 0$ for all $i = 1, \dots, m$, stop: w^k is an optimal solution of problem (7) — (9).

b) Otherwise, select

$$i_k = \arg\max \{(a^i, w^k) + b_i : i \in J_k\} \tag{13}$$

and go to Step 3.

STEP 3. Form the new polyhedral convex set

$$S_{k+1} = S_k \cap \{x : (a_{i_k}, x) + b_{i_k} \leqslant 0\}, \tag{14}$$

$$I_{k+1} = I_k \cup \{i_k\}.$$

Determine the set U_{k+1} of vertices and the set V_{k+1} of extreme directions of S_{k+1}, using the technique described in Section 2 (see Propositions

1 — 4, Rule A and Rule B). If $S_{k+1} = \phi$ is discovered then $D = \phi$ and **stop.** Otherwise, set $k \longleftarrow k+1$ and go to iteration $k+1$.

PROPOSITION 5. *The above algorithm stops after at most m iterations.*

Proof. Since at each iteration the current polyhedral convex set S_k is obtained from the previous one, S_{k-1}, by adding just one new constraint, since all these constraints are taken from the system (8), it is easily seen that the above algorithm stops after at most m iterations.

Remark 4. If step 2 occurs at some iteration k then by Lemma 2 it will occur at any iteration $h > k$. Therefore, at each iteration $h \geqslant k$, having found the set of vertices and extreme directions of S_{h+1}, one could return directly to step 2. Specifically, for the case where D is bounded, i.e. D is a polytope, each iteration of the algorithm consists only of steps 2 and 3, and the algorithm in this case reduces to the one developed in [4].

Remark 5. For convenience, we have restricted ourselves to problems with linear inequality constraints only, but with minor modifications the above algorithm also applies to the case of linear equality constraints. Indeed, if instead of (8) we have

$$(a^i, x) + b_i = 0, \; i \in I^0, \qquad (8')$$

$$(a^i, x) + b_i \leqslant 0, \; i \in I^-$$

with $I^0 = \{1, \ldots, r\}$ and $I^- = \{r+1, \ldots, m\}$, then (10) — (13) should be replaced by (10') — (13') respectively:

$$\alpha = \max \left\{ \max_{i \in J_k \cap I^0} \{ |(a^i, v)| \}, \max_{i \in J_k \cap I^-} \{ (a^i, v) \} \right\}, \qquad (10')$$

$$i_k = \arg\max \left\{ \max_{i \in J_k \cap I^0} \{ |(a^i, v)| \}, \max_{i \in J_k \cap I^-} \{ (a^i, v) \} \right\}, \qquad (11')$$

$$\beta = \max \left\{ \max_{i \in J_k \cap I^0} \{ |(a^i, w^k) + b_i| \}, \max_{i \in J_k \cap I^-} \{ (a^i, w^k) + b_i \} \right\} \qquad (12')$$

$$i_k = \arg\max \left\{ \max_{i \in J_k \cap I^0} \{ |(a^i, w^k) + b_i| \}, \max_{i \in J_k \cap I^-} \{ (a^i, w^k) + b_i \} \right\}. \qquad (13')$$

Also, if $i_k \in I^0$, (14) should be replaced by

$$S_{k+1} = S_k \cap \{ x : (a^{i_k}, x) + b_{i_k} = 0 \}. \qquad (14')$$

Remark 6. The proposed algorithm can be started from any polyhedral convex set $S_0 \supset D$, provided its vertices and extreme directions are known (or can easily be computed). We have chosen $S_0 = R^n_+$ but this is not necessary. The algorithm still works, if $f(x)$ is continuous relative to S_0 only.

4. APPLICATIONS

In this section we shall apply the above algorithm to some important problems of mathematical programming.

I. THE BILINEAR PROGRAMMING PROBLEM

Minimize $F(x,y) = c^T x + x^T Q^T y + d^T y$

subject to $x \in D = \{x \in R^n : Ax \geqslant a,\ x \geqslant 0\}$,

$$y \in E = \{y \in R^{n'} : B^T y \geqslant b,\ y \geqslant 0\},$$

where A is an m by n matrix, B^T an m' by n' matrix, Q^T an n by n' matrix; a, b, c and d are m, m', n, n' — vectors respectively.

