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OPTIMIZATION OF DISCRETE SYSTEMS

PHAM HUU SACH

Institute of Mathematics
Hanoi

INTRODUCTION

Let g be a real—valued function and let F_k $(k = 0, 1,\ldots, N-1)$ be N set-valued maps, all defined on the same space X. The problem we shall be considering is that of minimizing $g(x_N)$ over the trajectories $(x_0, x_1,\ldots, x_N)$ of the discrete inclusion

$$x_{k+1} \in F_k(x_k),\ k = 0,1,\ldots, N-1.$$

Our aim in this paper is to derive necessary conditions for an optimal solution of this problem.

An extensive theory of optimality conditions for problems of this kind has been developed in the last decade. In [1], assuming that the maps F_k have local sections, Boltianskii established an optimality criterion in the form of a Support Principle. Later, Sach [2—4] pointed out that the Support Principle still holds even if the maps F_k are assumed only to have smooth support functions. Recently, introducing the notion of adjoint cone to a subset of an Euclidean space, Morduhovitch [5] obtained an optimality condition for discrete systems in terms of the adjoint maps of F_k (i. e. the maps whose graphs are the adjoint cones to the graphs of F_k). In the present paper, we shall use for the investigation of the problem under consideration the notion of derivative of a set-valued map introduced earlier in [6]. Our main result will be Theorem 3.1 of Section 3. It should be noted that this Theorem differs from the result of Morduhovitch [5] in that it is established for any derivative of F_k. The advantage of our method is that a derivative of F_k in our sense can be easily found in many circumstances where the adjoint map in the Morduhovitch sense is difficult to construct. The paper is organized as follows. In Section 1 we recall the notion of derivative of set-valued map (see [6]) and give an auxiliary result. In Section 2 we discuss the conditions guaranteeing the existence of a derivative of a set-valued map. Combining the results of Section 2 and Theorem 3.1 we are able to obtain necessary optimality conditions for the given problem under various assumptions on g and F_k. In particular, we can recover the Support Principle for the case where the maps F_k have local sections (see [1]) as well as for the case where F_k have smooth support functions (see [2 — 4]). Section 4 is devoted to the proof of Theorem 3.1. In the last Section 5, using a result of Ioffe [7] we show that Theorem 3.1 remains valid for the infinite dimensional case provided that g and F_k are locally Lipschitzian.

§ 1. DERIVATIVE OF A SET-VALUED MAP

Let X and Y be two topological vector spaces and $T: X \to 2^Y$ a set-valued map from X to Y. The symbols dom T, ImT and graph T will denote the domain, the range and the graph of T respectively. Recall that

$$\operatorname{dom} T = \{x : T(x) \neq \varnothing\},$$

$$\operatorname{Im} T = \bigcup_{x \in X} T(x)$$

and

$$\operatorname{graph} T = \{(x,y) : y \in T(x)\},$$

where $\varnothing$ stands for the empty set.

We shall say that T is

— *proper*, if its domain is nonempty,

— *convex*, if its graph is a convex set.

Consider now a point $z_0 = (x_0, y_0) \in \operatorname{graph} T$. Denote by $N(x)$ the collection of all neighbourhoods of $x \in X$ and by $B(x, \delta)$ the ball of radius $\delta > 0$ around x.

DEFINITION 1.1. *A proper convex map $t: X \to 2^Y$ is said to be a* derivative *of T at z_0 if, for every $\hat{z} = (\hat{x}, \hat{y}) \in$ graph t and every $V \in N(0)$ there exist $\delta > 0$ and $U \in N(\hat{x})$ such that*

$$y_0 + \varepsilon \hat{y} \in T(x) + \varepsilon V \tag{1.1}$$

whenever

$$\varepsilon \in (0, \delta),\ x \in (x_0 + \varepsilon U) \cap \operatorname{dom} T. \tag{1.2}$$

Definition 1.1 is a generalization of the notion of the derivative of a single-valued map [8,9] to the case of a set-valued map. The interested reader is referred to [6] for more details.

For every integer k, consider the standard k-simplex:

$$P^k = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) : \sum_{i=1}^{k+1} \lambda_i = 1, \lambda_i \geqslant 0, i = 1, 2, \ldots, k+1\}.$$

Given $x_i \in X$, $i = 1, 2, \ldots, k+1$, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in P^k$, we set

$$x(\lambda) = \sum_{i=1}^{k+1} \lambda_i x_i. \tag{1.3}$$

We shall need the following concept, essentially due to Neustadt [8].

DEFINITION 1.2. *A nonempty set $A \subset X$ is said to be $(k+1)$ — contingent to B at $x_0 \in X$ if, for every finite set $\{x_1, x_2, \ldots, x_{k+1}\} \subset A$, every positive number δ and every neighbourhood $U \in N(0)$, there exist a number $\varepsilon \in (0, \delta)$ and a continuous map $\eta : P^k \to B$ such that*

$$\eta(\lambda) \in x_0 + \varepsilon(a(\lambda) + U) \tag{1.4}$$

whenever $\lambda \in P^k$.

Note that the point x_0 mentioned in the above definition may not belong to B.

Example 1.1. Let B be a convex set in a locally convex space X and $x_0 = 0$ a point belonging to the closure of B. Then, for every k, the set B is $(k+1)$—contingent to itself at x_0.

Example 1.2. For any k, the *hypertangent cone* to a set B at $x_0 \in B$ is $(k+1)$—contingent to itself at x_0. Recall [10] that this cone consists of the vectors x such that there exist $U \in N(x_0)$ and $\delta > 0$ with $x' + \varepsilon x \in$ B for all $x' \in C \cap U$ and $\varepsilon \in (0, \delta)$.

