ACTA MATHEMATICA VIETNAMICA

Vol. 9 No. 1 (1984)

TWO–DIMENSIONAL BOUNDARY VALUE PROBLEM OF THE DIFFUSION

NGUYEN VAN GIA

Institute of Mathematics

Hanoi

INTRODUCTION

The paper deals with a boundary value problem of the parabolic type for the diffusion equation describing the concentration of suspended sediments in long channels.

An initial condition and a boundary condition at the cross section $x = 0$ are given in a general form. A condition at the bottom $y = 0$ is assumed to be a function of one variable x. The coefficients of the equation are functions of one (or two) variable x (or x, y).

Certain simpler cases of this problem have been extensively investigated, but for this problem many questions may be raised. Some of these questions are examined and answered in this paper.

For the boundary value problem under the assumption that the vertical diffusion coefficient K_2 is constant we shall find asymptotic solutions and analyse influences of the flow velocity U and of the horizontal diffusion coefficient K_1 on them.

Under the additional assumptions that K_2 is an increasing function and the concentration at the bottom is constant, we shall find an estimate for the asymptotic solutions.

In a simpler case of constant coefficients of the diffusion equation we shall find analytic and asymptotic solutions. We shall analyse the stability of the sol tions in relation to the independence of the initial condition and of the boundary condition at the cross section $x = 0$.

In a still simpler case of constant concentration at the bottom we shall also analyse an influence of the coefficients of the diffusion equation on the asymptotic solutions.

Finally a comparison between theoretical and experimental results for the case of constant coefficients of the diffusion equation will be given.

1. PROBLEM FORMULATION

In the field of hydraulics (see e.g. [5], [12]), the differential equation describing a distribution of suspended sediments in long channels is given as follows

$$\frac{\partial C}{\partial t} + U_0 \frac{\partial C}{\partial x} = K_{01} \frac{\partial^2 C}{\partial x^2} + \frac{\partial}{\partial y}\left(K_{02} \frac{\partial C}{\partial y}\right) + V \frac{\partial C}{\partial y} \tag{1.1}$$

where C is a function of the variables x, y, t; $(x, y, t) \in \Omega_0 = \{x, y, t: 0 < x < \infty,\ 0 < y < H,\ 0 < t < \infty\}$ x, y; the space-coordinates of the channels; t: the time. K_{01}, K_{02} : the diffusion coefficients in the x and y directions, respectively; U_0 : the flow velocity in the x direction, V: the time averaged fall velocity of sediment particles in the flow ($V = \text{const}$).

The functions U_0, K_{01}, K_{02} are given in various forms as we look through the literature. However, in some practical and theoretical investigations they can be assumed to be constant.

An initial condition is given in the general form

$$C\Big|_{t=0} = Q_{00}\, f_{00}\,(x, y),\ (x, y) \in (0, \infty) \times (0, H) \tag{1.2}$$

and the boundary conditions are

$$C\Big|_{x=0} = Q_{01}\, f_{01}\,(y, t),\ (y, t) \in (0, H) \times (0, \infty) \tag{1.3}$$

$$C\Big|_{y=0} = Q_{02}\, f_{02}\,(x),\ (x, t) \in (0, \infty) \times (0, \infty) \tag{1.4}$$

$$(1.5)\quad \left(K_{02}\frac{\partial C}{\partial y}+VC\right)\Big|_{y=H}=0,\ (x,t)\in(0,\infty)\times(0,\infty)$$

Here Q_{0j}, $j=0, 1, 2$ are constants, which allow us to determine

$$(1.6)\qquad 0\leqslant f_{0j}\leqslant 1.$$

Furthermore, we assume

$$(1.7)\qquad C\leqslant M<\infty \quad \text{as} \quad x\to\infty \quad \text{or} \quad t\to\infty$$

$$(1.8)\qquad \left|\frac{\partial C}{\partial x}\right|\leqslant M<\infty \quad \text{as} \quad x\to\infty.$$

To find the distribution of suspended sediments in the flows we shall examine the boundary value problem (1.1),..., (1.8).

For the sake of convenience we convert this problem into a nondimensional form by using the transformations

$$(1.9)\qquad x'=\sqrt{\frac{K_2}{K_1}}\cdot\frac{x}{H},\quad y'=\frac{y}{H},\quad t'=\frac{K_2}{H_2}t$$

$$(1.10)\qquad C'=\frac{C}{Q_{02}}.$$

Without loss of generality the coefficient functions of the equation (1.1) can be written as

$$(1.11)\qquad U_0=U\,u(x',y'),\qquad 0<u(x'y')<1$$

$$(1.12)\qquad K_{01}=K_1\alpha_1(y'),\qquad 0<\alpha_1(y')<1$$

$$(1.13)\qquad K_{02}=K_2\alpha_2(y'),\qquad 0<\alpha_2(y')<1$$

where U, K_1, K_2 are positive constants, and $u(x', y')$, $\alpha_1(y')$, $\alpha_2(y')$ are assumed to belong to a class of enough smooth functions.

Via the transformations (1.9),..., (1.13) this problem takes on a new form written for the functions $C'(x', y', t')$ and variables x', y', t', which are nondimensional.

To simplify the notation, from now on we shall omit the prime in the functions C' and variables x', y', t' and denote them again by C, x, y, t, respectively.

In the new form the problem is formulated as follows

$$\frac{\partial C}{\partial t} + 2au(x, y)\frac{\partial C}{\partial x} = \alpha_1(y)\frac{\partial^2 C}{\partial x^2} + \alpha_2(y)\frac{\partial^2 C}{\partial y^2} + (2b + \alpha_2'(y))\frac{\partial C}{\partial y} \tag{1.14}$$

$$(x, y, t) \in \Omega = \{x, y, t: 0 < x < \infty, 0 < y < 1, 0 < t < \infty\}$$

$$C|_{t=0} = Q_0 f_0(x, y), \qquad Q_0 = \frac{Q_{00}}{Q_{02}}, \ (x, y) \in (0, \infty) \times (0,1) \tag{1.15}$$

$$C|_{x=0} = Q_1 f_1(y, t), \qquad Q_1 = \frac{Q_{01}}{Q_{02}}, (y, t) \in (0, 1) \times (0, \infty) \tag{1.16}$$

$$C|_{y=0} = f_2(x), \qquad (x, t) \in (0, \infty) \times (0, \infty) \tag{1.17}$$

$$\left(\alpha_2(y)\frac{\partial C}{\partial y} + 2bC\right)\Big|_{y=1} = 0, \ (x, t) \in (0, \infty) \times (0, \infty) \tag{1.18}$$

$$C \leqslant M < \infty \qquad \text{as } x \to \infty \qquad \text{or } t \to \infty \tag{1.19}$$

$$\left|\frac{\partial C}{\partial X}\right| \leqslant M < \infty \qquad \text{as } x \to \infty \tag{1.20}$$

$$Q_j = \text{const.}, \ 0 \leqslant f_j \leqslant 1, \ j = 0, 1, 2$$

where the constants a, b are defined by

$$a = \frac{UH}{2\sqrt{K_1 K_2}}, \qquad b = \frac{VH}{2K_2}. \tag{1.21}$$

It is this boundary value problem (1.14) through (1.20) which we proceed to study in the following sections.

2. SOME SOLUTION ESTIMATES

We first state two estimate theorems, which will be considered as the tools for finding the asymptotic solutions of this problem.

Denote by C^+ and C^- the solutions of the problem (1.14) through (1.20) satisfying the conditions (instead of (1.15))

$$C^+|_{t=0} = Q_0 \tag{2.1}$$

$$C^-|_{t=0} = Q_0 \cdot \min f_0(x, y), \ (x, y) \in (0, \infty) \times (0, 1), \tag{2.2}$$

respectively.

THEOREM 2.1. *Let C be a solution of the problem (1.14) through (1.20). Then there exists an estimate :*

$$C^- \leqslant C \leqslant C^+. \tag{2.3}$$

Proof. Define $\mathcal{C} = C^+ - C$. Then $\mathcal{C}$ is a solution of the equation (1.14) and satisfies the conditions :

(2.4) $\mathcal{C}\,|_{t=0} = Q_0\,(1 - f_0(x, y)) \geqslant 0, \qquad (x, y) \in (0, \infty) \times (0, 1)$

(2.5) $\mathcal{C}\,|_{x=0} = 0$

(2.6) $\mathcal{C}\,|_{y=0} = 0$

(2.7) $\left(\alpha_2(y)\,\frac{\partial \mathcal{C}}{\partial y} + 2b\,\mathcal{C}\right)\Big|_{y=1} = 0$

(2.8) $\mathcal{C} \leqslant M < \infty$ as $x \to \infty$, or $t \to \infty$.

Do not care about the trivial case $f_0(x,y) = 1$. Therefore, the solution $\mathcal{C}$ must be different from a constant, because the constant does not satisfy the condition (2.4). By virtue of the maximum theorem ([11], p. 173), the function $\mathcal{C}$ can attain its minimum over any cube $(0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant t \leqslant T)$ in only one of the planes $x = 0$, $y = 0$, or $y = 1$. If the minimum of the function $\mathcal{C}$ is located in the plane $y = 1$, then $\frac{\partial \mathcal{C}}{\partial y}\Big|_{y=1} \leqslant 0$, therefore, from (2.7) it follows that

(2.9) $$\mathcal{C}\Big|_{y=1} \geqslant 0$$

Using (2.5), (2.6) and (2.9) one concludes: $\mathcal{C} \geqslant 0$ in the above cube. Since X and T are arbitrary numbers, it follows that $\mathcal{C} \geqslant 0$ for $(x,y,t) \in \Omega$ or $C \geqslant C^+$, $(x, y, t) \in \Omega$. Similarly, $C \geqslant C^-$, (x, y, t). The proof is complete.

THEOREM 2.2.