This problem has been extensively studied in the literature during the last ten years. Most of the solution methods up to now developed for it require assumption about the boundedness of both D and E. In the sequel we shall assume only that the set E is bounded (the case of bounded D is similar). Our method is based upon the observation that the bilinear programming problem can be converted into a concave minimization problem (see [3]). Namely, let

$$f(x) = \min\{F(x,y) : y \in E\} = c^T x + \min\{(d + Qx)^T y : y \in E\}$$
$$= c^T x + \max\{b^T u : Bu \leqslant d + Qx,\ u \geqslant 0\}, \tag{15}$$

where the last equality follows from the Duality Theory in linear programming. Then $f(x)$ is a concave function defined throughout R^n and the bilinear programming problem now reduces to minimizing f over D. Since for every x the value of $f(x)$ can easily be computed by solving a linear program (depending upon x) over the polytope E, the algorithm developed in the previous sections applies and yields a finite procedure for solving the bilinear programming problem.

Note that in this case the checking of whether f is unbounded below over a ray emanating from a given point $\bar{x} \in S_k$ in an extreme direction v of S_k (with $S_k \supset D$ as described in the above algorithm) is a relatively easy task. Indeed, from (15) we have

$$f(\bar{x} + \theta v) = c^T(\bar{x} + \theta v) + \max\{b^T u : Bu \leqslant d + Q(\bar{x} + \theta v),\ u \geqslant 0\}.$$

To determine the largest value of θ for which $f(\bar{x}+\theta v) \geqslant f(\bar{x})$, it suffices to solve the linear program

$(L(v))$: Maximize θ subject to

$$\theta c^T v + b^T u \geqslant f(\bar{x}) - c^T \bar{x}$$
$$-\theta Q v + Bu \leqslant d + Q\bar{x}$$
$$\theta \geqslant 0, \ u \geqslant 0.$$

Let $\bar{\theta}$ be the optimal value in this program. If $\bar{\theta} = \infty$, $f(x)$ attains its minimum over the ray $\{\bar{x} + \theta v : \theta \geqslant 0\}$ at $\bar{x}$, i. e. f is bounded below on this ray; if not ($\bar{\theta} < \infty$), then by a well-known property of concave functions, f is unbounded below on this ray.

Note that at each given iteration, the linear programs $(L(v))$ associated with different $v \in S_k$ differ from one another just by the column coefficient of the variable θ. This property should be exploited by using reoptimization techniques in solving these linear programs $(L(v))$.

II. THE LINEAR COMPLEMENTARITY PROBLEM

Find an n-vector x, an n-vector y and a p-vector z satisfying

$$Ax + By + Cz + b = 0,$$
$$x^T y = 0;\ x, y, z \geqslant 0, \qquad (16)$$

where A, B are m by n matrices, C is an m by p matrix, b is an m-vector.

As was shown [3], $(\bar{x}, \bar{y}, \bar{z})$ is a solution of (16) if and only if $(\bar{x}, \bar{y}, \bar{z})$ is an optimal solution of the following problem

$$\min \{ l(x, y, z) = \sum_{i=1}^{n} \min (x_i, y_i) : Ax + By + Cz + b = 0,$$
$$x, y, z \geqslant 0\} \qquad (17)$$

with $l(\bar{x}, \bar{y}, \bar{z}) = 0$.

Since the objective function of (17) is clearly a concave function, (17) is a concave programming problem under linear constraints. Thus, instead of solving the linear complementarity problem (16), we can solve the corresponding concave minimization problem (17). If an optimal solution $(\bar{x}, \bar{y}, \bar{z})$ exists such that $l(\bar{x}, \bar{y}, \bar{z}) = 0$, it is a solution to the linear complementarity problem; otherwise the linear complementarity problem has no solution.

The idea of solving complementarity problems via concave programming was first implemented in [7], [9], [14]. However, since at that time algorithms for concave minimization were available only for problems with bounded constraint sets, the authors of [7], [9], [14] had to overcome this boundedness assumption which, of course, does not hold for the concave program equivalent to the original complementarity problem. Actually, it was the extension of the method in [7], [9], [14] that led to a general algorithm for concave minimization over arbitrary (possibly unbounded) closed convex sets (see [13]).