DEFINITION 1.3. *Given a set $B \subset X$ and a point $x_0 \in X$, the set*

$$K(B, x_0) = \{x \in X : \{x\} \text{ is 1-contingent to } B \text{ at } x_0\}$$

is called the Bouligand contingent cone to B at x_0.

The following definition is due to Aubin.

DEFINITION 1.4. *The contingent derivative of T at $z_0 = (x_0, y_0) \in$ graph T is the map whose graph coincides with the set K (graph T, z_0).*

Remark 1.1. Let t be a derivative of T at $z_o = (x_o, y_o)$. In general, the inclusion

$$\text{graph } t \subset K(\text{graph } T, z_o) \tag{1.2}$$

is not true. But it is easily seen that, if the condition

$$\text{dom } t \subset K(\text{dom } T, x_o) \tag{1.6}$$

holds, so does the above inclusion.

Remark 1.2. Let Y be a normed space and let $d(y, Q)$ denote the distance from y to $Q \subset Y$. Then the collection of all the derivatives of T at z_o which satisfy (1.6) coincides with the family of all the maps whose graphs are nonempty convex subsets of the set

$$\left\{ (\widehat{x}, \widehat{y}) : \quad \lim_{\substack{x \to \widehat{x} \\ \varepsilon \downarrow 0 \\ x_o + \varepsilon \widehat{x} \in \text{dom } T}} d\left(\widehat{y}, \frac{T(x_o + \varepsilon x) - y_o}{\varepsilon}\right) = 0 \right\}.$$

Denote by X^* and Y^* the continuous duals of X and Y respectively. To every set $K \subset X$ we can associate a set $K^* \subset X^*$ consisting of all the functionals $x^* \in X^*$ that are nonpositive on K.

DEFINITION 1.5. *Given a convex map $T : X \to 2^Y$, we define the adjoint of T by*

$$T^*(y^*) = \{x^* \in X^* : (x^*, -y^*) \in (\text{graph } T)^*\}.$$

DEFINITION 1.6. *Let g be an extended-real-valued function defined on X and x_o a point such that $|g(x_o)| < \infty$. Denote by G the map whose graph coincides with*

$$\text{epi } g \overset{df}{=} \{(x, r) \in X \times R : g(x) \leq r\},$$

where R stands for the real line. Assume that G' is a derivative of G at $(x_o, y_o) = z_o \in \text{epi } g$ ($y_o = g(x_o)$). We shall say that G' is an epi-derivative of g at x_o and that the set $\partial g(x_o) \overset{df}{=} \{x^* : (x^*, -1) \in (\text{graqh } G')^*\}$ is a generalized gradient of g at x_o.

THEOREM 1.1. *Let D be a nonempty set of a vector topological space X; Y the Cartesian product of two normed spaces Y_i, $i = 1,2$ ($\dim Y_1 = k < \infty$); $S : D \to 2^Y$ a continuous map with nonempty convex closed values and $\widetilde{x} \in D$ a point satisfying*

$$\text{int}\left\{ y_2 \in Y_2 : (0, y_2) \in S(\widetilde{x}) \right\} \neq \emptyset,$$

where int denotes the interior.

Let a derivative s of S at $(\tilde{x}, 0) \in$ graph S be given such that

1. dom s is $(k + 1)$ — contingent to D at $\tilde{x}$.

2. $0 \in int \{ S(x_o) + Ims \}$.

Then the set

$$\{x : x \in dom\, s,\ 0 \in S(\tilde{x}) + s(x) \}$$

is contained in the Bouligand cone to the set

$$\{ x : x \in D;\ 0 \in S(x) \}$$

at $\tilde{x}$.

Proof. Theorem 1. 1 can be proved by an argument analogous to that used for the proof of Theorem 2. 1 in [6].

Recall that $S : D \to 2^Y$ is said to be *upper semicontinuous* (resp. *lower semicontinuous*) if, for every $x_o \in D$ and $V \in N(O)$, there exists $U \in N(x_o)$ such that

$$S(x) \subset S(x_o) + V$$
$$(\text{resp. } S(x_o) \subset S(x) + V)$$

whenever $x \in U \cap D$. A map S is said to be *continuous* if it is both lower and upper semicontinuous.

§ 2. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF DERIVATIVES

Throughout this section, X and Y are assumed to be normed spaces. Consider now a map $T : X \to 2^Y$ and a point $z_o = (x_o, y_o) \in$ graph T. Denote by Z the Cartesian product of X and Y. We shall give some sufficient conditions for the existence of a derivative of T at z_o.

PROPOSITION 2. 1. *Assume that $x_o \in$ int dom T. Let A be the set of points $(\widehat{x}, \widehat{y})$ such that, for every $V \in N(0)$, there exist $W \in N(z_o)$, $U \in N(0)$ and $\delta > 0$ such that*

$$y + \varepsilon\widehat{y} \in T(x + \varepsilon(\widehat{x} + u)) + \varepsilon V \tag{2.1}$$

whenever

$$(x, y) \in W \cap \text{graph}\, T,\ u \in U,\ \varepsilon \in (0, \delta). \tag{2.2}$$

If A is nonempty, then the map whose graph coincides with A is a derivative of T at z_o.

Proof. It suffices to prove the convexity of A. Indeed, assume that $z_i = (x_i, y_i) \in A$, $i = 1, 2$, and $\lambda \in (0, 1)$. For $V \in N(O)$, we can choose two balls $W = B(z_o, \eta) \in N(z_o)$, $U \in N(O)$ and a number $\delta \in (0, 1)$ such that

$$y + \varepsilon y_i \in T(x + \varepsilon(x_i + u)) + \varepsilon V, \quad i = 1, 2, \qquad (2,3)_i$$

for all x, y, ε and u satisfying (2. 2). Let us set $W' = U \times V$. Since X and Y are normed spaces, we may assume δ to be so small that

$$\varepsilon(z_1 + W') \subset B(0, \eta/2)$$

for all $\varepsilon \in (0, \delta)$.