For the solution C of the problem (1.14) through (1.20) the following estimates hold

(2.10) $C_* \leqslant C \leqslant C^*$,

(2.11) $C^{--} \leqslant C \leqslant C^{++}$,

where C_*, C^*, C^{--}, C^{++} are the solutions of the same problem satisfying, respectively, the following conditions (instead of (1.16) or both (1.16) and (1.15)

(2.12) $C_*\Big|_{x=0} = Q_1 \cdot \min f_1(y, t),\ (y, t) \in (0, 1) \times (0, \infty)$

(2.13) $C^*|_{x=0} = Q_1$

(2.14) $\begin{cases} C^{--}|_{t=0} = Q_0 \cdot \min f_0(x, y),\ (x, y) \in (0, \infty) \times (0, 1) \\ C^{--}|_{x=0} = Q_1 \cdot \min f_1(y, t),\ (y, t) \in (0, 1) \times (0, \infty) \end{cases}$

(2.15) $C^{++}|_{t=0} = Q_0,\ C^{++}|_{x=0} = Q_1$

The proof of Theorem 2.2 is similar to that of Theorem 2.1.

3. APPROXIMATE SOLUTION OF THE PROBLEM FOR THE GENERAL CASE

In order to find an influence of functions of coefficients $u(x, y)$, $\alpha_1(y)$, $\alpha_2(y)$ on asymptotic solutions of the problem for large values of x and t, in this section we shall examine the special case of constant coefficients of the diffusion equation. In this case we assume

$$(3.1) \qquad u(x, y) = \alpha_1(y) = \alpha_2(y) = 1.$$

Under this assumption, equation (1.14) is written as

$$(3.2) \qquad L_0 C \equiv \frac{\partial C}{\partial t} + 2a\frac{\partial C}{\partial x} - \frac{\partial^2 C}{\partial x^2} - \frac{\partial^2 C}{\partial y^2} - 2b\frac{\partial C}{\partial y} = 0,$$

and the condition (1 18) changes into.

$$(3.3) \qquad \left(\frac{\partial C}{\partial y} + 2bC\right)\Big|_{y=1} = 0$$

DEFINITION.

We say that $\mathcal{C}(x, y, t)$ is an approximate solution of the equation

$$(3.4)\, LC \equiv \frac{\partial C}{\partial t} + 2au(x,y)\frac{\partial C}{\partial x} - \alpha_1(y)\frac{\partial^2 C}{\partial x^2} = \alpha_2(y)\frac{\partial^2 C}{\partial y^2} - (2b + \alpha_2'(y))\frac{\partial C}{\partial y} = 0$$

if $\mathcal{C}(x, y, t)$ satisfies the following equation:

$$(3.5) \qquad L\,\mathcal{C}(x, y, t) = 0(e^{-\lambda_1 x}) + 0(e^{-\lambda_2 t})$$

where $\lambda_1 > 0$, $\lambda_2 > 0$. If $\mathcal{C}(x, y, t)$ also satisfies (1.15) through (1.20), we call it the approximate solution of the problem (1.14) through (1.20)

THEOREM 3.1.

Let $\mathcal{C}(x, y, t)$ be a solution of the problem (3.1), (3.2), (1.15) through (1.20) Suppose that

$$(3.6) \qquad \mathcal{C}(x, y, t) = C_1(y) + 0(e^{-\lambda_1 x}) + 0(e^{=\lambda_2 t})$$

$$(3.7) \qquad \frac{\partial\, \mathcal{C}(x, y, t)}{\partial x} = 0\,(e^{-\lambda_1 x}) + 0\,(e^{-\lambda_2 t})$$

$$(3.8) \qquad \frac{\partial^2 \mathcal{C}(x, y, t)}{\partial x^2} = 0\,(e^{-\lambda_1 x}) + 0\,(e^{-\lambda_2 t}).$$

If

$$(3.9) \qquad \alpha_2(y) \equiv 1$$

then there exists an approxi · ate solution of the mentioned problem; this solution is expressed by the same from (3.6).

Proof.

One can write the operator L defined by (3.4) in the form

(3. 10) $L = L_o - 2a\ (1 - u\ (x,y)) \frac{\partial}{\partial x} + (1-\alpha_1\ (y)) \frac{\partial^2}{\partial x^2} + (1-\alpha_2\ (y)) \frac{\partial^2}{\partial y^2} -$

$- \alpha_2'(y) \frac{\partial}{\partial y}.$

Substituting the function $C\ (x,y,t)$ defined by (3.6) into (1.14), and using (3.2), (3.7), (3.8), (3.9) and (3.10) we obtain

$$L\ C\ (x,y,t) = 0\ (e^{-\lambda_1 x}) + 0\ (e^{-\lambda_2 t}).$$

It follows that $C\ (x,y,t)$ is an approximate solution of the problem (1.14) through (1.20).

THEOREM 3.2.

Suppose that for arbitrary functions f_o, f_1 *an asymptotic solution of the problem* (3. 1), (3. 2), (1. 15) *through* (1. 20) *can be written as*

(3. 11) $\qquad C(x,y,t) = e^{-2by} + 0\ (e^{-\delta x}) + 0\ (e^{-\eta_1^2 t})$

where $\delta > 0$. *Assume that the function* $C\ (x,y,t)$ *satisfies* (3. 7), (3. 8), *and* (3.12), (3. 13) :

(3. 12) $\qquad \partial C/\partial y = -2b\ e^{-2by} + 0\ (e^{-\delta x}) + 0\ (e^{-\eta_1^2 t})$

(3. 13) $\qquad \partial^2 C/\partial y^2 = 4b^2 e^{-2by} + 0\ (e^{-\delta x}) + 0\ (e^{-\eta_1^2 t}).$

If

(3.14) $\alpha_2\ (y) \in C^{(1)}[0,1],\ \alpha_2'(y) > 0$

then for large x *and large* t *a solution of the problem* (1. 14) *through* (1. 20) *satisfies the following estimates.*

(3. 15) $\qquad C\ (x,y,t) \lesssim e^{-2by} + 0\ (e^{-\delta x}) + 0\ (e^{-\eta_1^2 t})$

(3. 16) $\qquad C\ (x,y,t) \gtrsim e^{-2By} + 0\ (e^{-\Delta x}) + 0\ (e^{-\eta_1^2 t})$

where

(3. 17) $\qquad B = b + \lambda,\ \Delta > 0,\ \delta > 0$

(3. 18) $\lambda = max\ \{[b\ (1 - \alpha_2 (y)) + \alpha_2'\ (y)]/\alpha_2\ (y)\},\ y \in [0,1]$

In order to prove Theorem 3.2 we first establish the following lemma :

LEMMA 3.1.

Assume that

(3.19) $p(y), q(y), r(y) \in C^{(1)}[0,1]$

(3.20) $p(y) > 0,\ q(y) > 0,\ y \in [0,1]$.

Let $J(y)$ *be a solution of the problem* (3.21) *through* (3.24)

(3.21) $p(y)J''(y) + q(y)\ J'(y) = r(y)$

(3.22) $J(0) = 0$

(3.23) $\alpha\ J'(1) + \beta\ J(1) = \gamma$

(3.24) $\alpha > 0,\ \beta > 0.$

(i) *If*

(3.25) $r(y) > 0,\ y \in [0,1]$

(3.26) $\gamma < 0$

then $J(y) < 0,\ y \in [0,1]$.

(ii) If

(3.27) $r(y) < 0,\ y \in [0,1]$

(3.28) $\gamma > 0$

then $J(y) > 0,\ y \in [0,1]$.

Proof.

We first prove part (i). Suppose that $J(1) > 0$. Then it follows from (3.23), (3.24), (3.26) that $J'(1) < 0$, hence in a neighbourhood of the point $y = 1$, $J(y)$ is a decreasing function. Because $J(0) = 0$ (see (3.22)), there exists a maximum of the function $J(y)$ at a point $y^* \in (0,1)$, therefore $J'(y^*) = 0$, $J''(y^*) \leqslant 0$. From this and (3.20), (3.21) it follows that $r(y^*) \leqslant 0$. Thus the assumption $J(1) > 0$ leads to a contradiction with (3.25). Therefore we must have $J(1) \leqslant 0$. Now suppose that there exists point $y_* \in (0,1)$ such that $J(y_*) > 0$. Because $J(0) = 0$, $J(1) \leqslant 0$, the function $J(y)$ again has a maximum in the interval (0,1). As previously, we obtain a contradiction, therefore $J(y) \leqslant 0$.

The proof of the second part is similar to that of the first one.

Proof of Theorem 3.2.

Start with the proof of (3.15). Denote by $C(x,y,t)$ a solution of the problem (1.14) through (1.20) and put

(3.29) $$C(x,y,t) = C^+(x,y,t) + J(y)$$

where $C^+(x,y,t)$ is a solution of (3.2) satisfying (3.1), (3.7), (3.8), (3.12), (3.13) (1.17) through (1.20) and (3.30), (3.31)

(3.30) $$C^+|_{t=0} = Q_0 f_0(x,y) - J(y)$$

(3.31) $$C^+|_{x=0} = Q_1 f_1(y,t) - J(y)$$

According to the first assumption of Theorem 3.2 the function $C^+(x,y,t)$ is expressed in the form (3.11). Noting that $C(x,y,t)$ and $C^+(x,y,t)$ satisfy (1.17), (1.18) and (1.17), (3.3), respectively, we obtain from (3.29)

(3.32) $J(0) = 0,$

(3.33) $\alpha_2(1)\, J'(1) + 2bJ(1) = (1 - \alpha_2(1)) \cdot \left.\dfrac{\partial C^+}{\partial y}\right|_{y=1}$.

