ACTA MATHEMATICA VIETNAMICA
Vol. 9 No 2 (1984)

TOPOLOGICAL DEGREE OF PSEUDOPOSITIVE MAPPINGS

DUONG MINH DUC

The topological degree of analytic compact vector fields has been studied in [1, 2, 6, 9]. These elegant results suggest the possibility of defining the topological degrée for a larger class of mappings.

In this paper, using the techniques of Hirsch [10], we shall establish a topological degree for a class of mappings which contains all proper analytic Fredholm mappings of index 0.

The values of this topological degree are positive integer numbers, therefore we can extend the Cronin's theorem [2]. In [4, 8, 11] we can see the other methods of defining of topological degree and the Mod 2 — topological degree.

For the definitions of Fredholm mappings, proper mappings we refer to Smale's paper [8].

In this paper let E and F be two real separable Banach spaces and D be a nonvoid open subset of E. We shall use $\overline{A}$, ∂A and Card A to denote the closure, boundary and cardinal of A.

Let $L(F, F)$ be the set of all continuous linear mappings from E into F, and put

$$S(E, F) = \{T \in L(E, F): T \text{ is a Fredholm operator of index 0 and the dimension of Ker } T \text{ is even.}\}$$

DEFINITION 1. *Let $f: \overline{D} \to F$ be proper and continuous on $\overline{D}$ and twice continuously differentiable on D, then f is said to be pseudopositive on $\overline{D}$ if $f'(x) \in S(E, F)$ for every x in D. We denote the set of all pseudopositive mappings on $\overline{D}$ by $P(D)$.*

Remark 1. Suppose that E and F are two Banach spaces over the field C of complex numbers. If $f: \overline{D} \to F$ is continuous on $\overline{D}$, complex continuously differentiable on D and Fredholm of index 0 on D, we say that f is an *analytic Fredholm mapping* of index 0 on $\overline{D}$.

Now let f be a proper analytic Fredholm mapping of index 0 on $\overline{D}$, we shall show that f is pseudopositive on $\overline{D}$.

Indeed, by Theorem 8.1.5 in [5] f is twice continuous differentiable on D. For a given $x \in D$ we put $E_x = \operatorname{Ker} f'(x)$. We shall show that E_x, as a real vector linear space, has even dimension.

Put $Tx = ix$ for every $x \in E$. Because $f'(x)$ is a complex linear mapping, we have $T.f'(x) = f'(x).T$. Then $T(E_x) \subset E_x$. Suppose by contradiction that the dimension of E_x is odd. Since E_x is finite dimensional, it follows that T has a real eigenvalue. This contradiction shows that the dimension of E_x is even.

Therefore all proper analytic Fredholm mappings of index 0 on $\overline{D}$ are pseudopositive on $\overline{D}$.

We shall define the topological degree of pseudopositive mappings, at first we need the following lemma.

LEMMA 1. *Let a and b be two real numbers such that $a < b$, and let $H: (a, b) \times \overline{D} \to F$ be continuously differentiable on $(a, b) \times D$ such that $H(t,.) \in P(D)$ for every $t \in (a, b)$. Let K be a compact subset of $(a, b) \times D$ such that the derivative of H at (t, x), $DH(t, x)$, is surjective for every $(t, x) \in K$. Then there exists a positive real number d, such that for each (t, x) in K there exists an unique continuous mapping $g: I_t \to D$ such that $g(t) = x$, $(s, g(s)) \in (a, b) \times D$ and $H(s, g(s)) = H(t, x)$ for every $s \in I_t = (t - d, t + d)$.*

Proof.

Let $(t, x) \in K$, it is well known that there exists $c \in F$ such that

(1) $DH(t, x)(s, k) = sc + D_2H(t, x)k$ for every $(s, k) \in \mathbf{R} \times E$. where $D_2H(t, x)$ is the partial derivative of H with respect the second variable at (t, x).

Then

(2) $$F = DH(t, x)(\mathbf{R} \times E) = \mathbf{R}c + D_2H(t, x)(E).$$

In the other hand, because $D_2H(t, x)$ is a Fredholm operator of index 0 and the dimension of its kernel is even, we see that codim $(D_2H(t, x)(E))$ is even.

Therefore it follows from (2) that $D_2H(t, x)$ is a homeomorphism of E onto F. Applying the implicit mapping theorem, there exist a positive real number $d(t, x)$ and an unique continuous mapping $g: (t - d(t, x), t + d(t, x)) \to D$ such that $g(t) = x$, $(s, g(s)) \in (a, b) \times D$ and $H(s, g(s)) = H(t, x)$ for every s in $(t - d(t, x), t + d(t, x))$. The number $d(t, x)$ depends on $DH(t, x)$ and the continuity of DH, Since K is compact, we have the desired result.