If $z = (x, y) \in B(z_o, \eta/2) \cap$ graph T, $\varepsilon < \delta$ and $u \in U$, then, by means of $(2.\ 3)_1$, we can find $v \in -V = V$ such that the point $z' = z + \lambda\ \varepsilon(z_1 + v')$ belongs to the graph of T, where $v' = (u, v)$. Using $(2.3)_2$ with $z' = (x', y')$ and $(1 - \lambda)\ \varepsilon$ in place of z and ε resp., we get

$$y' + (1 - \lambda)\ \varepsilon\ y_2 \in T(x' + (1 - \lambda)\ \varepsilon\ (x_2 + u)) + (1 - \lambda)\ \varepsilon\ V$$

e. g. $\lambda\ z_1 + (1 - \lambda)z_2 \in A$. Q.E.D.

DEFINITION 2. 1. *A nonempty convex cone $K \subset X$ is said to be a* tangent cone *to a set $B \subset X$ at $x_o \in X$ if for every $\hat{x} \in K$ and every $U \in N(\hat{x})$ there exists $\delta > 0$ such that $(x_o + \varepsilon U) \cap B \neq \phi$ for all $\varepsilon \in (0, \delta)$.*

Denote by $h(A, B)$ the Hausdorff distance of A and B.

DEFINITION 2. 2. *We shall say that a map $T : X \to 2^Y$ is* locally Lipschitzian *at x_o if there exist $c > 0$ and $U \in N(x_o)$ such that $h(T(x), T(x')) \leqslant c \| x - x' \|$ for all $x, x' \in U$.*

PROPOSITION 2. 2. *Let K be a tangent cone to graph T at z_o. If T is locally Lipschitzian at x_o, then the map whose graph coincides with the closure of K is a derivative of T at z_o.*

DEFINITION 2. 3. *We shall say that a nonempty convex cone $K \subset X$ is an* interior cone *to a set $B \subset X$ at x_o if, for every $\hat{x} \in K$, there exist $\delta > 0$ and $U \in N(\hat{x})$ such that $x_o + \varepsilon U \subset B$ for all $\varepsilon \in (0, \delta)$.*

PROPOSITION 2. 3. *If K is an interior cone to graph T at z_o, then the map whose graph coincides with K is a derivative of T at z_o.*

The proof of Propositions 2.2 and 2.3 is omitted.

Consider now two extended-real-valued functions $g(z) = d(z, \text{graph } T)$ and $f(z) = f(x, y) = d(y, T(x))$. It is clear that

1. $\text{dom } g = Z$, $\text{dom } f = (\text{dom } T) \times Y$.
2. $g(z) \leqslant f(z)$ for all z.
3. $g(z_o) = f(z_o)$ if $z_o = (x_o, y_o) \in \text{graph } T$.

Denote by R_+ the positive half-line. For any extended-real-valued function φ and any point $z_o \in Z$ where φ is finite we set

$$\tilde{d}\varphi(z_o, \hat{z}) = \limsup_{\substack{z \to \hat{z} \\ \varepsilon \downarrow 0 \\ z_o + \varepsilon z \in \text{dom } \varphi}} \frac{\varphi(z_o + \varepsilon z) - \varphi(z_o)}{\varepsilon},$$

$$\Phi(z_o, \hat{z}) = \sup_{0(.) \in Q} \limsup_{\varepsilon \downarrow 0} \frac{\varphi(z_o + \varepsilon(\hat{z} + 0(\varepsilon))) - \varphi(z_o)}{\varepsilon}$$

where Q stands for the collection of all the maps $0(.) : R_+ \to Z$ such that

$$\lim_{\lambda \to 0} \frac{0(\lambda)}{\lambda} = 0.$$

It is clear that, for all z, we have

$$\tilde{d}g(z_o, z) = \tilde{d}f(z_o, z).$$

PROPOSITION 2.4. *Let h be a closed convex positively homogeneous function such that $\tilde{d}f(z_o, z) \leqslant h(z)$ for all z. Then the map t defined by*

$$\text{graph } t = \{ z = (x, y) : h(z) = 0\} \tag{2. 4}$$

is a derivative of T at $z_o \in$ graph T.

The proof of this proposition is left to the reader.

Before going further, we need the following

LEMMA 2.1. *If, for $b \in R$ and $\hat{z} \in Z$, we have $\Phi(z_o, \hat{z}) \leqslant b$, then $\tilde{d}\varphi(z_o, \hat{z}) \leqslant b$.*

Proof. Suppose the contrary and let δ be a positive number satisfying $b + \delta < \tilde{d}\varphi(z_o, \hat{z})$. By definition, we can find a decreasing sequence $\varepsilon_i \to 0$ and a sequence $h_i \to 0$ such that

$$b + \delta \leqslant \frac{\varphi(z_o + \varepsilon_i(\hat{z} + h_i)) - \varphi(z_o)}{\varepsilon_i} \tag{2. 5}$$

Consider now the map $h(\,.\,): \mathrm{R}_+ \to Z$ defined by setting $h(\varepsilon_i) = h_i$ and by linear interpolation for $\varepsilon \in (\varepsilon_{i+1}, \varepsilon_i)$. It is easily seen that $h(\varepsilon) \to 0$ as ε tends to zero. Let us set $0(\varepsilon) = \varepsilon h(\varepsilon)$. In view of the inequality $\Phi(z_0, \widehat{z}) \leqslant b$, there exists a number $\varepsilon_0 > 0$ satisfying

$$\sup_{0 < \varepsilon < \varepsilon_0} \frac{\varphi(z_0 + \varepsilon \widehat{z} + 0(\varepsilon)) - \varphi(z_0)}{\varepsilon} < b + \frac{\delta}{2},$$

which contradicts to (2. 5). Q.E.D.