Substituting (3.29) into (1.14) and using the form (3.10) of the operator L we get

$$L\,C = L_0\, C^+ - 2a(1 - u(x,y)) \frac{\partial C^+}{\partial x} + (1 - \alpha_1(y)) \frac{\partial^2 C^+}{\partial x^2} + (1 - \alpha_2(y)) \frac{\partial^2 C^+}{\partial y^2} -$$

$$- \alpha_2'(y) \frac{\partial C^+}{\partial y} - \alpha_2(y)\, J''(y) - (2b + \alpha_2'(y))\, J'(y) = 0.$$

Using (3.2), (3.7), (3.8), (3.11), (3.12), (3.13) we obtain from the above expression

(3.34) $\alpha_2(y)\, J''(y) + (2b + \alpha_2'(y))\, J'(y) = 2b[(1 - \alpha_2(y))\, 2b + \alpha_2'(y)]\, e^{-2by} +$

$$+ 0\,(e^{x\partial} + 0\,(e^{-\eta_1^2 t}).$$

For large x and large t, the conditions (1.13), (3.14) allow us to apply Lemma 3.1 to the function $J(y)$ defined by (3.32), (3.33), and (3.34). It follows that $J(y) \leqslant 0$, hence:

(3.35) $C(x, y, t) \leqslant C^+(x, y, t).$

Therefore we get the estimate (3.15). To prove (3.16) we put

(3. 36) $C(x, y, t) = C^-(x, y, t) + J(y)$

where $C^-(x, y, t)$ is a solution of

(3.37) $L_0^*\, C^- \equiv \dfrac{\partial C^-}{\partial t} + 2a \dfrac{\partial C^-}{\partial x} - \dfrac{\partial^2 C^-}{\partial x^2} - \dfrac{\partial^2 C^-}{\partial y^2} - 2B \dfrac{\partial C^-}{\partial y} = 0$

satisfying (1.17), (1.19), (1.20), (3.30), (3.31) and

(3.38) $(\partial C^-/\partial y + 2BC^-)|_{y=1} = 0,\ B = b + \lambda.$

Using the first assumption of Theorem 3. 2 we have

(3.39) $C^-(x, u, t) = e^{-2By} + 0\,(e^{-\Delta x}) + 0\,(e^{-\eta^2 t})$

where $\Delta > 0$, $\eta > 0$. Proceeding in a similar way to find the function $J(y)$ as in the previous part. We see that the function $J(y)$ satisfies the equation and boundary conditions

(3.40) $\alpha_2(y) J''(y) + (2b + \alpha_2'(y)) J'(y) =$

$$= 2B [(1 - \alpha_2(y)) 2B - 2\lambda + \alpha_2'(y)] e^{-2By} + 0 (e^{-\Delta x}) + 0 (e^{-\eta^2 t})$$

(3. 41) $J(0) = 0,$

(3.42) $\alpha_2(1) J'(1) + 2bJ(1) = [2\lambda - (1 - \alpha_2(1))] 2Be^{-2B} + 0 (e^{-\Delta x}) + 0(e^{-\eta^2 t}).$

If x and t are large enough and λ is chosen such that (3.18) is satisfied, we can apply Lemma 3.1 to problem (3.40), (3.41), (3. 42).

We then have

(3.43) $J(y) \geqslant 0$

hence

(3.44) $C(x, y, t) \geqslant C^-(x, y, t).$

Thus (3.11), (3.35), (3.39), (3.44), completing the proof.

Remark.

The assumptions (3.6), (3.7), (3.8), (3.11), (3.12), (3.13) of Theorems 3,1, 3,2 are admissible. This will be examined in the next sections.

4. DIFFUSION PROBLEM WITH CONSTANT COEFFICIENTS

In this section we assume that the coefficient functions U_0 and K_{01}, K_{02} are constant, i.e.

(4.1) $u(x, y) = \alpha_1(y) = \alpha_2(y) = 1.$

For the initial and boundary conditions we make the following assumption:

(4.2) $f_0(x, y) = f_1(y, t) = 1.$

To convert the problem (1.14) through (1.20) under assumptions (4.1), (4.2) to the canonical form we introduce here a transformation

(4.3) $E = C . e^{-ax+by+ct}$, $C = C(x, y, t)$

where a, b are defined by (1.21) and the constant c is

(4.4) $c = a^2 + b^2$

Using the transformation (4.3) and assumption (4.1) one can write the diffusion equation (1.14) as follows

(4.5) $$\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}, \ (x, y, t) \in \Omega.$$

After the transformation (4.3) the initial and boundary conditions for the function E become

(4. 6) $$E\Big|_{t=0} = Q_0 e^{-ax+by}, \ (x, y) \in (0, \infty) \times (0,1)$$

(4. 7) $$E\Big|_{x=0} = Q_1 e^{by+ct}, \ (y, t) \in (0, 1) \times (0, \infty)$$

(4. 8) $$E\Big|_{y=0} = f_2(x)\, e^{-ax+ct}, \ (x, t) \in (0, \infty) \times (0,\infty)$$

(4. 9) $$\left(\frac{\partial E}{\partial y} + b\,E\right)\Big|_{y=1} = 0, \ (x, t) \in (0, \infty) \times (0, \infty)$$

and

(4. 10) $$E \to 0, \ \frac{\partial E}{\partial x} \to 0 \quad \text{as} \quad x \to \infty$$

(4. 11) $$E = 0\,(e^{ct}) \text{ as } t \to \infty.$$

The problem (4. 5) through (4. 11) will henceforth be referred to as problem (A). We shall find a solution of problem (A), which belongs to the class $C^{(2,\,1)}(\Omega)$.

5. APPLICATIONS OF FOURIER AND LAPLACE TRANSFORMATIONS

By assumption C is bounded, hence the function E determined by (4, 3) is integrable with respect to $x \in (0, \infty)$ and it is sufficient to apply the Fourier transformation with respect to x

(5. 1) $$F = \int_0^\infty E \sin \xi x \, dx, \ F = F(\xi, y, t)$$

where ξ is a transformation parameter, $0 < \xi < \infty$.

From (4. 11) and (5. 1) it follows that $F = 0\,(e^{ct})$ as $t \to \infty$, so we can use the Laplace transformation for the function F with respect to t,

(5. 2) $$G = \int_0^\infty F e^{-pt} \, dt, \quad G = G(\xi, y, p)$$

where p is a complex variable. The integral (5.2) converges if $Re\, p > c$. By virtue of (5.1), (5.2) the problem (A) is formulated as follows

(5.3) $$d^2 G / dy^2 - k^2 G = -\lambda e^{by}$$

(5.4) $$G\big|_{y=0} = f(\xi) / (k^2 - k_0^2)$$

(5.5) $$(dG / dy + bG)\big|_{y=1} = 0$$

where we used the notations:

(5.6) $$k^2 = p + \xi^2$$

(5.7) $$k_0^2 = c + \xi^2$$

(5.8) $$\lambda = \xi (Q_0 / (\xi^2 + a^2) + Q_1 / (k^2 - k_0^2))$$

(5.9) $$f(\xi) = \int_0^\infty f_2 (X) e^{-aX} \sin \xi X \, dX$$

and k is a new complex variable.

Remark 5.1.

If $a > 0$ and $f_2(X)$ is bounded on $[0, \infty]$, then $f(\xi) \in C^\infty(0, \infty)$ and $\lim_{\xi \to \infty} f(\xi) = 0$.

The latter conclusion is based on the Riemann–Lebesgue theorem (see [4]).

Solving the equation (5.3) yields a solution satisfying the conditions (5.4), (5.5). This solution is

(5.10) $$G = \Gamma(k) \frac{D(k,y)}{D(k)} - (\Gamma_0(k) + \Gamma_1(k)) \left(\frac{D(k,y) + B(k,y)}{D(k)} - e^{by} \right)$$

where we used the symbols:

(5.11) $$\Gamma(k) = f(\xi) / (k^2 - k_0^2)$$

(5.12) $$\Gamma_0(k) = Q_0 \xi / (\xi^2 + a^2)(k^2 - b^2)$$

(5.13) $$\Gamma_1(k) = Q_1 \xi / (k^2 - k_0^2)(k^2 - b^2)$$

(5.14) $B(k,y) = 2be^b \, (e^{ky} - e^{-ky})$

(5.15) $D(k,y) = (k + b) \, e^{k(1-y)} + (k - b) \, e^{-k(1-y)}$

(5.16) $D(k) \; = (k + b) \, e^k + (k - b) \, e^{-k}$.

6. SOME PROPERTIES OF FUNCTION G ON THE K—COMPLEX PLANE

To find the inverse Laplace transformation of the function G we must study this function on the k or p complex plane.

PROPOSITION 6.1.

On the k-complex plane the function G determined by (5.10) through (5.16) has simple poles only at the points:

(6.1) $k = \pm k_o$

(6.2) $k = \pm i\eta_n$, $n = 1, 2, 3, \ldots$

where η_n *are the positive roots of the following equation*

(6.3) $\eta = - b tg\, \eta.$

Proof.

It is clear that $k = \pm k_o$ are simple poles of the function G. To find other poles we solve the algebraic equation $D(k) = 0$ with $k = \xi + i\eta$; so finally we get the solution (6.2) and (6.3). From (6.3) it follows that the sequence η_n can be ordered

(6.4) $\pi/2 < \eta_1 < \eta_2 < \ldots < \eta_n < \eta_{n+1} < \ldots, \eta_n \approx (2n - 1)\, \pi/2.$

Note that the points $k = 0$ and $k = \pm b$ are not poles of the function G, because the limits of the function G exist as k tends to zero or to $\pm b$.

COROLLARY 6.1.

For each value of $\xi \in [0, \infty)$ the function G defined by (5.10) through (5.16) possesses only real simple poles, which are not greater than c on the p-complex plane, where $p = k^2 - \xi^2$.

This conclusion directly follows from (6.1), (6.2) and (5.6), (5.7).

COROLLARY 6.2.

The function G defined by (5.10) through (5.16) is analytic on the p-complex plane. If re $p > c$, *the modulus of this function has the order of* $|p|^{-1}$ *as* $|p| \to \infty$.

This corollary follows readily from Corollary 6.1, Remark 5.1, and the form of the function G.

PROPOSITION 6.2.

Let $G_0(k)$ *be the function*

(6.5) $$G_0(k) = k\, e^{k^2 t}\, G \;/\; (k^2 - \xi^2)$$

where G is defined by (5.10) *through* (5.16), *and let* Σ_n^1 *be an integral curve*

$k = \xi + i\pi n,\ 0 \leqslant \xi \leqslant \sqrt{(\pi n)^2 + k_*^2}\,,\ k_*^2 = const, < \infty$

then for $(\xi, y, t) \in \Omega = \{\xi, y, t : 0 < \xi < \infty, 0 < y < 1, 0 < t < \infty\}$ *there exists the limit*

$$(6.6) \qquad \lim_{n\to\infty} \int_{\Sigma_n^1} G_0(k)\, dk = 0.$$

Proof.

