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UNIQUE DETERMINATION OF ANY HARMONIC FUNCTION FROM ITS VALUES GIVEN ON THE POINTS OF TWO CONVERGENT SEQUENCES

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This paper is in the setup of real numbers. Let $u(x, y)$ be a harmonic function (i.e., $u_{xx}(x, y) + u_{yy}(x, y) = 0$) in an open disk D with center at the origin (0, 0). It is well known [1, p.98], [2, p.212] that $u(x, y)$ is the real part of an analytic function $F(z)$ of a complex variable z where $z = x + iy$ with $(x, y) \in D$. As such $u(x, y)$ has a power series expansion in x and y valid in D given by:

$$u(x, y) = \mathrm{Re} \sum_{n=0}^{\infty} (a_n + ib_n)(x + iy)^n \tag{1}$$

It can be readily verified that (1) implies that $u(x, y)$ can be expressed as a sum of infinitely many homegeneous polynomials $p_n(x, y)$ of degree n with $n = 0, 1, 2, \ldots$ where the coefficients of the terms x^n, $x^{n-1}y, \ldots, y^n$ in $p_n(x, y)$ occur according to a rather simple pattern. Indeed, expanding the right side of the equality sign in (1) according to the «binomial rule» and preserving the real terms after performing the necessary multiplications, we obtain:

$$\begin{aligned} u(x, y) = a_0 &+ (a_1 x - b_1 y) + (a_2 x^2 - 2b_2 xy - a_2 y^2) \\ &+ (a_3 x^3 - 3b_3 x^2 y - 3a_3 xy^2 + b_3 y^3) \\ &+ (a_4 x^4 - 4b_4 x^3 y - 6a_4 x^2 y^2 + 4b_4 xy^3 + a_4 y^4) \end{aligned} \tag{2}$$

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$$+(a_5x^5 - 5b_5x^4y - 10a_5x^3y^2 + 10b_5x^2y^3 + 5a_5xy^4 - b_5y^5)$$

$$+(a_6x^6 - 6b_6x^5y - 15a_6x^4y^2 + 20b_6x^3y^3 +$$

$$+ 15a_6s^2y^3 - 6b_6xy^5 - a_6y^6)$$

$$+ \dots\dots$$

$$+(a_nx^n - \binom{n}{1}b_nx^{n-1}y - \binom{n}{2}a_nx^{n-2}y^2 +$$

$$+ \binom{n}{3}b_nx^{n-3}y^3 + \dots + B_ny^n)$$

$$+ \dots\dots$$

so that

(3) $u(x,y) = p_0(x, y) + p_1(x, y) + \dots + p_n(x, y) + \dots$

where

(4) $p_n(x,y) = (a_nx^n - \binom{n}{1}b_nx^{n-1}y - \binom{n}{2}a_nx^{n-2}y^2 + \binom{n}{3}b_nx^{n-3}y^3 +$

$+ \dots + B_ny^n)$

with

(5) $B_n = a_n$ *if* $n = 0, 4, 8, \dots$ *and* $B_n = -a_n$ *if* $n = 2, 6, 10, \dots$

and

(6) $B_n = b_n$ *if* $n = 3, 7, 11, \dots$ *and* $B_n = -b_n$ *if* $n = 1, 5, 9, \dots$

Clearly, as (2) or (3) and (4) show, $u(x,y)$ is uniquely determined by the values of $a_0, a_1, b_1, a_2, b_2, \dots$. Let us also observe that as (4) shows in (3) every polynomial $p_n(x,y)$ with $n > 0$ is uniquely determined by the values of only two coefficients a_n and b_n.

Next, let e be a transcendental real and $(p_k, 0)$ and (q_k, eq_k) with $k = 1, 2, \dots$ be two sequences in D each converging to $(0, 0)$, respectively along the x-axis and along the line of slope e passing through (0,0).

Thus,

(7) $\lim_{k\to\infty} (p_k, 0) = \lim_{k\to\infty} (q_k, eq_k) = (0,0)$ *with* $p_k \neq 0 \neq q_k$

Also, let the following real numbers

(8) $u(p_1, 0), u(p_2, 0), u(p_3, 0), \dots, u(q_1, eq_1), u(q_2, eq_2), u(q_3, eq_3), \dots$

be given which represent the values of $u(x,y)$ at the points of the two sequences mentioned in (7).