DEFINITION 2.4. *(Pshenitchnyi). A closed convex positively homogeneous function h is said to be an* upper approximation *of an extended-real-valued function φ at z_0 if $\Phi(z_0, \widehat{z}) \leqslant h(\widehat{z})$ for all $\widehat{z} \neq 0$.*

PROPOSITION 2.5. *Assume that, for the function $\varphi = f$, we have $\Phi(z_0, o) = 0$. Let h be an upper approximation of f at z_0. Then the map t defined by (2. 4) is a derivative of T at z_0.*

Proof. This follows from Proposition 2. 4 and Lemma 2.1.

PROPOSITION 2.6. *Assume that h is an upper approximation of g at $z_0 = (x_0, y_0)$ and that T is locally Lipschitzian at x_0. Then the map t defined by (2. 4) is a derivative of T at z_0.*

Proof. T being locally Lipschitzian at x_0, we see that

1. There exists a constant $q > 0$ satisfying
$$f(z) \leqslant q\, g(z) \qquad \text{for all } z.$$

2. f is locally Lipschitzian at z_0, hence $\tilde{d}f(z_0, o) = 0$.

Putting $h_1(z) = q\, h(z)$ we have $\tilde{d}f(z_0, z) \leqslant h_1(z)$ for all z. Applying Proposition 2. 4, we obtain the desired result.

Consider now the case $\dim Y < \infty$. If, for every $x \in \operatorname{dom} T$, the set $T(x)$ is convex and compact, then, by the well-known Minimax Theorem, we have

$$f(z) = f(x, y) = \min_{y' \in T(x)} \| y - y' \| = \max_{\| y^* \| \leqslant 1} q(y^*, z),$$

where

$$q(y^*, z) = \langle y^*, y \rangle - c^T(y^*, x),$$
$$c^T(y^* x) = \max \{ \langle y^*, y \rangle : y \in T(x) \}.$$

Putting $q(z) = \max \{ q(y^*, z) : \| y^* \| = 1 \}$, we find that $q(z) \leqslant f(z)$ for all $z \in Z$.

PROPOSITION 2. 7. *Assume that* $\dim Y < \infty$ *and that, for every* $x \in \operatorname{dom} T$, *the set* $T(x)$ *is convex and compact.* Let h be *a convex function such that* $\tilde{d}q(z_o, z) \leqslant h(z)$ *for all* z *and the set*

$$\{ z : h(z) \leqslant 0 \} \tag{2.6}$$

is nonempty. Then the map whose graph coincides with (2. 6) is a derivative of T *at* z_o.

Proof. We begin by noting that, $f(z) \leqslant a$ for every $a > 0$ satisfying $q(z) \leqslant a$. Taking $\eta > 0$ and $\widehat{z} = (\widehat{x}, \widehat{y}) \in$ graph t, we can find $\delta > 0$ and $U \in N(\widehat{z})$ such that

$$q(z_o + \varepsilon z) - \dot{q}(z_o) < \varepsilon\eta$$

whenever

$$o < \varepsilon < \delta,\ z \in U,\ z_o + \varepsilon z \in \operatorname{dom} q = (\operatorname{dom} T) \times Y.$$

By the above remark and the fact that $q(z_o) \leqslant f(z_o) = 0$ we deduce from (2. 6) that $f(z_o + \varepsilon z) < \varepsilon\eta$, which proves the Proposition.

To state a corollary of Proposition 2. 7, we consider, for every fixed y^*, the one-sided directional derivative $c'(y^*, x_o; x)$ of the function $c(y^*, x) = c^T(y^*, x)$ at x_o. Put

$$S = \{ y^* \in Y^* : \| y^* \| = 1 \}$$

and denote by S_o the collection of all vectors $y^* \in S$ normal to $T(x_o)$ at y_o in the usual sense of Convex Analysis. Let

$$A = \{ (x, y) : \max_{y^* \in S_o} [\langle y^*, y \rangle - c'(y^*, x_o; x)] \leqslant 0 \}.$$

COROLLARY 2. 1. *Assume that*

1. y_o *is a boundary point of the set* $T(x_o)$ *in the finite dimensional space* **Y**.
2. *For all* x, (x) *is a nonempty convex compact set.*
3. *For all* y^*, *the derivative* $c'(y^*, x_o^-$ *is a concave function.*
4. *For all* x,

$$c'(y^*, x_o; x) = \lim_{\substack{x' \to x \\ \varepsilon \downarrow 0}} \frac{c(y^*, x_o + \varepsilon x') - c(y^*, x_o)}{\varepsilon}$$

uniformly on S.

5. $A \neq \phi$.

Then the map defined by A is a derivative of T at z_o.

Proof. This follows from Proposition 2. 7 and Theorem 3 in [12; p. 224—225].

Remark 2. 1. In the above corollary y_o is assumed to be a boundary point of $T(x_o)$. If $y_o \in \text{int } T(x_o)$ and if T is lower semicontinuous, then the map whose graph coincides with Z is a derivative of T at x_o.