We divide Σ_n^1 into two parts Σ_n^{11}, Σ_n^{12}, $k = \xi + i\pi n$

$\Sigma_n^{11}: 0 \leqslant \xi \leqslant \pi n - \delta, \delta = \text{const} > 0$

$\Sigma_n^{12}: \pi n - \delta \leqslant \xi \leqslant \sqrt{(\pi n)^2 + k_*^2}$

On the line segment Σ_n^{11} one has an estimate

$$\left| e^{k^2 t} \right| = e^{(\xi^2 - (\pi n)^2)t} \leqslant e^{-2\pi n \delta t + \delta^2 t}$$

From Proposition 6.1 it follows that if n is sufficiently large, so the function G does not have any simple pole on Σ_n^1. Therefore we get

$$| G_0 (k) | = | G\ k\ e^{k^2 t} / (k^2 - \xi^2) | \leqslant e^{-2\pi n \delta t + \delta^2 t} \cdot M,\ M = \textit{const.} < \infty$$

Because $t > 0$, $\delta > 0$, we have

$$(6.7) \quad \left| \int_{\Sigma_n^{11}} G_o (k)\, dk \right| \to 0 \qquad \text{as } n \to \infty.$$

Later we shall examine the expression (6.6) on the line segment Σ_n^{12}. We have

$$| e^{k^2 t} | = | e^{(\zeta^2 - (\pi n)^2)t} | \leqslant e^{k_*^2 t}$$

hence

$$| G k e^{k^2 t} / (k^2 - \xi^2) | \leqslant e^{k_*^2 t}\ | k/(k^2 - \xi^2) |\ | G | = 0\,(1/n^3)$$

Thus

$$(6.8) \quad \int_{\Sigma_n^{12}} G_o (k)\, dk \to 0 \qquad \text{as} \quad n \to \infty$$

The expression (6.6) follows from (6.7), 6.8).

COROLLARY 6.3.

Let $G_o(k)$ *be defined by* (6.5) *and* Σ_n^2, $\Sigma_n^3 \cup \Sigma_n^4$ *be integral curves defined, respectively, by*

$k = \zeta + i\pi n, -\sqrt{(\pi n)^2 + k_*^2} \leqslant \zeta \leqslant 0$

$k = \zeta - i\pi n, -\sqrt{(\pi n)^2 + k_*^2} \leqslant \zeta \leqslant \sqrt{(\pi n)^2 + k_*^2}$

$k_*^2 = \textit{const} . < \infty, (\xi, y, t \in \Omega$

Then

$$\lim_{n\to\infty} \int_{\Sigma^2_n} G_o(k)dk = \lim_{n\to\infty} \int_{\Sigma^3_n \cup \Sigma^4_n} G_o(k)dk = 0$$

The proof of this corollary is quite similar to the proof of the Proposition 6.2.

PROPOSITION 6.3.

Let I_1, I_2 and J_1, J_2 be two couple points of central symmetry in the k-complex plane

$$I_{1,2} = \left(\pm \sqrt{(\pi n)^2 + h_*^2},\ \mp \pi n\right)$$

$$J_{1,2} = \left(\pm \sqrt{(\pi n)^2 + k_*^2},\ \pm \pi n\right)$$

and $G_o(k)$ be the function determined by (6.5). Then the following equality holds

(6.9) $$\int_{\Gamma_{I_1 J_1}} G_o(k)dk = \int_{\Gamma_{I_2 J_2}} G_o(k)dk$$

where $\Gamma_{I_j J_j}$ is an integral curve connecting the two points $I_j J_j$, $j = 1.2.$

Proof.

The function $G_o(k)$ defined by (6.5), (5.10) through (5.16) is antisymmetric; so changing k to-k on the left-hand side of (6.9), we obtain the equality.

7. INVERSE LAPLACE TRANSFORMATION OF THE FUNCTION G

In view of Corollary 6.2 the function G satisfies the condition on the existence of the inverse Laplace transformation. So its form is given by (see [4])

(7.1) $$F = \frac{d}{t}\left(\frac{1}{2\pi i}\int_{\Gamma_\infty} G\,\frac{e^{pt}}{p}\,dp\right)$$

wheve

(7.2) $$\Gamma_\infty = \lim_{R\to\pm\infty} \Gamma_R;\ \Gamma_R = p_* + iR,\ p_* = Re p > C$$

To evaluate this integral we make the change of variable $p = k^2 - \xi^2$, the k-complex plane is denoted by

(7.3) $$k = \xi + i\eta.$$

Thus the integral curve Γ_R on the k-plane is (this follows from (7.2), (7.3))

$$(7.4)\quad \Gamma_R: \quad \begin{cases} \zeta^2 - \eta^2 = k_*^2, & k_*^2 = p_* + \xi^2 = \text{const.} \\ 2\eta\zeta = R, & R - \text{parameter} \end{cases}$$

To calculate (7.1) on the k-complex plane we choose a closed integral curve and consider an integral $\oint G_0(k)\,dk$, where $G_0(k)$ is given by (6.5) Then we have

$$(7.5)\quad \Big(\int_{\Gamma_R^+} + \int_{\Sigma_n^1 U \Sigma_n^2} + \int_{\Gamma_R^-} + \int_{\Sigma_n^3 U \Sigma_n^4}\Big) G_0(k)\,dk = 2\pi i \sum_j \text{res } G_0(\pm k_j)$$

where Σ_n^j, $j = 1, 2, 3, 4$ are defined as in Proposition 6.2, Corollary 6.3, and $\Gamma_R^{\pm}$ (see (7.4)) are $\Gamma_{I_k J_k}$, $k = 1, 2$, respectively, as in Proposition 6.3.

By letting $R \to \infty$ and using Proposition 6.2, 6,3, and Corollary 6.3, we obtain from (7.5)

$$(7.6)\quad \int_{\Gamma_\infty} G_0(k)\,dk = 2\pi i \sum_{j=0}^{\infty} \text{res } G_0(\pm k_j).$$

We now calculate all the residues of the function $G_0(k)$. On the basis of Proposition 6.1 we see that simple poles of the function $G_0(k)$ consist of:

$$k = \pm k_0, \quad \pm i\eta_n, \quad \pm \xi; \qquad n = 1, 2, 3, \ldots$$

So we have

$$(7.7)\quad \text{res } G_0(\pm k_0) = \frac{e^{k_0^2 t}}{2(k_0^2 - \xi^2)}\left[f(\xi)\,\frac{D(k_0 y)}{D(k_0)} - \frac{Q_1 \xi}{\xi^2 + a^2}\left(\frac{D(k_0 y) + B(k_0 y)}{D(k_0)} - e^{by}\right)\right]$$

$$(7.8)\quad res\ G_0(\pm i\eta_n) = \frac{e^{-\eta_n^2 t}}{2(\xi^2 + \eta_n^2)}\left[\frac{f(\xi)}{\xi^2 + k_n^2}\cdot \mu_n(y) - \left(\frac{Q_0}{\xi^2 + a^2} - \frac{Q_1}{\xi^2 + k_n^2}\right)\cdot \xi\rho_n(y)\right]$$

in which we denote

$$(7.9)\quad \mu_n(y) = \frac{2b\eta_n \sin(\eta_n y)}{b + \cos^2 \eta_n}$$

(7.10) $\rho_n(y) = \mu_n(y)(1 - 2e^b \cos\eta_n) / (b^2 + \eta_n^2)$

(7.11) $k_n^2 = \eta_n^2 + c$

At last we find:

$$(7.12)\ res\ G_0(\pm\xi) = \frac{e^{\xi^2 t}}{2}\left[-\frac{f(\xi)D(\xi, y)}{c\,D(\xi)} - \frac{\xi}{\xi^2 - b^2}\left(\frac{Q_0}{\xi^2 - a^2} - \frac{Q_1}{c}\right)\times \times\left(\frac{D(\xi, y) + B(\xi, y)}{D(\xi)} - e^{by}\right)\right]$$

Introducing $p = k^2 - \xi^2$ and putting (7.6) into (7.1), we obtain

$$(7.13)\quad F = 2\frac{d}{dt}\left[e^{-\xi^2 t}\left(\sum_{j=0}^{\infty} res\ G_0(k_j) + res\ G_0(\xi)\right)\right]$$

Substituting (7.7), (7.8), (7.12) into (7.13) and noting that this series is uniformly convergent with respect to t, $t \in [0,\infty)$, we may interchange the order of differentiation and summation to obtain

$$F = f(\xi)\left(e^{ct}\frac{D(k_o y)}{D(k_o)} - \sum_{n=1}^{\infty}\frac{e^{-(\eta_n^2+\xi^2)t}}{\xi^2 + k_n^2}\cdot\mu_n(y)\right) + Q_0\frac{\xi\, e^{-\xi^2 t}}{\xi^2 + a^2}\sum_{n=1}^{\infty}\rho_n(y)\, e^{-\eta_n^2 t}$$

$$- Q_1\xi\left(\frac{e^{ct}}{\xi^2 + a^2}\right)\frac{D(k_o y) + B(k_o y)}{D(k_o} - e^{by}\left(+\sum_{n=1}^{\infty}\rho_n(y)\frac{e^{-(\eta_n^2+\xi^2)t}}{\xi^2 + k_n^2}\right).$$

8. PROPERTIES OF THE FUNCTION F

To find the inverse Fourier transformation of the function F, it is necessary to establish some of its properties and some auxiliary formulas.

LEMMA 8. 1.

Let

$$(8.1)\ \Phi(x,t) = \int_0^{\infty}\frac{\xi\, e^{-\xi^2 t}}{\xi^2 + a^2}\sin\xi x\, d\xi$$

$a > 0$, $(x,t) \in \omega_2 = \{x,t : 0 < x < \infty, 0 < \varepsilon \leqslant t < \infty\}$. Then

$\Phi(x,t) = O(e^{-ax})$ *as* $x \to \infty$

Proof

Let us consider the following integral

$$I = \oint \Phi_o(k)\, dk$$

where

(8.2) $$\Phi_o(k) = \frac{ke^{-k^2 t}}{k^2 + a^2}\, e^{ikx}$$

and a closed integral curve is chosen as follows

$$k = \zeta + i\eta$$

I: $-R \leqslant \zeta \leqslant R,\ \eta = 0$; II: $\zeta = R,\ 0 \leqslant \eta \leqslant \sqrt{c}$;
III: $-R \leqslant \zeta \leqslant R,\ \eta = \sqrt{c}$ IV: $\zeta = -R,\ 0 \leqslant \eta \leqslant \sqrt{c}$.