Applying the preceding lemma, we have the following propositions.

PROPOSITION 1. *Let a and b be two real numbers such that $[0,1] \subset (a,b)$. Suppose that $H: (a, b) \times \overline{D} \to F$ is twice continuously differentiable on $(a, b) \times D$, and proper on $[0, 1] \times \overline{D}$, and that $D_2 H(t, x) \in S(E, F)$ for every $(t, x) \in (a, b) \times D$ Let p be a regular value of H such that $p \notin H([0, 1] \times \partial D)$. Then the cardinals of $H(0, .)^{-1}(\{p\})$ and $H(1, .)^{-1}(\{p\})$ are finite and equal.*

Proof.

Since H is proper on $[0, 1] \times \overline{D}$, p is a regular value of H, and $p \notin H([0, 1] \times \partial D)$, it follows that $K = H^{-1}(\{p\}) \cap [0, 1] \times D$ is compact. By Lemma 1, K is a disjoint union of paths, which start in $\{0\} \times D$ and end in $\{1\} \times D$. Then

$$\text{Card } H(0,.)^{-1}(\{p\}) = \text{Card } H(1,.)^{-1}(\{p\})$$

Arguing as in the proof of Lemma 1, we can show that p is a regular value of $H(0,.)$ Then $H(0,.)^{-1}(\{p\})$ is a compact subset of D and consists only of isolated points, hence it is finite.

PROPOSITION 2. *Let a and b be two real number, such that $[0,1] \subset (a, b)$. Suppose that $H: (a, b) \times \overline{D} \to F$ is continuous on $(a, b) \times \overline{D}$, twice continuously differentiable on $(a, b) \times D$ and proper on $[0, 1] \times \overline{D}$, and that $D_2 H(t, x) \in S(E, F)$ for every $(t, x) \in (a, b) \times D$. Put $f = H(0,.)$ and $g = H(1,.)$. Let p be a regular value of f and g and suppose that $p \notin H([0, 1] \times \partial D)$. Then the cardinals of $f^{-1}(\{p\})$ and $g^{-1}(\{p\})$ are equal and finite.*

Proof.

By (1) we see that H is a Fredholm mapping of index 1. Then by the Sard-Smale's theorem (Cf[8]), the set of regular values of H is dense in F. In the other hand, since f and g are proper and p is a regular value of f and g, by the inverse mapping theorem, there exists a positive real number r such that if $q \in F$ and the norm of $q-p$ is less than r, then

$$\text{Card } g^{-1}(\{q\}) = \text{Card } g^{-1}(\{p\}) \text{ and Card } f^{-1}(\{q\}) = \text{Card } f^{-1}(\{p\})$$

Therefore we can suppose that p is a regular value of H. Then by Proposition 1 we have the desired results.

QED.

PROPOSITION 3. *Let $f \in P(D)$ and p, q be two regular values of f. Suppose that p and q lie in the same component of $F \setminus f(\partial D)$, the complement of $f(\partial D)$ in F. Then Card $f^{-1}(\{p\}) =$ Card $f^{-1}(\{q\})$.*

Proof.

Let $h: [0, 1] \to F \setminus f(\partial D)$ be continuous on $[0, 1]$ such that $h(0) = p$ and $h(1) = q$. Since h is uniformly continuous and $h([0, 1])$ is compact, there exist $r > 0$ and $t_1 = 0 < t_2 < \dots < t_n = 1$ such that

$$h([0, 1]) \subset \bigcup_{1}^{n} B(f(t_i), r)$$

and

$$B(f(t_i), 4r) \subset F \setminus f(\partial D) \text{ for every } i \in \{1, \dots, n\}$$

$$B(f(t_i), r) \cap B(f(t_{i+1}), r) \neq \phi \text{ for every } i \in \{1, \dots, n-1\}$$

where $B(a, r)$ is the open ball of radius r and centered at a in F.