§ 3. OPTIMIZATION OF A DISCRETE SYSTEM

We now turn to the main problem. Given a real-valued function $g : X \to R$ and N maps $F_k : X \to 2^X$, $k = 0, 1, \ldots, N-1$, where X is a finite-dimensional space, we are interested in the necessary conditions for an optimal solution to the problem

$$\min \{g(x_N) : x_{k+1} \in F_k(x_k), k = 0, 1, \ldots, N-1\}. \tag{3.1}$$

The main result to be established is Theorem 3.1, which can be combined with the Propositions of Section 2 or the examples of derivatives of a set-valued map given in [6] to obtain necessary optimality conditions for Problem (3.1) under various assumptions on g and F_k. In particular, by that way we can recover the Support Principle for the case where the maps F_k have local sections [1] or have smooth support functions [2 — 4].

Assume that

$$(\overset{o}{x}_o, \overset{o}{x}_1, \ldots, \overset{o}{x}_N) \tag{3.2}$$

is a solution of Problem (3.1). Let us set

$$X_k = X, \quad Z = \prod_{k=0}^{N} X_k, \quad Y = \prod_{k=1}^{N} X_k,$$

$$z = (x_o, x_1, \ldots, x_N) \in Z, \; y = (y_1, y_2, \ldots, y_N) \in Y,$$

$$F(z) = \prod_{k=o}^{N} F_k(x_k), \; f(z) = (x_1, x_2, \ldots, x_N),$$

$$\widehat{g}(z) = g(x_N).$$

It is clear that Problem (3.1) can be rewritten as

$$\min \{\widehat{g}(z) : f(z) \in F(z)\} \tag{3.1'}$$

and that $\overset{o}{z} = (\overset{o}{x}_o, \overset{o}{x}_1, \ldots, \overset{o}{x}_N)$ yields a solution of Problem (3.1)'.

To formulate necessary optimality conditions for the problem under consideration we need the following

Definition 3.1. *We shall say that the system*

$$x_{k+1} \in F_k(x_k), \quad k = 0, 1, \dots, N-1, \tag{3.3}$$

is consistent if there exists at least a point $(x_0, x_1, \dots, x_N)$ *satisfying* (3.3). *System (3.3) is said to be* nondegenerate *if we can find a positive number* ρ *such that, for every point* $\xi = (\xi_0, \xi_1, \dots, \xi_{N-1})$ *of the Cartesian product* X^N *satisfying* $\| \xi_i \| \leqslant \rho$ *for all i, the perturbed system*

$$x_{k+1} \in F_k(x_k) + \xi_k, \quad k = 0, 1, \dots, N-1,$$

is consistent.

It is not difficult to verify that system (3.3) is nondegenerate if and only if

$$0 \in \text{int} \{ -f(z) + F(z) : z \in Z \}.$$

Example 3. 1. System (3, 3) is nondegenerate if dom $f_0 \neq \phi$ and if there exists a positive number ρ such that $F_k(X) + B(o, \rho) \subset \text{dom} f_{k+1}$ for all $k = 0, 1, \dots, N-2$. In particular, system (3. 3) is nondegenerate if dom $F_k = X$ for all $k = 0, 1, \dots, N-1$.

Assume that

a. In some neighbourhood of $\overset{o}{x}_k$, F_k is a continuous map with nonempty convex closed values.

b. There exist an epi-derivative t_g of g at $\overset{o}{x}_N$ and a derivative t_k of F_k at $\left(\overset{o}{x}_k, \overset{o}{x}_{k+1} \right)$ such that

b1. dom $t_g = X$.

b2. The system (3. 3) with $f_k(x) \overset{\text{df}}{=} -\overset{o}{x}_{k+1} + F_k\left(\overset{o}{x}_k \right) + t_k(x)$ in place of $F_k(x)$ is nondegenerate.

Note that by Definition 1. 6 we can associate to t_g a generalized gradient $\partial g(\overset{o}{x}_N)$ of g at $\overset{o}{x}_N$.

Remark 3.1. Throughout the forthcoming, we shall assume that the graphs of the derivatives to be considered are cones. This assumption is not restrictive since a map whose graph coincides with the cone generated by the graph of a derivative is also a derivative.

We are now in a position to formulate our main result.

THEOREM 3.1. *Under the stated assumptions, if (3.2) is a solution of Problem (3.1), then there exist functionals $x_i^* \in X^*$, $i = 0, 1, \ldots, N$, such that*

$$x_k^* \in t_k^*(x_{k+1}^*), \ k = 0, 1, \ldots, N-1, \tag{3.4}$$

$$x_N^* \in \partial g(\overset{o}{x}_N), \tag{3.5}$$

$$x_o^* = 0, \tag{3.6}$$

$$\left\langle x_{k+1}^*, \overset{o}{x}_{k+1} \right\rangle = \min_{x \in F_k(\overset{o}{x}_k)} \left\langle x_{k+1}^*, x \right\rangle, \ k = 0, 1, \ldots, N-1. \tag{3.7}$$

§ 4. PROOF OF THEOREM 3.1.

We begin by proving necessary optimality conditions for the problem

$$\min \{g(x) : f(x) \in F(x)\} \tag{4.1}$$

where $F : X \to 2^Y$, $f : X \to Y$, $g : X \to R$ are maps defined on X. (The spaces X and Y are assumed to be finitedimensional).

Let $\overset{o}{x}$ be a solution of Problem (4.1). Put $\overset{o}{f} = f(\overset{o}{x})$, $\overset{o}{g} = g(\overset{o}{x})$ and assume that

1. f is Frechet differentiable at $\overset{o}{x}$.