Now, based on (7) and (8), we determine (uniquely) the values of a_k, s and b_k, s in (2), which in turn, determine uniquely the values of $u(x,y)$ in the entire D.

To determine a_0 let us take from both sides of the equality (2) limit

(9) $\quad$ as $k \to \infty$ with $(x, y) = (p_k, 0)$

Since $u(x,y)$ is harmonic in D, clearly $\lim_{k \to \infty} u(P_k, 0)$ in uniquely determined (in fact is equal to $u(0,0)$) by its values $u(p_1, 0)$, $u(p_2, 0)$, $u(p_3, 0)$, ... as given in (8). Also, in view of (7), it follows that the limit (according to (9)) of the series immediately to the right of a_0 in (2) is equal to 0 since $y = 0$ throughout (9). Hence

(10) $$a_0 = \lim_{k \to \infty} u(p_k, 0)$$

and therefore a_0 is uniquely determined by (8).

To determine a_1 let us subtract a_0 from both sides of the equality (2) and then divide both sides by x and then take from both sides limit according to (9). From (7) it follows that the limit (according to (9)) of the product of x^{-1} and the series immediately to the right of $a_1 x$ in (2) is equal to 0 since $y = 0$ throughout (9). Hence (using (10)),

(11) $$a_1 = \lim_{k \to \infty} \frac{u(p_k, 0) - a_0}{p_k}$$

and therefore a_1 is uniquely determined by (8).

To determine b_1 let us subtract $a_0 + a_1 x$ from both sides of the equality (2) and then divide both sides by $-y$ and then take from both sides limit

(13) $\quad$ as $k \to \infty$ with $(x, y) = (q_k, eq_k)$

From (7) it follows that the limit (according to (13)) of the product of y^{-1} and the series immediately to the right of $-b_1 y$ in (2) is equal to 0 since $xy^{-1} = e^{-1}$ throughout (13). Hence (using (10) and (11)),

(14) $$b_1 = \lim_{k \to \infty} \frac{u(q_k, eq_k) - a_0 - a_1 q_k}{-eq_k}$$

and therefore b_1 is uniquely determined by (8).

To determine a_2 let us subtract $a_0 + a_1 x - b_1 y$ from both sides of the equality (2) and then divide both sides by x^2 and then take from both sides limit according to (9). From (7) it follows that the limit (according to (9)) of the product of x^{-2} and the series immediately to the right of $a_2 x^2$ in (2) is equal to 0 since $y = 0$ throughout (9). Hence (using (10) and (11)),

$$(15) \qquad a_2 = \lim_{k \to \infty} \frac{u(p_k, 0) - a_0 - a_1 p_k}{p_k^2}$$

and therefore a_2 is uniquely determined by (8).

We observe that a_2 which is already determined occurs twice in $(a_2 x^2 - 2b_2 xy - a_2 y^2)$ which appears in (2). Thus, to determine b_2 let us subtract $a_0 + a_1 x - b_1 y + a_2 x^2 - a_2 y^2$ from both sides of the equality (2) and then divide both sides by $-2xy$ and then take from both sides limit according to (13). From (7) it follows that the limit (according to (13)) of the product $(-2xy)^{-1}$ and the series immediately to the right of $-2b_2 xy$ in (2) is equal to 0 since $xy^{-1} = e^{-1}$ and $yx^{-1} = e$ throughout (13). Hence (using ((10, (11), (14), (15)),

$$(16) \qquad b_2 = \lim_{k \to \infty} \frac{u(q_k, eq_k) - a_0 - a_1 q_k + b_1 e q_k - a_2 q_k^2 + a_2 e^2 q_k^2}{-2eq_k^2}$$

and therefore b_2 is uniquely determined by (8).

To determine a_3 we proceed analogously to the case of a_2 and obtain:

$$(17) \qquad a_3 = \lim_{k \to \infty} \frac{u(p_k, 0) - a_0 - a_1 p_k - a_2 p_k^2}{p_k^3}$$

and therefore a_3 is uniquely determined by (8).