Now fix an $i \in \{1, \dots, n-1\}$ and let p_i be a regular value of f in $B(f(t_i), r) \cap B(f(t_{i+1}), r)$. For each $(t, x) \in (-1, 2) \times \overline{D}$ we put

$$H(t, x) = f(x) - p_i - t(p_{i+1} - p) + p.$$

The mapping H is proper on $[0, 1] \times \overline{D}$ (cf. the following Remark 4) and satisfies all the other conditions of Proposition 2 Hence it follows that

$$\text{Card}\, f^{-1}(\{p_i\}) = \text{Card}\, H(0,.)^{-1}(\{p\}) = \text{Card}\, H(1,.)^{-1}(\{p\}) =$$
$$= \text{Card}\, f^{-1}(\{p_{i+1}\}).$$

Therefore

$$\text{Card}\, f^{-1}(\{p\}) = \text{Card}\, f^{-1}(\{q\})$$

QED.

Now by Proposition 3 and the theorem Sard-Smale we can define the topological degree of pseudopositive mappings as follows

DEFINITION 2. *Let $f \in P(D)$ and G be a component of $F \setminus f(\partial D)$. If $p \in G$, we choose a regular value q of f in G and put*

$$\deg(f, D, p) = \text{Card}\, f^{-1}(\{q\}).$$

And $\deg(f, D, p)$ is called the topological degree of f on D at P.

Remark 2. Let f be a compact vector field on $\overline{D}$ (into E), and $p \in E \setminus f(\partial D)$. If f is twice continuously differentiable on D and the dimension of Ker $f'(x)$ is even for every $x \in D$, then f is pseudopositive on D and by the Leray-Shauder index theorem (cf. [6]) we have

$$\text{Deg}(f, D, p) \equiv \deg(f, D, p) \pmod 2$$

where $\text{Deg}(f, D, p)$ is the Leray-Schauder degree of f on D at p.

Remark 3. Suppose that E is a complex Banach space, and f is an analytic compact vector field on $\overline{D}$, then by Remark 1, f is pseudopositive on $\overline{D}$. By results in [2, 9] we have

$$\text{Deg}(f, D, p) = \deg(f, D, p) \text{ for every } p \in E \setminus f(\partial D).$$

By Definition 2, Proposition 3 and by a standard procedure we have the following basic properties of the topological degree of pseudopositive mappings

THEOREM 1. *Let $f \in P(D)$ and $p \in F \setminus f(\partial D)$, we have:*

(i) If $\deg(f, D, p) \neq 0$ then there exists $x \in D$ such that

$$f(x) = p$$

(ii) If $D_1, \dots, D_n$ are n pairwise disjoint open subsets of D such that

$$p \notin f(\overline{D} \setminus \bigcup_{1}^{n} D_i), \text{ then}$$

$$deg(f, D, p) = \sum_{1}^{n} deg(f, D_i, p)$$

(iii) If f is one-to-one on D, then

$$deg(f, D, p) = \begin{cases} 1 & if\ p \in f(D) \\ 0 & if\ p \notin f(D) \end{cases}$$

THEOREM 2. *Let a and b be two real number such that [0,1] is contained in (a, b). Suppose that* $H: (a, b) \times \overline{D} \longrightarrow F$ *is continuous on* $(a, b) \times \overline{D}$, *twice continuously differentiable on* $(a, b) \times D$ *and proper on* $[0,1] \times \overline{D}$, *and that* $D_2 H(t, x) \in S(E, F)$ *for every* $(t, x) \in (a, b) \times D$. *Let* $p \in F \setminus H([0,1] \times \partial D)$, *we have* $deg(H(0, .), D, p) = deg((H(1, .), D, p)$.

Remark 4. Let $f: \overline{D} \longrightarrow F$ be a proper analytic Fredholm mapping of index 0 on $\overline{D}$, and $g: \overline{D} \longrightarrow F$ be analytic on $\overline{D}$ and suppose that $g(\overline{D})$ is relatively compact. For $(t, x) \in (-1, 2) \times \overline{D}$ we put

$$H(t, x) = f(x) + tg(x)$$

Then H is twice continuously differentiable on $(-1, 2) \times D$, and for each $(t, x) \in (-1, 2) \times D$, $D_2 H(t, x) = f'(x) + tg'(x)$.

Since $f'(x)$ is Fredholm of index 0 and $g'(x)$ is a compact operator, by Corollary 2 of Theorem 6 in page 202 of [7], $D_2 H(t, x)$ is Fredholm of index 0. Then by Remark 1, $D_2 H(t, x) \in S(E, F)$ for every $(t, x) \in (-1, 2) \times D$. We shall show that H is proper on $[0,1] \times \overline{D}$.