2. In some neighbourhood of $\overset{o}{x}$, F is a continuous map with nonempty convex closed values.

3. There exist an epi-derivative t_g of g at $\overset{o}{x}$ and a derivative $t_F : X \to 2^Y$ of F at $(\overset{o}{x}, \overset{o}{f})$ such that

3.1. $\text{dom } t_g = X.$ (4.2)

3.2. $0 \in int \left\{ - f(\overset{o}{x}) + F(\overset{o}{x}) + (-f' + t_F)(E) \right\},$ (4.3)

where $E = \text{dom } t_F$ and f' stands for the Frechet derivative of f at $\overset{o}{x}$.

PROPOSITION 4.1. *Assume that $\overset{o}{x}$ is a solution of Problem (4.1). and that conditions 1-3 are fulfilled. Then there exist $x^* \in X^*$, $y^* \in Y^*$ such that*

$$x^* + f'^* y^* \in t_F^* (y^*), \tag{4.4}$$

$$- x^* \in \partial g(\overset{o}{x}), \tag{4.5}$$

$$\langle y^*, f(\overset{o}{x})\rangle = \min_{y \in F(\overset{o}{x})} \langle y^*, y\rangle. \tag{4.6}$$

Proof. Let us set

$$C = \left\{ x : g(x) < g(\overset{o}{x}) \right\}, \tag{4.7}$$

$$P = \left\{ x : \exists\, \mu \in t_g(x) \text{ such that } \mu < 0 \right\}. \tag{4.8}$$

In the case where $P = \phi$ we have $-\mu \leqslant 0$ for all $(x, \mu) \in$ graph t_g. This means that $0 \in \partial g(\overset{o}{x})$. Hence, putting $x^* = 0$, $y^* = 0$, we see that conditions (4.4) — (4.6) are satisfied.

Consider now the case where $P \neq \phi$. We are going to prove that, for every $\widehat{x} \in P$, there exist $V \in N(0)$ and $\delta > 0$ such that

$$\overset{o}{x} + \varepsilon(\widehat{x} + u) \in C \tag{4.9}$$

whenever

$$\varepsilon \in (0, \delta), u \in V. \tag{4.10}$$

Indeed, let $\widehat{\mu} \in t_g(\widehat{x})$ be a negative number. Take $\eta > 0$ such that $\widehat{\mu} + \eta < 0$. Let G be the map whose graph is epi g. By definition, we can choose $\delta > 0$ and $V \in N(0)$ such that, for all ε and u satisfying (4.10), we have

$$\overset{o}{g} + \varepsilon\mu \in G(\overset{o}{x} + \varepsilon(\widehat{x} + u)) + \varepsilon B(0, \eta),$$

hence (4.9).

Let $U \in N(\overset{o}{x})$ be a neighbourhood of $\overset{o}{x}$ such that F is continuous and, for every $x \in U$, $F(x)$ is a nonempty convex closed set. Define a map S from U into Y by $S(x) = -f(x) + F(x)$. It is easily seen that, for every derivative φ of F at $(\overset{o}{x}, y) \in$ graph F, the map $-f' + \varphi$ is a derivative of S at $(\overset{o}{x}, -\overset{o}{f} + y)$ $\in$ graph S and dom $(-f' + \varphi) =$ dom φ. In particular, $S' \overset{df}{=} -f' + t_E$ is a derivative of S at $(\overset{o}{x}, 0)$ and dom $S' =$ dom t_F. Setting

$$S''(x) = -\overset{o}{f} + F(\overset{o}{x}) + S'(x), \tag{4.11}$$

we deduce from Theorem 1.1 that the set

$$Q \overset{df}{=} \{x : x \in E,\ 0 \in S''(x)\}$$

is contained in the Bouligand cone to the set

$$\{x : x \in D,\ 0 \in S(x)\}.$$

Further, x being a solution of Problem (4.1), we have $P \cap Q = \emptyset$. Since P is a nonempty convex cone and Q is a convex set containing $0 \in Y$, we can find a nonzero functional $x_1^* \in X^*$ such that

$$\langle x_1^*,\ x \rangle \geqslant 0 \geqslant \langle x_1^*,\ x' \rangle$$

for all $x \in P$, $x' \in Q$. From this we conclude that the two following systems have no solution:

$$x \in E,\ 0 \in -\overset{\circ}{f} + F(\overset{\circ}{x}) + S'(x),\ 0 \in -\langle x_1^*, x \rangle + R_+, \tag{4.12}$$

$$x \in \operatorname{dom} t_g = X,\quad 0 \in \langle x_1^*,\ x \rangle + R_+,\quad 0 \in t_g(x) + R_+. \tag{4.13}$$

By condition (4.3) and the inconsistency of system (4.12) there exists [13] $y_1^* \in Y^*$ such that

$$\langle y_1^*,\ y \rangle + \langle x_1^*,\ x \rangle \leqslant 0 \tag{4.14}$$

whenever $(x, y) \in$ graph S''.

From (4.14) we get

$$\sup\ \{ \langle y_1^*, y \rangle : y \in -\overset{\circ}{f} + F(\overset{\circ}{x})\} \leqslant 0. \tag{4.15}$$

The converse inequality is plain since $\overset{\circ}{f} \in F(\overset{\circ}{x})$. Hence

$$\langle y_1^*, f(\overset{\circ}{x}) \rangle = \max_{y \in F(\overset{\circ}{x})} \langle y_1^*,\ y \rangle.$$

Taking into account the fact that graph $S' \subset$ graph S''_r, we obtain from (4.14)

$$x_1^* - f'^*\ y_1^* \in F^*(-y_1^*). \tag{4.16}$$

Consider now system (4.13) and note that the nonzero functional x_1^* maps X on R. By a well-known result [13] we can find a nonnegative number λ such that the inequality

$$-\lambda \langle x_1^*,\ x \rangle - \mu \leqslant 0$$

holds for all $(x, \mu) \in$ graph t_g. In other words,

$$-\lambda x_1^* \in \partial g(\overset{\circ}{x}).$$

To complete the proof it remains to put $y^* = -\lambda y_1^*$ and $x^* = \lambda x_1^*$. Q.E.D.