Using the residue theorem (see [7]), one can write

(8.3) $$\Big(\int_{I} + \int_{II \cup IV} + \int_{III}\Big) \Phi_o(k)dk = 2\pi i\, res\, \Phi_o(ia).$$

Now we calculate all the components of this equality.

a) On the line segment I, $k = \xi$. Because of the symmetry of the function $\Phi_0(k)$ we have

(8.4) $$\int_I \Phi_0(k)dk = 2i \int_0^R \frac{\xi e^{-\xi^2 t}}{\xi^2 + a^2} \sin \xi x\, d\xi$$

b) On the line segments II and IV, $k = \pm R = i\eta$, $0 \leqslant \eta \leqslant \sqrt{c}$, so we have the estimate

$$|\Phi_0(k)| \leqslant \frac{e^{-R^2 t}}{R}$$

from which it follows that

(8.5) $$\lim_{R\to\infty} \int_{II \cup IV} \Phi_0(k)\, dk = 0$$

c) On the line segment III, $k = \zeta + i\sqrt{c}$, $-R \leqslant \zeta \leqslant R$. Hence

(8.6) $$\int_{III} \Phi_0(k)\, dk = -e^{ct - x\sqrt{c}} \int_{R}^{-R} \frac{(\xi + i\sqrt{c})\ e^{\xi (x - 2\sqrt{c}t) i - \xi^2 t}}{(\xi^2 + a^2 - c) + 2\xi\sqrt{c}\, i}\, d\xi.$$

After some algebraic calculations, we denote by letting $R \to \infty$

(8.7) $$6(x, t) = \frac{2}{\pi} \int_0^\infty \psi(x, t, \xi) d\xi$$

$$(8.8)\qquad \psi(x,t,\xi)=\frac{\xi(\xi^2+c+a^2)\sin\varphi-\sqrt{c}\,(\xi^2+c-a^2)\cos\varphi}{(\xi^2+a^2-c)^2+4c\,\xi^2}\;e^{-\xi^2 t}$$

$$(8.9)\qquad \varphi=\xi(x-2\sqrt{c}\,t).$$

Then we obtain from (8.6)

$$(8.10)\qquad \int_{III}\Phi_0(k)\,dk=-i\pi e^{ct-x\sqrt{c}}\,\sigma(x,t).$$

Finally, for the residue of $\Phi_0(ia)$ we get:

$$(8.11)\qquad \operatorname{res}\Phi_0(ia)=\frac{1}{2}\;e^{a^2t-ax}.$$

Now, letting $R\to\infty$ in (8.4) and substituting it along with (8.5), (8.10), (8.11) into (8.3) we obtain

$$(8.12)\qquad \int_0^\infty\frac{\zeta e^{-\zeta^2 t}}{\zeta^2+a^2}\sin\zeta x d\zeta=\frac{\pi}{2}\left(e^{a^2t-ax}+e^{ct-\sqrt{c}\,x}\quad\sigma(x,t)\right).$$

On the basis of the Riemann—Lebesgue theorem (see [7]) we find $\sigma(x,t)\to 0$ as $x\to\infty$, $0<t<\infty$. Then (8.12) completes the proof.

LEMMA 8. 2.

If $k_n>\sqrt{c}$ *and* $(x,t)\in\omega_2=\{x,t:0<x<\infty,\ 0<\varepsilon\leqslant t<\infty\}$

then

$$(8.13)\qquad \int_0^\infty\frac{\zeta e^{-\zeta^2 t}}{\zeta^2+k_n^2}\sin\zeta x d\zeta=\frac{\pi}{2}\,e^{ct-\sqrt{c}\,x}\,.\,\sigma_n(x,t)$$

and

$$(8.14)\qquad \int_0^\infty\frac{e^{\zeta^{-2}t}}{\zeta^2+k_n^2}\cos\zeta x d\zeta=\frac{\pi}{2}\,e^{ct-\sqrt{c}\,x}\,.\,\delta_n(x,t)$$

where we used the notations

$$(8.15)\qquad \sigma_n(x,t)=\frac{2}{\pi}\int_0^\infty\psi_n(x,t,\zeta)\,d\zeta$$

(8.16) $$\psi_n(x, t, \zeta) = \frac{\zeta(\zeta^2 + c + k_n^2)\sin\varphi - \sqrt{c}\,(\zeta^2 + c - k_n^2)\cos\varphi}{(\zeta^2 + k_n^2 - c)^2 + 4c\zeta^2}\, e^{-\zeta^2 t}$$

(8.17) $$\delta_n(x, t) = \frac{2}{\pi}\int_0^\infty \theta_n(x, t, \zeta)\, d\zeta$$

(8.18) $$\theta_n(x, t, \zeta) = \frac{(\zeta^2 - c + k_n^2)\cos\varphi + 2\sqrt{c}\,\zeta\,\sin\varphi}{(\zeta^2 + k_n^2 - c)^2 + 4c\zeta^2}\, e^{-\zeta^2 t}$$

and φ *is defined by (8.9).*

The proof of Lemma 8.2 is similar to the proof of Lemma 8.1.

Remark 8.1.

The first component in the right-hand side of (8.12) does not appear in (8.13) or (8.14), because in Lemma 8.2 we assume $k_n > \sqrt{c}$, so that no simple pole exists on the domain of the integral and for this case the right-hand side of (8.3) is equal to zero. This explains the difference between (8.12) and (8.13) or (8.14).

Remark 8.2.

The function $\sigma(x, t)$, $\sigma_n(x,t)$ defined in Lemmas 8.1, 8.2 are of class $C^\infty(\Omega)$. Moreover, $\sigma(x,t) \to 0$ $\sigma_n(x,t) \to 0$ $\delta_n(x,t) \to 0$ as $x \to \infty$, $0 < t < \infty$; and as $t \to \infty$, $0 < x < \infty$. The first conclusion is evident. The second is based on the Riemann-Lebesgue theorem (see [7]),

Remark 8.3.

For the integral (8.1) if $t = 0$, we have the formula

(8.19) $$\int_0^\infty \frac{\zeta}{\zeta^2 + a^2}\sin\zeta x\, d\zeta = \frac{\pi}{2} e^{-ax} \quad (x > 0).$$

LEMMA 8.3.

Let

(8.20) $$\Phi_1(x,y) = \int_0^\infty \frac{\zeta}{\zeta^2 + a^2}\,\frac{D(k_0 y)}{D(k_0)}\sin\zeta x\, d\zeta$$

where $D(k,y)$, $D(k)$, k_0 *are given by (5.15), (5.16), (5.17), respectively* $a > 0$, $(x,y) \in \omega_0 = \{x,y: 0 < x < \infty, 0 < y < 1\}$. *Then*

(8.21) $$\Phi_1(x,y) = 0(e^{-ax}) \text{ as } x \to \infty.$$

Proof.

Define a function $\Phi_2(k)$ of the complex variable k as follows

(8.22) $$\Phi_2(k) = \frac{k}{k^2+a^2}\,\frac{D(K,y)}{D(K)}\,e^{ikx},\ K = k^2 + c,\ \sqrt{c} > a$$

and consider an integral $\oint \Phi_2(k)\,dk$ with a closed integral contour given by

$$k = \{-R \leqslant \zeta \leqslant R,\ \eta = 0;\ \Gamma_R : k = R\,e^{i\varphi},\ 0 \leqslant \varphi \leqslant \pi\}$$

The radius of the circle Γ_R is chosen such that

(8.23) $$R = (R_{n+1} + R_n)/2,$$

$$R_n^2 = (\pi^2/4)(2n-1)^2 + c.$$

From Proposition 6.1 it follows that the function $\Phi_2(k)$ defined by (8.22) consists of simple poles only at the points

(8.24) $$k = \pm ia, \pm ik_n$$

(8.25) $$k_n = \sqrt{\eta_n^2 + c}\ ,\ \forall n:\ k_n > a.$$

Then using the Residue Theorem (see [7]) one can write:

(8.26) $$\left(\int_{-R}^{R} + \int_{\Gamma_R}\right) \Phi_2(k)dk = 2\pi i \left(\operatorname{res} \Phi_2(ia) + \sum_{j=1}^{n} \operatorname{res} \Phi_2(ik_j)\right).$$

For the first integral on the left-hand side of (8. 26) $k = \zeta$, the real part of $\Phi_2(\zeta)$ is antisymmetric, while the imaginary part is symmetric. Therefore

(8.27) $$\int_{-R}^{R} \Phi_2(k)\,dk = 2i \int_0^R \frac{\xi}{\xi^2 + a^2}\,\frac{D(k_0, y)}{D(k_0)} \sin\xi x d\xi.$$

For the second integral we have the estimate on Γ_R

$$\left|\frac{k}{k^2+a^2}\,\frac{D(K,y)}{D(K)}\right| \leqslant \frac{1}{R}$$

This allows us to apply the Jordan Lemma (see [7]). We thus obtain

(8.28) $$\lim_{R\to\infty} \int_{\Gamma_R} \Phi_2(k)\,dk = 0$$

Finally, let us calculate the residue on the right-hand side of (8.26), we have

(8.29) $$\operatorname{res} \Phi_2\ (ia) = \frac{1}{2}\,e^{-ax-by}$$

(8.30) $$\operatorname{res} \Phi_2(i k_\eta) = -\frac{1}{2}\,\omega_\eta(y)\,e^{-k_\eta x}$$

where

(8.31) $$\omega_n(y) = 2b\eta_n \sin(\eta_n y) / (b^2 + \eta_n^2)(b + \cos^2 \eta_n)$$

At last, substituting (8.27) through (8.30) into (8.26) and letting $R \to \infty$, we obtain

(8.32) $$\Phi_1(x, y) = \frac{\pi}{2}\left(e^{-ax-by} - \sum_{n=1}^{\infty} \omega_n(y)\, e^{-k_n x}\right)$$

because $\forall n : k_n = \sqrt{\eta_n^2 + c} > a$; so (8.32) completes the proof.