Since a_3 is already determined, we use

$$(18) \qquad (a_3 x^3 - 3b_3 x^2 y - 3a_3 xy^2 + b_3 y^3),$$

which appears in (2), to determine b_3. We observe that the part of (18) which involves b_3 is given by:

$$(19) \qquad x^2 y \left(-3 + \left(\frac{y}{x}\right)^2\right) b_3$$

To determine b_3 we use the obvious procedure (suggested by the case of b_1) and after the necessary subtraction from both sides of (2) and then division of both sides by x^2y and then taking from both sides limit according to (13), in view of (13) and (19), we obtain:

$$(20)\quad \lim_{k\to\infty} H(q_k, eq_k) = \lim_{k\to\infty} \left(-3+\left(\frac{eq_k}{q_k}\right)^2\right) b_3 = (-3 + e^2)\, b_3$$

where $H(q_k, eq_k)$ is a well defined expression (see (21)) below). However, since e is transcendental, we see that $(-3 + e^2) \neq 0$. Hence, dividing both sides of the equality signs in (20) by the nonzero real $(-3 + e^2)$ we obtain b_3. It can be readily verified that

$$(21)\quad b_3 = \lim_{k\to\infty} \frac{u(q_k, eq_k) - a_0 - a_1 q_k + b_1 eq_k - a_2 q_k^2 + 2b_2 eq_k^2 + a_2 e^2 q_k^2 - a_3 q_k^3 + 3a_3 e^2 q_k^3}{(-3 + e^2)\, eq^3}$$

and therefore b_3 is uniquely determined by (8).

From the above considerations it can be readily verified that based on (7) and (8), for $n = 0, 1, 2, \dots$ our procedure determines recursively the a_n's which appear in (2), as follows:

$$a_0 = \lim_{k\to\infty} u(p_k, 0)$$

$$(22)\quad a_n = \lim_{k\to\infty} \frac{u(p_k, 0) - a_0 - a_1 p_k - a_2 p_k^2 - \dots - a_{n-1} p_k^{n-1}}{p_k^n} \qquad \text{for } n > 0$$

and therefore a_n's are uniquely determined by (8).

Next, we give a formula for determining recursively the b_n's appearing in (2). First however, let us observe that as (21) shows b_3 is determined based on the previously determined a_0, a_1, b_1, a_2, b_1, a_3 and the crucial fact that $(-3 + e^2) \neq 0$. Let us consider $(a_4 x^4 - 4b_4 x^3 y - 6a_4 x^2y^2 + 4b_4 xy^3 + a_4 y) = p_4(x, y)$ which appears in (2). We observe that the part of $p_4(x, y)$ which involves b_4 is given by $x^3 y \left(-4 + 4\left(\frac{y}{x}\right)^2\right) b_4$. Since e is transcendental we see that $(-4 + 4e^2) \neq 0$. Therefore, procedure of determining b_4 (by taking limit according to (13)) determines b_4 uniquely in terms of already determined a_0, a_1, ..., b_3, a_4. Again, let us consider $(a_5 x^5 - 5b_5 x^4 y - 10a_5 x^3y^2 + 10b_5 x^2y^3 + 5a_5 xy^4 - b_5 y^4) = p_5(x, y)$ which appears in (2).

We observe that the part of $p_5(x, y)$ which involves b_5 is given by $x^4 y \left(-5 + 10\left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)^4\right) b_5$. Since e is transcendental we see that $(-5 + 10e^2 - e^4) \neq 0$. Therefore, our procedure of determining b_5 (by taking limit according to (13)) determines b_5 uniquely in terms of already determined $a_0, a_1, \ldots, b_4, a_5$.