Let us set $0.\ \emptyset = 0$. By a standard argument we can draw from Proposition 4.1 the following

COROLLARY 4.1. *Assume that conditions 1, 2 and 3.1 are fulfilled. If $\overset{\circ}{x}$ is a solution of Problem (4.1), then there exist $y^* \in Y^*$, $x^* \in X^*$ and $\lambda \geqslant 0$ such tha*

1. $(y^*, \lambda) \neq 0$.
2. *Conditions (4.4), (4.6) and the inclusion*

$$-x^* \in \lambda\, \partial g(\overset{\circ}{x})$$

ar: satisfied.

PROOF OF THEOREM 3.1. Theorem 3.1 follows from applying Proposition 4.1 to Problem (3.1)' and using the following properties (whose proofs are omitted):

(I) The map $t_F : Z \to 2^Y$ whose graph coincides with the set

$$\Big\{ (z, y) = (x_0, x_1, \dots, x_N;\ y_1, y_2, \dots, y_N) :$$
$$(x_k, y_{k+1}) \in \text{graph } t_k \text{ for all } k = 0, 1, \dots, N-1 \Big\}$$

is a derivative of F at $(\overset{0}{z}, \overset{0}{y})$ with $\overset{0}{y} = (\overset{0}{x}_1, \overset{0}{x}_2, \dots, \overset{0}{x}_N)$.

(II) $z^* \in t_F^*(y^*) \Leftrightarrow x_N^* = 0,\ x_k^* \in t_k^*(y_{k+1}^*)$, $k = 0, 1, \dots, N-1$,

where

$$z^* = (x_0^*, x_1^*, \dots, x_N^*)\ ,\quad y^* = (y_1^*, y_2^*, \dots, y_N^*).$$

(III) If $y_k^* \in X^*$, $k = 1, 2, \dots, N$, and if $y^* = (y_1^*, y_2^*, \dots, y_N^*) \in Y^*$, then

$$f'^* y^* = (0, y_1^*, y_2^*, \dots, y_N^*) \in Z^*.$$

(IV) $z^* = (x_0^*, x_1^*, \dots, x_N^*) \in \partial \widehat{g}(\overset{\circ}{z}) \Leftrightarrow x_N^* \in \partial g(\overset{\circ}{x}_N),\ x_k^* = 0,\ k = 0, 1, \dots, N-1.$

§ 5. CASE WHERE X IS A BANACH SPACE

While the theory of discrete optimal systems in finitedimensional space has been extensively developed in the last decade, little attention has been paid up to now to the infinitedimensional case. The only paper dealing with this case is the one by Dzyuba and Psenitchnyi [14]. It should be noted, however, that the optimality criterion given in [14] suffers from the defect that the conditions for its validity are difficult to check. In this section, we shall show that Theorem 3.1 still holds for the infinitedimensional case, assuming only that g and F_k are locally Lipschitzian. Unlike Theorem 3.1 the following result is established without nondegeneracy assumption.

THEOREM 5.1. *Assume that*

1. X is a Banach space.

2. g is locally Lipschitzian at $\overset{\circ}{x}_N$.

3. For every k, F_k is locally Lipschitzian (at $\overset{\circ}{x}_k$) map with closed convex values.

Denote by $\partial g(\overset{\circ}{x}_N)$ the Clarke generalized gradient of g at $\overset{\circ}{x}_N$ and by t_k the map whose graph coincides with the Clarke tangent cone of graph F_k at $(\overset{\circ}{x}_k, \overset{\circ}{x}_{k+1})$.

If (3.2) is a solution of Problem (3.1), then there exist functionals $x^*_i \in X^*$, $i = 0, 1, \ldots, N$, satisfying conditions (3.4), (3.5), (3.6), (3.7).

Recall from [10] that the Clarke tangent cone to a set $C \subset X$ at $x_0 \in C$ is the collection of vectors x such that, for every $V \in N(x)$, there exist $U \in N(x_0)$ and $\delta > 0$ satisfying the condition $(x' + \varepsilon V) \cap C \neq \emptyset$ whenever $x' \in C \cap U$ and $\varepsilon \in (0, \delta)$. This definition has been known [10] to be equivalent to Clarke original definition [15].

The Clarke generalized gradient $\partial g(\overset{\circ}{x})$ of a locally Lipschitzian function g at x [16] is the subdifferential (in the sense of Convex Analysis) of the function $g^0(\overset{\circ}{x}; .)$ defined by

$$g^0(\overset{\circ}{x}; x) = \limsup_{\substack{x' \to \overset{\circ}{x} \\ \varepsilon \downarrow 0}} \frac{g(x' + \varepsilon x) - g(x')}{\varepsilon}$$

In other words,

$$\partial g(\overset{\circ}{x}) = \{ x^* \in X^* : g^0(\overset{\circ}{x}; x) \geqslant \langle x^*, x \rangle \text{ for all } x \}.$$

Proposition 2.2 shows that the map G' whose graph coincides with the Clarke tangent cone to $G \overset{\text{df}}{=} \text{epi } g$ at $\overset{0}{x} \in G$ is an epiderivative of g at $\overset{0}{x}$. In that case, it follows from Proposition 3.17 in [15] that the Clarke generalized gradient $\partial g(\overset{\circ}{x})$ is a generalized gradient in the sense of Definition 1.6.