With our approach the algorithm in Section 3 can be applied to the problem (17) directly. Let us mention some particular feature of the algorithm when applied to this problem.

a) Since $l(x, y, z) \geqslant 0$ for all points $(x, y, z) \in R^{2n+p}_{+}$ and since S_k (polyhedral convex set containing the constraint set D of (17) at iteration k of the algorithm) is contained in R^{2n+p}_{+}, $l(x, y, z)$ is bounded below on every ray of S_k. Therefore, in solving the problem (17) by our algorithm step 1 has never to be executed (even though S_k is unbounded). By Remark 4, each iteration will then consist only of steps 2 and 3.

b) If at some iteration k

$$\min \{ l(x, y, z) : (x, y, z) \in U_k \} > 0$$

(where, it will be recalled, U_k is the vertex set of S_k), then

$$\begin{aligned} &\min \{ l(x, y, z) : (x, y, z) \in D\} \\ &\geqslant \min \{ l(x, y, z) : (x, y, z) \in S_k \} \\ &= \min \{ l(x, y, z) : (x, y, z) \in U_k \} > 0, \end{aligned}$$

which implies that the linear complementarity problem (16) has no solution. In this case we stop the algorithm at iteration k.

An important special case of the problem (16) is when $m = n$, $B = -E$ (E is a unit matrix) and $C = 0$. Then the problem (16) becomes: Find $x \in R^n$, $y \in R^n$ such that

$$y = Ax + b \geqslant 0, \ x \geqslant 0, \ x^T y = 0. \tag{18}$$

It is often in this form that the linear complementarity problem has been treated in the literature. However, as mentioned in [14], most of the existing methods solve the problem under some additional assumptions about the matrix A. By contrast our algorithm can be applied in all cases where the problem is solvable. The concave minimization problem equivalent to (18) is:

$$\text{Minimize } f(x), \ s.t. \ Ax + b \geqslant 0, \ x \geqslant 0, \tag{19}$$

where

$$f(x) = \sum_{i=1}^{n} \min \{ x_i, \sum_{j=1}^{n} a_{ij} x_j + b_i \}. \tag{20}$$

If an optimal solution $\overline{x}$ exists such that $f(\overline{x}) = 0$, it is a solution to the linear complementarity problem (18); otherwise (18) has no solution.

In solving problem (19) by our algorithm, the operation of checking whether f is unbounded below on a ray emanating from a given point u in a direction v, can be performed as follows. We have from (20)

$$f(u + \theta v) = \sum_{i=1}^{n} \min\{u_i + \theta v_i,\ \sum_{j=1}^{n} a_{ij}u_j + b_i + \theta \sum_{j=1}^{n} a_{ij}v_j\}$$

$$= \sum_{i=1}^{n} \min\{u_i + \theta v_i,\ \alpha_i + \theta\beta_i\},$$

where $\alpha_i = \sum_j a_{ij}u_j + b_i$, $\beta_i = \sum_j a_{ij}v_j$. Therefore, for all $i = 1,\ldots, n$ with large enough θ we have

$$\min\{u_i + \theta v_i,\ \alpha_i + \theta\beta_i\} = \begin{cases} u_i + \theta v_i \text{ if } v_i < \beta_i, \text{ or} \\ \qquad v_i = \beta_i \text{ and } u_i < \alpha_i \\ \alpha_i + \theta\beta_i \text{ otherwise.} \end{cases}$$

Hence, for large enough θ:

$$f(u + \theta v) = \sum_{i\in K}(u_i + \theta v_i) + \sum_{i\notin K}(\alpha_i + \theta\beta_i) = \lambda + \theta\mu,$$

where $K = \{\ i : v_i < \beta_i \text{ or } v_i = \beta_i,\ u_i < \alpha_i\ \}$, $\lambda = \sum_{i\in K} u_i + \sum_{i\notin K}\alpha_i$ and $\mu = \sum_{i\in K} v_i + \sum_{i\notin K}\beta_i$. Therefore, if $\mu < 0$ then $f(u + \theta v) \longrightarrow -\infty$ as $\theta \longrightarrow \infty$; otherwise f is bounded from below over the ray $\{u + \theta v : \theta \geqslant 0\}$.

III. CONCAVE MINIMIZATION WITH SPECIAL STRUCTURE

In a recent work [12] H. Tùy has developed a decomposition method for solving the following class of concave minimization problems under linear constraints

minimize $f(x)$, sudject to (21)

$$Ax + By + c \leqslant 0, \tag{22}$$

$$x \in X, \quad y \in Y, \tag{23}$$

where X, Y are polyhedral convex sets in R^p, R^q respectively, A an m by p matrix, B an m by q matrix, c an m-vector and $f(x)$ a continous concave function over X. Here p is assumed to be small as compaired to $n = p + q$.

The basic idea of the method is to reduce the original problem (21) — (23) to a finite sequence of linearly constrained concave minimization subproblems in the variable x, such that each subproblem is obtained from the previous one by adding just a new linear constraint. Therefore, to solve these problems our algorithm could be used advantageously.