To determine b_n in general, let us consider $p_n(x, y)$ which appears in (2) and which is given by (4). From (4), (5), (6) it follows that the part of $p_n(x, y)$ which involves b_n is given by:

(23) $$x^{n-1}y\left(-\binom{n}{1} + \binom{n}{3}\left(\frac{y}{x}\right)^2 - \binom{n}{5}\left(\frac{y}{x}\right)^4 + \binom{n}{7}\left(\frac{y}{x}\right)^6 - \ldots + (-1)^{t+1}\binom{n}{2t+1}\left(\frac{y}{x}\right)^{2t}\right) b_n$$

where t is the largest integer such that (24) $(2t + 1) \leqq n$

Since e is transcendental we see that

(25) $$\left(-\binom{n}{1} + \binom{n}{3}e^2 - \binom{n}{5}e^4 + \binom{n}{7}e^6 - \ldots + (-1)^{t+1}\binom{n}{2t+1}e^{2t}\right) \neq 0$$

Therefore our procedure of determining b_n (by taking limit according to (13)) determines b_n uniquely in terms of already determined $a_0, a_1, \ldots, b_{n-1}, a_n$.

Indeed, from the above considerations it can be readily verified that based on (7) and (8), for $n = 1, 2, \ldots$ our procedure determines recursively the b_n's which appear in (2), as follows:

(26) $$b_n = \lim_{k \to \infty} \frac{u(q_k, eq_k) - a_0 - a_1 q_k + b_1 eq_k - \ldots + (-1)^{s+1}\binom{n}{2s} a_n e^{2s} q_k^n}{\left(-\binom{n}{1} + \binom{n}{3}e^2 - \binom{n}{5}e^4 + \binom{n}{7}e^6 - \ldots + (-1)^{t+1}\binom{n}{2t+1}e^{2t}\right) eq_k^n}$$

for $n > 0$.

where s *is the largest integer such that* $2s \leqslant n$

and where t *is the largest integer such that* $2t + 1 \leqslant n$

Therefore, b_n's are also uniquely determined by (8).

Clearly, (22) and (26) imply the proof of the following:

THEOREM 1. *Let $u(x,y)$ be a harmonic function in an open disk D with center at (0,0). Then $u(x,y)$ is uniquely determined in D by its values at the points of two sequences in D, one converging to (0,0) along the x-axis and the other converging to (0,0) along a line with a transcendental slope.*

Let us observe that we chose e to be transcendental in order to make nonzero the expression appearing in (25) which also appears in the denominator of the expression defining b_n in (26). Thus, any other nonzero real number r for which (25) holds when e is replaced by r could be used (instead of e) to determine b_n as given by (26).

DEFINITION. *A real number is called an A-number if and only if it is not a root of a polynomial $A_n(x)$ given by:*

$$(27)\quad A_n(x) = -\binom{n}{1}+\binom{n}{3}x^2-\binom{n}{5}x^4+\binom{n}{7}x^6 - \dots + (-1)^{t+1}\binom{n}{2t+1}x^{2t}$$

for any $n > 0$ where t is the largest integer such that $(2t+1) \leqslant n$.

Clearly, 0 and every transcendental real is an A-number and there may be some nonzero nontranscendental A-numbers.

Based on (25), (26), Theorem I and the above Definition, we have:

THEOREM 2. *Let $u(x,y)$ be a harmonic function in an open disk D with center at (0,0). Then $u(x,y)$ is uniquely determined in D by its values at the points of two sequences in D, one converging to (0,0) along the x-axis and the other converging to (0,0) along a line whose slope is a nonzero A-number.*

Finally, let us observe that for the polynomial $A_n(x)$ given by (27), we have:

$$(28)\qquad A_n(x) = -\frac{1}{x}\,\mathrm{Im}\,(1+ix)^n$$

Motivated by the fact that x in $A_n(x)$ represents the slope of a line passing through (0,0), we let $x = \tan\theta$ with $-\pi/2 < \theta < \pi/2$. But then (28) implies

$$(29)\quad A_n = (-\cot\theta)\,\frac{\sin n\theta}{\cos^n\theta}\ \textit{with}\ -\pi/2 < \theta < \pi/2$$

From the above it follows that if θ is not a rational multiple of π then $A_n \neq 0$ for every $n > 0$. Since in Theorem 2 the x-axis can be replaced by any line passing through (0,0), from Theorem 2 and (29) we have:

THEOREM 3. *Let $u(x,y)$ be a harmonic function in an open disk D with center at $(0,0)$. Then $u(x,y)$ is uniquely determined in D by its values at the points of two sequences in D each converging to $(0,0)$ along a line passing through $(0,0)$ such that the angle (in radians) between the two lines is not a rational multiple of π.*

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