LEMMA 8. 4.

Let $B(k_0, y)$, $D(k_0, y)$ be the functions defined by (5.14), (5.16), (5.7). Then

(8.33) $$\int_0^\infty \frac{\xi}{\xi^2 + a^2} \frac{B(k_0 y)}{D(k_0)} \sin(\xi x)\, d\xi =$$

$$= \frac{\pi}{2}\left[e^{-ax}\left(e^{by} - e^{-by}\right) + 2e^b \sum_{n=1}^{\infty} \omega_n(y) \cos\eta_n\, e^{-k_n x}\right]$$

where $\omega_n(y)$, η_n, k_n are given by (8.31), (6.3), (8.25), respectively, $(x, y) \in \omega_0$. The proof of this lemma is similar to the proof of Lemma 8.3.

LEMMA 8. 5.

Let $\mathcal{D}(\xi)$ be a function satisfying the following conditions
1. $\mathcal{D}(\xi)$ is an even function

(8.34) $$\mathcal{D}(k) = \mathcal{D}(-k).$$

2. On the k-complex plane, $\mathcal{D}(k)$ has simple poles only at the points $k = \pm ik_n$, which are ordered by $0 < a < k_1 < \dots k_n < k_{n+1} \dots$, $k_n \to \infty$ as $n \to \infty$, and

(8.35) $$\forall n \; |\operatorname{res} \mathcal{D}(ik_n)| \leqslant M < \infty.$$

3. On the segments $\Gamma^1_{R_n}$ defined by $\Gamma^1_{R_n} = \{k : k = R_n e^{i\varphi}, R_n = (k_{n+1} + k_n)/2, \varphi \in \left[\frac{\pi}{2} - \varphi_0, \frac{\pi}{2} + \varphi_0\right], 0 < \varphi_0 < \frac{\pi}{2}\}$, the following holds;

(8.36) $$|\mathcal{D}(k)| \leqslant M < \infty \qquad \text{as } R_n \to \infty.$$

4. *On the segments* $\Gamma^2_{R_n}$ *defined by* $\Gamma^2_{R_n} = \{k : k = R_n e^{i\varphi},$

$R_n = (k_{n+1} + k_n)/2, \varphi \in \left[0, \varphi_o\right] U \left[\frac{\pi}{2} + \varphi_o, \pi\right], 0 < \varphi_o < \frac{\pi}{2}\}$, *the following holds:*

(8.37) $\forall_m :\ |\mathcal{D}(k)k^m| \to 0$ *as* $R_n \to \infty$.

If $\mathcal{D}_1(u) = \int_0^\infty \mathcal{D}(\xi) \cos \xi u d\xi$, *then*

(i) $|\mathcal{D}_1(u)| = O(e^{-k_1 u})$ *as* $u \to \infty$.

(ii) $\mathcal{D}_1(u) \in C^\infty\ (0 < \varepsilon \leqslant u < \infty)$.

Proof.

Define an auxiliary function $\mathcal{D}_2(k) = \mathcal{D}(k) e^{uki}$. Integrating the function $\mathcal{D}_2(k)$ on the closed contour as in the proof of Lemma 8.3. On the basis of the second assumption of the lemma, and using the residue theorem we obtain

$$(8.38)\quad \left(\int_{-R}^{R} + \int_{\Gamma^1_{R_n}} + \int_{\Gamma^2_{R_n}}\right) \mathcal{D}_2(k)\, dk = 2\pi i \sum_{j=1}^{n} \text{res}\ \mathcal{D}_2(ik_j).$$

Let us calculate all the components of this equality. For the first integral on the left-hand side of (8.38), it follows from the condition (8.34) that

$$(8.39)\quad \int_R^R \mathcal{D}_2(k)dk = 2\int_0^R \mathcal{D}(\xi) \cos \xi u\, d\xi.$$

On the segment $\Gamma^1_{R_n}$, using (8.36) one has the estimate

$$|\mathcal{D}_2(k)| = |\mathcal{D}(k)|\ e^{-R_n u, \sin\varphi} \leqslant M . e^{-R_n u \sin\varphi_o}$$

Therefore,

$$\left|\int_{\Gamma^1_{R_n}} \mathcal{D}_2(k)dk\right| \leqslant M \int_{\Gamma^1_{R_n}} e^{-R_n u \sin\varphi_o} . R_n . d\varphi.$$

Since $0 < \varepsilon \leqslant u < \infty$, $0 < \varphi_o < \pi/2$, this implies that

$$(8.40)\quad \int_{\Gamma^1_{R_n}} \mathcal{D}_2(k)dk \to 0 \quad \text{as} \quad R_n \to \infty.$$

On the segment $\Gamma^2_{R_n}$ we have the estimate

$$|\mathcal{D}_2(k)| \leqslant |\mathcal{D}(k)|\ e^{-R_n u . \sin\varphi}$$

hence

$$\left| \int_{\Gamma^2_{R_n}} \mathcal{D}_2(k)dk \right| \leqslant \int_{\Gamma^2_{R_n}} |\mathcal{D}(k)| R_n d\varphi$$

From (8.37) it follows that

(8.41) $$\left| \int_{\Gamma^2_{R_n}} \mathcal{D}_2(k)dk \right| \to 0 \qquad \text{as } R_n \to \infty.$$

Finally, the residues in (8.38) have the form

(8.42) $$\text{res}\ \mathcal{D}_2(ik_n) = \text{res}\ \mathcal{D}(ik_n)\ e^{-uk_n}$$

Putting (8.39) through (8.42) into (8.38) and letting $n \to \infty$ we obtain:

(8.43) $$\int_0^\infty \mathcal{D}(\xi) \cos \xi u\, d\xi = e^{-uk_1} \pi i \sum_{n=1}^{\infty} res\ \mathcal{D}(ik_n)\, e^{-u(k_n - k_1)}$$

From (8.35) it follows that the previous series is uniformly convergent with respect to $u (0 < \varepsilon \leqslant u < \infty)$, therefore, the first conclusion of the Lemma is proved by (8.43). The second conclusion of the Lemma is directly justified by (8.37).

LEMMA 8. 6.

Let

(8.44) $$F_1(x) = e^{ax} \int_0^\infty f(\xi)\mathcal{D}(\xi) \sin \xi x\, d\xi$$

(8.45) $$f(\xi) = \int_0^\infty f_2(\chi) e^{-a\chi} \sin \xi \chi\, d\chi.$$

If $\mathcal{D}(\xi)$ satisfies the four conditions of Lemma 8.5 and if

(8.46) $$f_2(\chi) \leqslant M < \infty,\ \chi \in [0, \infty]$$

Then

(8.47) $$|F_1(x)| \leqslant M < \infty \qquad as\ x \to \infty$$

(8.48) $$F_1(x) \in C^\infty\ (0 < \varepsilon \leqslant x < \infty)$$

(8.49) $$\forall m\ \ |d^m F_1(x)/dx^m| \leqslant M < \infty\ \ as\ x \to \infty.$$

Proof.

Substituting (8.45) into (8.44), and using the assumption (8.37) we may interchange the order of integrations, so

$$F_1(x) = e^{ax}\int_0^\infty f_2(X)\, e^{-ax}\,(\int_0^\infty \mathcal{D}(\xi)\, \sin \xi x\, \sin \xi X\, d\xi)\, dX =$$

$$= \frac{e^{ax}}{2}\left(\int_0^\infty f_2(X)\, e^{-aX}\left(\int_0^\infty \mathcal{D}(\xi)\, \cos(x-X)\xi\, d\xi\right) dX + \right.$$

$$+ \int_x^\infty f_2(X)\, e^{-aX}(\int_0^\infty \mathcal{D}(\xi)\, \cos(X-x)\xi\, d\xi) dX -$$

$$- \int_0^\infty f_2(X) e^{-aX}(\int_0^\infty \mathcal{D}(\xi)\, \cos(X+x)\,\xi\, d\xi)\, dX).$$

Making the change of variables $u = x - X$ for the first integral, $u = X - x$ for the second one, and choosing a constant X_0 such that $0 < X_o < x$, we find

$$2E1(x) = \int_0^{X_0} (f_2(x-u)\, e^{au} + f_2(x+u)\, e^{-au})\, (\int_0^\infty \mathcal{D}(\xi)\, \cos \xi u d\zeta)\, du +$$

$$+ \int_{X_0}^{x} f_2(x-u)\, e^{au} \int_0^\infty \mathcal{D}(\xi)\, \cos\xi u d\xi\, .\, du \quad +$$

$$+ \int_{X_0}^{\infty} f_2(x+u)\, e^{-au} \int_0^\infty \mathcal{D}(\xi)\, \cos\xi u d\xi\, .\, du \quad -$$

$$- e^{ax} \int_0^\infty f_2(X)\, e^{-aX} \int_0^\infty \mathcal{D}(\xi)\, \cos(X+x)\,\zeta\, d\zeta\, .\, dX$$

where for the second integral: $0 < X_0 \leqslant u \leqslant x$. for the third: $0 < X_0 \leqslant u \leqslant \infty$, and for the fourth: $0 < \varepsilon \leqslant X + x = \infty$. This allows us to apply Lemma 8.5 to the above formulations (see (8.43)), hence

$$2F_1(x) = \int_0^{X_0} [(f_2(x-u)e^{au} + f_2(x+u)e^{-au}) \int_0^\infty \mathcal{D}(\xi)\, \cos\xi u d\xi\,]\, du +$$

$$+ \int_{X_0}^{x} f_2(x-u)\, e^{-(k_1-a)u}(\pi i \sum_{n=1}^{\infty} \operatorname{res}\mathcal{D}(ik_n)e^{-u(k_n - k_1)})\, du +$$

(8.50)
$$+ \int_{X_0}^{\infty} f_2(x+u)\, e^{-(k_1+a)u}(\pi i \sum_{n=1}^{\infty} \operatorname{res}\mathcal{D}(ik_n)e^{-u(k_n - k_1)})\, du -$$

$$- e^{-(k_1-a)x} \int_0^\infty f_2(X)\, e^{-(k_1+a)X} \times$$

$$\times\ (\pi i \sum_{n=1}^{\infty} \operatorname{res}\mathcal{D}(ik_n)e^{-(X+x)(k_n - k_1)})\, dX.$$

From the condition (8.35) for the function $\mathcal{D}(\xi)$ it follows that the series appearing in (8.50) are absolutely convergent on the path of integrations. Since $f_2(X)$ is bounded (see (8.46)), which implies that all the integrals are bounded, the first conclusion of the lemma is proved provided $k_1 - a > 0$.