5. A SIMPLE ILLUSTRATIVE EXAMPLE

Here a two-dimensional example will illustrate how the algorithm might perform on problems with unbounded constraint set.

We consider the problem :

$$\text{Minimize } f(x) = \frac{x_1 x_2}{x_1 + x_2} - \frac{0.05\,(x_1 - x_2)^2}{x_1 + x_2}$$

subject to

$$\begin{aligned} -3x_1 + x_2 - 1 &\leqslant 0 \quad (1) \\ -3x_1 - 5x_2 + 23 &\leqslant 0 \quad (2) \\ x_1 - 4x_2 - 2 &\leqslant 0 \quad (3) \\ -x_1 + x_2 - 5 &\leqslant 0 \quad (4) \\ x_1 &\geqslant 0 \\ x_2 &\geqslant 0 \end{aligned}$$

Fig. 1 depicts the constraint set D (note that $f(x)$ is a concave function, defined and continuous on $R^2_+ \supset D$).

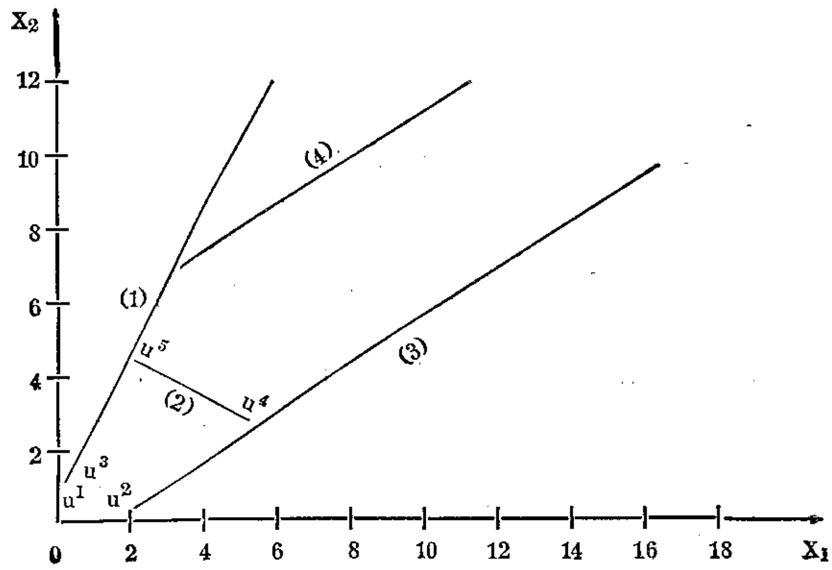

Fig. 1. The constraint set D

The algorithm starts from $S_o = R^2_+$ with vertex set $U_o = \{u^1\}$ and extreme direction set $V_o = \{v^1, v^2\}$, where $u^1 = (0, 0)$ with $f(u^1) = 0$, $v^1 = (1, 0)$ and $v^2 = (0, 1)$.

ITERATION 0. On the ray $\{x = tv^1 = (t, 0) : t \geqslant 0\}$ we have

$$f(x) = -0.05t \longrightarrow -\infty \text{ as } t \longrightarrow +\infty,$$

i. e. f is unbounded below over this ray. According to (10) we compute

$$\alpha = \max\{-3, -3, 1, -1\} = 1 > 0$$

and select $i_o = 3$. Thus

$$S_1 = S_o \cap \{x : x_1 - 4x_2 - 2 \leqslant 0\}.$$

$$U^- = \{u^1\},\ U^+ = \phi,\ V^- = \{v^2\},\ V^+ = \{v^1\}.$$

Applying Rule A, the pair $(u^1, v^1) \in U^- \times V^+$ generates a new vertex $u^2 = (2,0)$ with $f(u^2) = -0.1$. Applying Rule B, the pair $(v^2, v^1) \in V^- \times V^+$ generates a new extreme direction $v^3 = (4,1)$. So we have, by Proposition 3, $U_1 = \{u^1, u^2\}$ and, by Proposition 4, $V_1 = \{v^2, v^3\}$.

ITERATION 1. On the ray $\{x = tv^3 = (4t, t) : t \geqslant 0\}$ we have

$$f(x) = \frac{4t^2}{5t} - \frac{0.05 \times 9t^2}{5t} = 0.71t \rightarrow \infty \text{ as } t \rightarrow \infty,$$

i.e. $f(x)$ attains its minimum at the origin of this ray.