Let C be a compact subset of F and $\{(t_n, x_n)\}$ be a sequence in $H^{-1}(C) \cap \{[0,1] \times \overline{D}\}$. Put $y_n = f(x_n)$, $z_n = g(x_n)$ and $u_n = H(t_n, x_n)$. We can suppose that the sequences $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are convergent. Then $\{y_n\}$ is also convergent in F.

Because f is proper on $\overline{D}$, $\{x_n\} = f^{-1}(\{y_n\})$ is relatively compact. Then $\{[0,1] \times \overline{D}\} \cap H^{-1}(C)$ is compact. Thus H is proper on $[0,1] \times \overline{D}$, and we have the following corollary

COROLLARY 1. *Let* $f: \overline{D} \longrightarrow F$ *be a proper analytic Fredholm mapping of index 0 and* $p \in F \setminus f(\partial D)$. *Let* $g: \overline{D} \longrightarrow F$ *be analytic on* $\overline{D}$ *and suppose that* $g(\overline{D})$ *is relatively compact and*

(4) $$f(x) + tg(x) \neq p \text{ for every } (t, x) \in [0,1] \times \partial D$$

Then $deg(f + g, D, p) = deg(f, D, p)$.

Proof.

For every $(t, x) \in (-1,2) \times \overline{D}$ we put

$$H(t, x) = f(x) + tg(x)$$

By (4) and Remark 4, H satisfies all conditions of Theorem 2. Then we have

$$\deg (f, D, p) = \deg (H (0,.), D, p) = \deg (H(1,.), D, p) = \deg (f + g, D, p)$$

QED.

DEFINITION 3. Let $f: \overline{D} \to F$ *be continuously differentiable on D. We shall say that f has the Riesz property on D, if there exists* $T \in L(E, F)$ *such that*

(i) T is a homeomorphism of E onto F.

(ii) 0 is not the accumulation point of $\{c \in \mathbf{C} : \ker (cT - f'(x)) \neq \{0\}\}$ *for every* $x \in D$.

(iii) For each $x \in D$ *there exists a projection* P_x *of E onto a finite dimensional vector subspace* E_x *of E such that* $P_x(E)$ *contains* $\ker f'(x)$ *and* P_x *commutes with* $T^{-1}. f'(x)$.

Remark 5. If $F = E$ and f is an analytic compact vector field on $\overline{D}$, by the properties of the spectrum of compact vector field (cf. [3]) f has the Riesz property on D with $T = I$.

Now we can extend the Cronin' s theorem [2] to the case of proper analytic Fredholm mapping of index 0.

THEOREM 3. *Let* $f: \overline{D} \to F$ *be a proper analytic Fredholm mapping of index 0 on* $\overline{D}$, *and* $p \in F \setminus f(\partial D)$. *Suppose that f has the Riesz property on D. Then*

$$\mathrm{Card}\, f^{-1}(\{p\}) \leqslant \deg (f, D, p).$$

Proof.

Let T be as in Definition 3, let S be a Fredholm operator of E into F. Since T is a homeomorphism of E onto F, ind $T^{-1}. S$ is equal to ind S. Then we can (and shall) suppose that $F = E$ and $T = I$. In the other hand, since $f^{-1}(\{p\})$ is compact, we shall suppose that D is bounded.

By Definition, $\deg (f, D, p) = k \geqslant 0$. The proof of the theorem will be by contradiction. Suppose that there exist $(k+1)$ distinct points $z_1, \ldots, z_{k+1}$ in D such that $\{z_1, \ldots, z_{k+1}\}$ is contained in $f^{-1}(\{p\})$. We shall show that this will imply a contradiction.

Indeed, let $E_j = E_{z_j}$ and $P_j = P_{z_j}$ be the projection of E onto E_j as in Definition 3, and $Q_j = I - P_j$ for every j in $\{1, \ldots, k+1\}$.

Using the Hahn — Banach theorem, one shows that there exists a complex continuous linear functional f on E such that

$$f(z_j - z_{j'}) \neq 0 \qquad \text{for every } j \neq j'.$$

Put

$$a_j(x) = \prod_{j' \neq j} (f(x - z_{j'}))^2 \quad \text{for every } j \in \{1, \ldots, k+1\}. \quad x \in E$$

For each $c > 0$ we put

$$f_c(x) = f(x) - c \sum_{j=1}^{k+1} a_j(x)\, P_j\, (x - z_j).$$

Since $\overline{D}$ is bounded and $P_j(E)$ is finite dimensional, we see that $(\sum_j a_j(.)\, P_j(. - z_j))\,(\overline{D})$ is relatively compact.