The second conclusion is directly proved by assumption (8.37) and the form of the function $F_1(x)$.

For the third conclusion (8.49) the result follows in a manner similar to that used in the first one.

LEMMA 8.7.

Let $E_n(x, t)$ *be*

(8.51) $$E_n(x,t) = e^{ax}\int_0^\infty f(\xi)\,\frac{e^{-\xi^2 t}}{\xi^2 + k_n^2}\sin\xi x\, d\xi$$

(8.52) $$f(\xi) = \int_0^\infty f_2(X)\, e^{-aX}\sin\xi X\, dX$$

(8.53) $|f_2(X)| \leqslant M < \infty,\ X \in [0, \infty]$

$0 < a < k_1 < \dots < k_n < k_{n+1} < \dots, k_n \to \infty$

$(x, t) \in \omega_2 = \{ x, t : 0 < \varepsilon \leqslant x < \infty,\ 0 < \varepsilon \leqslant t < \infty\}.$

Then

(8.54) $|E_n(x,t)| \leqslant M < \infty$ *as* $x \to \infty,\ 0 < t < \infty$; *0Г* $t \to \infty, 0 < x < \infty$.

(8.55) $E_n(x, t) \in C^\infty(\omega_2)$

(8.56) $\forall_m : \partial^m E_n(x,t) / \partial x^m,\ \partial^m E_m(x,t) / \partial t^m$ *are bounded as* $x \to \infty,\ 0 < t < \infty$, *or* $t \to \infty,\ 0 < x < \infty$.

Proof.

Adopting the approach used in the proof of Lemma 8.6 we obtain

$$2E_n(x,t) = \int_0^{X_0} (f_2(x-u)\,e^{au} + f_2(x+u)\,e^{-au}) \int_0^\infty \frac{e^{-\xi^2 t}}{\xi^2 + k_n^2}\cos\xi u\, d\xi\, du +$$

$$+ \int_{X_0}^{x} f_2(x-u)\,e^{au} \int_0^\infty \frac{e^{-\xi^2 t}}{\xi^2 + k_n^2}\cos\xi u\, d\xi\, du +$$

$$+ \int_0^\infty f_2(x+u)\,e^{au} \int_0^\infty \frac{e^{-\xi^2 t}}{\xi^2 + k_n^2}\cos\xi u\, d\xi\, du -$$

$$- e^{au}\int_0^\infty f_2(X)\,e^{-ax}\int_0^\infty \frac{e^{-\xi^2 t}}{\xi^2 + k_n^2}\cos(X+x)\,\xi\, d\xi\, dX.$$

By using Lemma 8.2 (see (8.14)) it can be written:

$$2E_n(x, t) = \int_0^{X_0} (f_2(x-u)e^{au} + f_2(x+u)e^{-au}) \int_0^{\infty} \frac{e^{-\xi^2 t}}{\xi^2 + k_n^2} \cos\xi u d\xi du +$$

$$+ e^{ct} [\int_{X_0}^{x} f_2(x-u)e^{-(\sqrt{c}-a)u} \sigma_n(u, t) du +$$

$$+ \int_{X_0}^{\infty} f_2(x+u) e^{(\sqrt{c}+a)u} \cdot \delta_n(u, t) du -$$

$$- e^{-(\sqrt{c}-a)x} \int_0^{\infty} f_2(X) e^{-(\sqrt{c}+a)X} \cdot \delta_n(X+x, t) dX]$$

where $X_o = \text{const.}$, $0 < X_o < x$, $\sqrt{c} - a > 0$, $\delta_n(u, t)$ defined by (8.17), (8.18), (8.9). Since $f_2(X)$ and $\delta_n(u, t)$ are bounded for $0 < t < \infty$ from the above expression it follows that $E_n(x, t)$ is bounded as $x \to \infty$.

The conclusion (8.56) for the case of $x \to \infty$, $0 < t < \infty$ will be proved in the same way as previously. For the case $t \to \infty$, $0 < x < \infty$, (8.54) and (8.56) are directly proved by (8.51). Finally, (8.55) is an immediate consequence of (8.51) through (8.53). This completes the proof.

9. SOLUTION OF THE PROBLEM (A)

In this section we find the inverse transformation of the function given by (7. 14). For $0 < y < 1$, $0 < t < \infty$, $0 < \xi < \infty$, the function F is bounded, $F \to 0$ as $\xi \to \infty$, hence the inverse Fourier transformation exists, and we have

(9.1) $$E = \frac{2}{\pi} \int_0^{\infty} F \sin \xi x \, d\xi.$$

To find the concentration function C that is a solution of the problem (1. 14) through (1. 20), (4. 1) (4. 2) put (4. 3) into (9. 1)

(9.2) $$C = \frac{2}{\pi} e^{ax - by - ct} \int_0^{\infty} F \sin \xi x \, d\xi.$$

For convenience some notations will be introduced here

(9.3) $$\alpha = k_1 - a = \sqrt{\eta_1^2 + a^2 + b^2} - a$$

(9.4) $$\delta = \sqrt{c} - a = \sqrt{a^2 + b^2} - a$$

(9.5) $$\beta_n = k_n - k_1$$

(9.6) $$\nu_n = \eta_n^2 - \eta_1^2$$

We also denote

(9.7) $$C_1(x, y) = \frac{2}{\pi} e^{ax - by} \int_0^\infty f(\xi) \frac{D(k_0, y)}{D(k_0)} \sin \xi x \, d\xi$$

(9.8) $$C_2(x, y, t) = \frac{2}{\pi} e^{-by - (c+\eta_1^2)t} \sum_{n=1}^\infty e^{-\nu_n t} \mu_n(y) E_n(x,t)$$

where $f(\xi)$, $D(k, y)$, $D(k)$, k_0, $\mu_n(y)$, $E_n(x, t)$ are defined by (5.9), (5.15), (5.16), (5.7), (7.9), (8.51), respectively,

(9.9) $$C_3(y, t) = e^{-by-(\eta_1^2 + b^2)t} \sum_{n=1}^\infty e^{-\nu_n t} \rho_n(y)$$

(9.10) $$C_4(x, y, t) = e^{-\delta x - by - \eta_1^2 t} \sigma(x, t) \sum_{n=1}^\infty e^{-\nu_n t} \rho_n(y)$$

in which $\rho_n(y)$, $\sigma(x, y)$ are given by (7.10) and (8.7) through (8.9), respectively. Finally we denote

(9.11) $$C_5(x, y) = e^{-\alpha x - by} \sum_{n=1}^\infty e^{-\beta_n x} \rho_n(y)$$

(9.12) $$C_6(x, y, t) = e^{-\delta x - by - \eta_1^2 t} \sum_{n=1}^\infty e^{-\nu_n t} \rho_n(y) \sigma_n(x, t)$$

On the basis of (9.3) through (9.12), substituting the function F defined by (7.14) into (9.2) and using Lemmas 8.1 through 8.4, and Remark 8.3, we get

(9.13) $$C(x, y, t) = C_1(x, y) - C_2(x, y, t) + Q_0 (C_3(y, t) + C_4(x, y, t)) + \\ + Q_1 (C_5(x, y) - C_6(x, y, t)).$$

The last point is to check whether the function $C(x, y, t)$ satisfies the conditions (1.19) and (1.20) because they are not equivalent to the conditions (4.10), (4.11) for the function E.

The remaining conditions and the equation (1.14) are automatically satisfied by the function C, because the transformations $C \leftrightarrow E \leftrightarrow F \leftrightarrow G$ are bijective.

Using Lemmas 8.6, 8.7 we establish that the function $C(x, y, t)$ determined by (9.13) actually satisfies these conditions and it is therefore the solution of the mentioned problem.

THEOREM 9.1. *The solution of the problem (1.14) through (1.20), (4.1), (4.2) exists, is unique and belongs to the class of* $C^\infty(\Omega)$, *where* $\Omega = \{x, y, t : 0 < \varepsilon \leqslant x < \infty,\ 0 < y < 1,\ 0 < \varepsilon \leqslant t < \infty\}$.

Proof.

First consider the differentiability of the functions C_j, $j = 1, \dots, 6$, with respect to x. The functions $f(\xi)$, $D(k_0, y) / D(k_0)$ in (9.7) satisfy all the conditions for the function $f(\xi)$ and $\mathcal{D}(\xi)$, respectively, of Lemma (8.6) ; it follows that $C_1(x, y)$ belongs to the class $C^\infty(0 < \varepsilon < x < \infty)$. On the basis of Lemma 8.7 we conclude that $C_2(x, y, t)$ belongs to the class $C^\infty(0 < \varepsilon \leqslant x < \infty)$ too. Further it is evident that $C_j \in C^\infty(0 < \varepsilon \leqslant x < \infty)$, $J = 3, 4, 5, 6$. Similarly we arrive at the conclusion $C(x, y, t) \in C^\infty(\Omega)$.

Secondly, we show the uniqueness of the solution. Suppose that C^o and C^{oo} are two solutions of this problem, then $C^* = C^o - C^{oo}$ is a solution of the same problem, in which $Q_o = Q_1 = f_2(x) = 0$. Once more, we use the transformations $C^* \leftrightarrow E^* \leftrightarrow F^* \leftrightarrow G^*$. Repeating the procedure of finding the function G^* we obtain $G^* = 0$, hence $F^* = 0$, $E^* = 0$ and consequently $C^* = 0$. This completes the proof.

CONSEQUENCE 9.1.

In the problem (1.14) through (1.20), (4.1), (4.2) the continuity of the boundary conditions is sufficient but not necessary for the continuity of the solution in the interior of the domain.