On the ray $\{x = tv^2 = (0,t) : t \geqslant 0\}$ we have

$$f(x) = -0.05t \rightarrow -\infty \text{ as } t \rightarrow +\infty,$$

i.e. $f(x)$ is unbounded below on this ray. Compute

$$\alpha = \max\{1, -5, 1\} = 1 > 0.$$

We select $i_1 = 1$ Thus

$$S_2 = S_1 \cap \{x : -3x_1 + x_2 - 1 \leqslant 0\}.$$

$$U^- = \{u^1, u^2\},\ U^+ = \phi,\ V^- = \{v^3\},\ V^+ = \{v^2\}.$$

The pair $(u^1, v^2) \in U^- \times V^+$ generates a new vertex $u^3 = (0,1)$ with $f(u^3) = -0.05$, while $(u^2, v^2) \in U^- \times V^+$ generates no vertex and $(v^3, v^2) \in V^- \mathrm{x} V^+$ gives a new extreme direction $v^4 = (1, 3)$. So we have

$$U_2 = \{u^1, u^2, u^3\},\ V_2 = \{v^3, v^4\}.$$

ITERATION 2. $f(x)$ is bounded on the rays $\{tv^3 : t \geqslant 0\}$ and $\{tv^4 : t \geqslant 0\}$. So we have

$$\min\{f(x) : x \in S_2\} = \min\{f(u^1), f(u^2), f(u^3)\} =$$
$$= \min\{0, -0.1, -0.05\} = -0.1$$

and $w^2 = u^2 = (2,0)$. According to (12) we compute

$$\beta = \max\{17, -7\} = 17 > 0.$$

Select $i_2 = 2$ and

$$S_3 = S_2 \cap \{x : -3x_1 - 5x_2 + 23 \leqslant 0\}.$$

$$U^- = \phi, U^+ = \{u^1, u^2, u^3\}, V^- = \{v^3, v^4\}, V^+ = \phi.$$

$(u^2, v^3) \in U^+ \times V^-$ gives a new vertex $u^4 = (6,1)$ with $f(u^4) = 0.67857143$ and $(u^3, v^4) \in U^+ \times V^-$ gives a new vertex $u^5 = (1,4)$ with $f(u^5) = 0.71$ (the other pairs in $U^+ \times V^-$ give no vertex). We thus have

$$U_3 = \{u^4, u^5\} \text{ and } V_3 = \{v^3, v^4\}.$$

ITERATION 3. There is no extreme ray of S_3 over which $f(x)$ is unbounded below, so

$$\min\{f(x) : x \in S_3\} = \min\{f(u^4), f(u^5)\} = 0.67857143$$

and $w^3 = u^4 = (6, 1)$. According to (12)

$$\beta = -10 < 0.$$

Hence, the optimal solution $u^4 = (6,1)$ is found with the objective function value $f(u^4) = 0.67857143$.

6. COMPUTATIONAL EXPERIENCE

The above algorithm was coded in FORTRAN IV and has been run on a IBM 360/50. It was tested on a number of concave minimization problems with bounded constraint sets and with negative quadratic, piecewise linear concave, linear fixed-charge and exponential objective functions. The largest problem so far treated using this algorithm is a 16-variable, 14-constraint problem having a linear fixed-charge objective function. The algorithm solves the problem after three iterations having generated 497 vertices. The computer time was 6.11 minutes. Preliminary results presented below show that the average number of iterations required for obtaining an optimal solution is about $m/2$ (m is the number of linear equality and inequality constraints, not including non-negativity constraints). Therefore this is a viable method for concave minimization problems with a moderate size. Some other computational experiments carried out by Ng. V. Thoai [5] also demonstrated the efficiency of this algorithm when applied to the decomposition method developed in]12]. Additional computational experience for problems with unbounded constraint set will be reported in a subsequent paper.

Problem	Size of A	Objective function	Number of iterations	Maximal number of generated vertices	CPU time (Minute)
1	4 × 2	Quadratic	2	4	0.01
2	4 × 2	—	2	5	0.02
3	8 × 2	—	3	5	0.02
4	5 × 3	—	5	8	0.02
5	9 × 3	—	5	10	0.05
6	2 × 4	—	1	8	0.02
7	4 × 5	—	4	24	0.04
8	4 × 8	—	1	25	0.03
9	6 × 8	—	3	46	0.07
10	6 × 8	Piecewise linear concave	3	48	0.10
11	5 × 12	Quadratic	3	48	0.82
12	11 × 10	—	4	116	0.61
13	11 × 10	Exponential	5	231	2.24
14	14 × 16	Fixed-charge	3	497	6.11

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