Then if c is enough small, by Corollary 1 we have

(5) $$\deg(f_c, D, p) = \deg(f, D, p) = k.$$

For every $j \in \{1, \ldots, k+1\}$ we have

(6) $$f_c(z_j) = p$$

(7) $$f'_c(z_j) = f'(z_j) - ca_j(z_j)\, P_j.$$

Then if there exists $x \in E$, such that $x \neq 0$ and $f_c'(z_j)\,x = 0$, we have

$$f'(z_j)\,x - ca_j(z_j)\, P_j\, x = 0$$

Thus

(8) $$f'(z_j)\,(1\text{-}ca_j(z_j))\, P_j x = -f'(z_j) Q_j\, x.$$

or

$$P_j\, [(1\text{-}ca_j(z_j) f'(z_j)(x)] = -Q_j\, f'(z_j)(x)$$

It follows that

$$0 = f'(z_j)\,(1 - ca_j(z_j)) p_j x = -f'(z_j)) Q_j x.$$

Since $P_j(E)$ contains $\ker f'(zj)$, $f'(z_j) Q_j x = 0$ implies that $Q_j x = 0$. Hence $P_j x \neq 0$ and by (8) we have

$$f'(z_j) P_j x = ca_j(z_j) P_j\, x.$$

It follows that $ca_j(z_j) \in C_j = \{s \in \mathbb{C} : \ker(cI - f'(z_j)) \neq \{0\}\}$ Because 0 is not the accumulation point of C_j and $a_j(z_j) \neq 0$, there exists $r > 0$ such that if $c \in (0, 2r)$, then we have (5) and $ca_j(z_j) \notin C_j$ for every $j \in \{1, \ldots, k+1\}$.

Therefore $f'_r(z_j)$ is a homeomorphism of E onto E (we recall that $f'_r(z_j)$ is a Fredholm operator of index 0), for every j. Then we can find a positive number s such that $f_r \mid B(z_j, 2s)$, the restriction of f_r on $B(z_j, 2s)$, is a homeomorphism of $B(z_j, 2s)$ onto an open in E for every j, and that $\{B(z_j, s)\}_j$ is a family of $k+1$ pairwise disjoint subsets of D.

By Theorem 1 we have

$$(9) \quad \deg f_r, D, p) = \deg (f_r, D \setminus \overline{\bigcup_j B(z_j, s)}, p) + \sum_{j=1}^{k+1} \deg (f_r, B(z_j, s), p).$$

and

$$(10) \qquad \deg (f_r, B(z_j, s), p) = 1$$

Hence it follows that

$$(11) \qquad \deg (f_r, D, p) \geqslant k + 1$$

This contradicts (5). Then

$$\text{Card } f^{-1} (\{p\}) \leqslant \deg (f, D, p)$$

QED.

Received June 14, 1984

REFERENCES

[1] F. E. Browder, *Non-linear mappings of analytic type in Banach spaces*, Math. Ann. 185 (1970), 259 — 278.

[2] J. Cronin, *Analytic functional mappings*, Ann. Math. 58 (1953). N. 1

[3] R. E. Edwards, *Functional analysis, theory and applications*, Holt, Rinehart and Winston, New York, 1965,

[4] K. D. Elworthy and A. J. Tromba, *Degree theory on Banach manifolds*, Proc. Sympos. Pure Math., vol. 18, part 1, 1970, 86 — 94.

[5] F. Hille, *Methods in classical and functional analysis*, Addison-Wesley Publ., company, Reading, 1972.

[6] M.A. Krasnosel'skii and P.P. Zabreiko, *Geometrical methods of nonlinear analysis* (Russian), Nauka, Moscu, 1975,

[7] S. Lang, *Analysis II*. Addison Wesley, Reading, 1969.

[8] S. Smale, *An infinite dimensional version of Sard's theorem*. Amer. J. Math. 87 (1965), 861 — 866.

[9] J. Schwartz, *Compact analytical mappings of B-spaces and a theorem of Jane Cronin*, Comm, Pure Appl. Math. 16, 1963, 253 — 260.

[10] M. W. Hirsch, *A proof of the nonretractability of a cell onto its boundary*, Proc. amer. Math. Soc. 14 (1963), 364 — 365.

[11]. A. J. Tromba, *The Eucler characteristic of vector fields on Banach manifolds and a globalization of Leray—Schder degree*. Adv. Math. 28, 1978, № 2, 148 — 173.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HO CHI MINH CITY, VIETNAM.