Proof.

In Theorem 9. 1 we only assumed $f_2(x)$ to be a bounded function, besides, at the points $x = 0$, $y = 0$ the boundary condition may be discontinuous.

CONSEQUENCE 9.2.

From the solution (913) we obtain the following asymptotic solutions for this problem

1. *for large* t:

(9. 14) $C_T(x,y,t) = C_1(x,y) + Q_1\, C_5(x,y) + 0\,(e^{-\eta_1^2 t})$

2. *for large* x:

(9. 15) $C_X(x,y,t) = C_1(x,y) - C_2(x,y,t) + Q_o\, C_3(y,t) + 0\,(e^{-\delta x})$

3. *for large* x *and* t:

(9. 16) $C_{XT}(x,y,t) = C_1(x,y) + 0\,(e^{-\eta_1^2 t}) + 0\,(e^{-\delta x})$

10. AN ASYMPTOTIC SOLUTION FOR THE MORE GENERAL BOUNDARY CONDITIONS

In this section we remove the assumption (4.2), i e. we consider the problem (1. 14) through (1. 20), (4.1). We shall prove two theorems.

THEOREM 10.1.

An asymptotic solution of the problem (1.14) through (1.20), (4.1) for large x and t is expressed just by the formula (9. 16).

Proof.

Denote by $C^{--}(x,y,t)$, $C^{++}(x,y,t)$ the solutions of the problem (1. 14) through (1. 20), (4. 1) under the additional assumption (2.14) and (2.15), respectively. If $C(x,y,t)$ is a solution of the mentioned problem, then by virtue of Theorem 2,2 there holds the estimate $C^{--} \leqslant C \leqslant C^{++}$. On the other hand, from Consequence 9.2 we see that the asymptotic forms of C^{--} and C^{++} for large x and t are expressed by the same formula (9. 16). This completes the proof.

THEOREM 10.2.

Denote by $C_X(x, y, t)$, $C_T(x, y, t)$ the asymptotic solutions of the problem (1.14) through (1.20), (4.1) for large x and t, respectively.

(i) If $f_0(x, y) \equiv 1$, $(x, y) \in (0, \infty)$ x $(0, 1)$, then $C_X(x, y, t)$ is expressed by (9.15).

(ii) If $f_1(y, t) \equiv 1$, $(y, t) \in (0, 1)$ x $(0, \infty)$, then $C_T(x, y, t)$ is expressed by (9.14).

The proof of this theorem is similar to the proof of Theorem 10.1.

11. AN APPLICATION

In this section we use the results of § 9 and § 10 to examine the distribution of suspended sediments in the channels, which have a natural structure of a bottom (grains of sand) so that the following condition (in nondimensional form) is satisfied

$$C|_{y=0} = f_2(x) \equiv 1 \tag{11.1}$$

This condition has previously been used in [2].

Using the condition (11.1) in the general formula (9.13), with the help of Lemmas 8.1, 8.3 we obtain

$$C(x, y, t) = e^{-2by} - e^{-\alpha x - by} \sum_{n=1}^{\infty} \omega_n(y)\, e^{-\beta_n x} -$$

$$- e^{-by-(b^2+\eta_1^2)\, t} \sum_{n=1}^{\infty} \omega_n(y)\, e^{-\nu_n t} -$$

$$- e^{-\delta x - by - \eta_1^2 t} \sum_{n=1}^{\infty} \omega_n(y)\, e^{-\nu_n t}\, (\delta(x, t) - \sigma_n(x, t)) + \quad (11.2)$$

$$+ Q_0 [C_3(y, t) + C_4(x, y, t)] + Q_1 [C_5(x, y) - C_6(x, y, t)].$$

where $\omega_n(y)$, $\sigma(x, t)$, $\sigma_n(x, t)$, $C_3(y, t)$, $C_4(x, y, t)$, $C_5(x, y)$ and $C_6(x, y, t)$ are given by (8.31), (8.7), (8.15), (9.9), (9.10), (9.11) and (9.12), respectively.

On the basis of Theorems 10.1, 10.2, from (11.2) we obtain asymptotic forms of the solution for the more general case of initial and boundary conditions as follows

(i) For large x and t, and for all the initial conditions and all the boundary conditions of the source ($x = 0$), i.e. $f_0(x, y)$ and $f_1(y, t)$ are arbitrary functions, we get

$$C_{XT}(x, y, t) = e^{-2by} + O(e^{-\delta x}) + O(e^{-\eta_1^2 t}) \quad (11.3)$$

(ii) for large x, $f_1(y, t)$ being arbitrary function, $f_0(x, y) = 1$, we get

$$C_X(x, y, t) = e^{-2by} - e^{-by-(b^2+\eta_1^2)t} \sum_{n=1}^{\infty} \omega_n(y) e^{-\nu_n t} + \quad (11.4)$$

$$+ Q_0 C_3(y, t) + O(e^{-\delta x}).$$

(iii) for large t, $f_0(x, y)$ being arbitrary function, $f_1(y, t) = 1$, we get

$$C_T(x, y, t) = e^{-2by} - e^{-\alpha x - by} \sum_{n=1}^{\infty} \omega_n(y)\, e^{-\beta_n x} - \quad (11.5)$$

$$- Q_1 C_5(x, y) + O(e^{-\eta_1^2 t})$$

where $\bar{\omega}_n(y)$, $C_3(y, t)$, $C_5(x, y)$ are given by (8.31) (9.9), (9.11), respectively.

1. Influence of the coefficients of the diffusion equation on the asymptotic solutions

Assuming that the coefficients of the diffusion equation are constant we have found the analytic and asymptotic solutions for the diffusion problem (see (11.2),..., (11.5)). The function $C_{XT}(x, y, t)$ given by (11.3) satisfies all the assumptions of Theorem 3.1, therefore the asymptotic solution of the more general problem (1.14) through (1.20), (3.9) (in relation to the coefficients $u(x, y)$, $\alpha_1(y)$) is expressed by the same form (11.3), because the nondimensional coefficient b appearing in (11.3) does not depend on the flow velocity U and the diffusion coefficient K_1(see (1.21)). We thus come to the following conclusion:

If the function of the flow velocity U_0 changes in some interval of values such that the condition (11.1) at the bottom of channels is valid and if t and x are large enough, then the distribution of suspended sediments in the flow depends, in an asymptotic sense, neither on the flow velocity nor on the diffusion coefficient K_1.

The property that the particle distribution is independent of the flow velocity is justified by the experimental data given in [1].

2. Influences of the initial condition and the boundary condition at the inlet of the channel ($x = 0$) on the asymptotic distribution of suspended sediments in the flows.

From Theorem 10.1 we may conclude that, if t is large enough, then the initial distribution of particles in the flow does not cause nearly any influence on the later distribution of suspended sediments. Similarly, at a great distance from the source located at the cross section $x = 0$, the distribution of suspended sediments in the flow does not almost depend on the source intensity of sediments expressed by the function $f_1(y, t)$.

3. Stability of the distribution of suspended sediments in the flows

From the asymptotic solution (9.16) it follows that there exists a state of the distri bution of particles in the flow, asymptotically expressed by the function $C_1(x, y)$ it is independent of functions $Q_0 f_0(x, y)$ and $Q_1 f_1(y, t)$ (see (9.7)). So in relation to the relaxation of influences of the initial condition and of the inlet condition of sediments at the cross section $x = 0$, we shall call it the stability of the asymptotic distribution of suspended sediments in the flow.

The state of stable distribution of sediments in the flow examined in § 11 is expressed by (see (11. 3))

$$C = c^{-2by}, \quad b = VH/2K_2. \tag{12.1}$$

The formula (12.1) coincides with the formula (2) in [1] (p.64), which is consistent with the experiments for the suspended sediments of grains of sand diameter between 0,1 and 0,6 mm.

From the stability of the distribution of suspended sediments in the flow we may conclude that the ability of transport of suspended sediments of the flow for any channel can be strictly determined; it is limited and does not depend on the source intensity.

(Received January, 14, 1984)

REFERENCES

[1] Antsyferov, S. M. and Debol'skiy, V.K.: *On laws governing distribution of suspended sediments in open flow.* Meteorologiya i Gidrologiya 1972, No. 5, 63 – 68.

[2] Apmann, R. P. and Rumer, R. R.: *Diffusion of sediment in developing flow.* J. of the Hydraulics Division, ASCE 96, No. HY1 (Proc. Paper 7018) (1970), 109–123.

[3] Aronsson, G.: *Diffusion of sediment in long channels. J. of the Hydraulics Division,* ASCE 104, No. HY6 (Proc. Paper 13845) (1978), 821–837.

[4] Ditkin, V. A. and Prudnikov, A. P.: *Integral transformations and operational calculus.* Moscow, 2nd ed., 1974.

[5] Graf, W.H.: *Hydraulics of sediment transport,* New York: McGraw Hill Book Co., Inc., 1971.

[6] Ladyzhenskaya O. A., Solonnikow V. A. and Uraltseva N.N.. *Linear and Quasi–linear Equations of Parabolic Type* (in Russian), Moscow, «Nauka» 1967.

[7] Lavrentiev. M.A. and Sabat, B.V.: *Methods for the theory of complex variable functions.* Moscow, 1965.

[8] Nguyen Van Gia: *A solution of boundary value problem for an equation of the parabolic type. J. Phuong Phap Toan Ly (J. Methods of Physics–Mathematics)* (Hanoi) 1977, No.2, 21–35.

[9] Nguyen Văn Gia: *Rozwiazanie rownania dyfuzji zawiesiny czasteczek w kanale.* IFTR Reports No. 10 (Warszawa 1981).

[10] Nguyen Van Gia: *A solution of the diffusion equation for the distribution of suspended sediments in long channels,* Archiwum Hydrhtechniki, T. XXIX, № 1 – 2, 1982, 77 – 90.

[11] Protter, M.H. and Weinberger, H. F. *Maximum principles in differential equations.* Prentice Hall, N. J., 1967.

[12] Sayre, W.W: *Dispersion of mass in open-channel flow. Hydrology Papers* No. 75, Colorado State University, Fort Collnis, Col., Aug. 1975.