PROOF OF THEOREM 5.1. Let us set

$$X_k = X\ ,\quad Z = \prod_{k=0}^{N} X_k\ ,$$

$$z = (x_0\ , x_1\ ,\dots, x_N\ ),\quad \| z \| = \sum_{i=0}^{N} \| x_i \|\ .$$

Since F_k is a map with closed values, we can assert that Problem (4.1) is equivalent to the following one

$$\min \{ \widehat{g}(z) : b(z) = 0 \} \tag{5.1}$$

where

$$\widehat{g}(z) = g(x_N),$$

$$b(z) = \sum_{i=0}^{N-1} d(x_{k+1}\ , F_k(x_k)).$$

We are going to show that $\overset{\circ}{z} = (\overset{\circ}{x}_0\ , \overset{\circ}{x}_1\ ,\dots, \overset{\circ}{x}_N)$ is a regular point for b in the sense of Ioffe(1). Indeed, let q be the Lipschitzian constant for F_k in the neighbourhood

$$V_k = \{ x : \| x - \overset{\circ}{x}_k \| \leqslant \delta \} \tag{5.2}$$

of $\overset{\circ}{x}_k$. Let us set

$$q_1 = q + 1,\ K_1 = \sum_{k=0}^{N-1} q_1^i\ , K = 2K_1\ .$$

By the continuity of b and the fact that $b(\overset{\circ}{z}) = 0$ we can choose a positive number $\varepsilon < \frac{\delta}{2}$ such that

$$b(z) < \frac{\delta}{4K_1} \tag{5.3}$$

---

1) Recall that $\overset{\circ}{z}$ is a *regular point* for b [17] if there exist $k > o$ and $U \in N(\overset{\circ}{z})$ such that $d(z, Q) \leqslant k \mid b(z) - b(\overset{\circ}{z}) \mid$ for all $z \in U$ where $Q = \{z : b(z) = b(\overset{\circ}{z})\}$.

whenever

$$\| z - \overset{\circ}{z} \| < \varepsilon. \quad (5.4)$$

To prove the regularity of $\overset{\circ}{z}$, it suffices to check that, for every z satisfying (5. 4). there exists a point $y \in Z$ such that $b(y) = 0$ and $\| y - z \| \leqslant K \; b(z)$. To this end, let us construct by induction the points $y_0, y_1, \ldots, y_N$ satisfying

$$\left\| y_i - \overset{\circ}{x}_i \right\| < \delta, \; i = 0, 1, \ldots, N, \quad (5.5)_i$$

$$y_i \in F_{i-1}(y_{i-1}) \text{ (e. g. } d(y_i, F_{i-1}(y_{i-1})) = 0, \quad (5.6)_i$$

$$i = 1, 2, \ldots, N,$$

$$\left\| y_i - x_i \right\| \leqslant 2q_1^{i-1} \left[d(x_1, E_0(x_0)) + q_1^{i-2} \; d(x_2, F_1(x_1)) + \right.$$

$$\left. + \ldots + q_1^0 \; d(x_i, F_{i-1}(x_{i-1})) \right], \; i = 1, 2, \ldots, N. \quad (5.7)_i$$

Indeed, if $z = (x_0, x_1, \ldots, x_N) \in B(\overset{\circ}{z}, \varepsilon)$, then

$$\left\| x_i - \overset{\circ}{x}_i \right\| < \varepsilon < \frac{\delta}{2} \text{ for all } i = 0, 1, \ldots, N. \quad (5.8)$$

Setting $y_0 = x_0$, we get $(5.5)_0$. Now, since $F_0(x_0)$ is closed. we can pick $y_1 \in F_0(x_0)$ such that

$$\left\| y_1 - x_1 \right\| \leqslant 2 \, d(x_1, F_0(x_0)).$$

We then have

$$\left\| y_1 - x_1 \right\| \leqslant 2 \, d(x_1, F_0(x_0)) \leqslant 2b(z) < 2 \frac{\delta}{4K_1} < \frac{\delta}{2};$$

and this together with (5.8) implies $(5.5)_1$. We have thus constructed $y_i, i = 0, 1$ satisfying $(5.5)_0$, $(5.5)_1$, $(5.6)_1$, $(5.7)_1$. Assume now that $(k + 1)$ points y_0, $y_1, \ldots, y_k$ $(k < N-1)$ satisfying $(5.5)_i$, $(5.6)_i$, $(5.7)_i$ for all $i \leqslant k$ have been constructed. Let us find a pionty$_{k+1}$ such that conditions $(5.5)_{k+1}$, $(5.6)_{k+1}$, $(5.7)_{k+1}$ hold. Indeed, in view of $(5.5)_k$ and $\left\| x_k - \overset{\circ}{x}_k \right\| < \varepsilon < \delta$ we get

$$h(F_k(x_k), F_k(y_k)) \leqslant q \left\| x_k - y_k \right\| < q_1 \left\| x_k - y_k \right\|. \quad (5.9)$$

The set $F_k(x_k)$ being closed, there exists a point

$$\overline{y}_{k+1} \in F_k(x_k) \quad (5.10)$$

such that

$$\| x_{k+1} - \bar{y}_{k+1} \| \leqslant 2 d(x_{k+1}, F_k(x_k)). \qquad (5.11)$$

Using (5.9) and (5.10) we can choose y_{k+1} satisfying $(5.6)_{k+1}$ and

$$\| \bar{y}_{k+1} - y_{k+1} \| < q_1 \| x_k - y_k \| . \qquad (5.12)$$

Hence,

$$\| x_{k+1} - y_{k+1} \| \leqslant q_1 \| x_k - y_k \| + 2 d(x_{k+1}, F_k(x_k)) \leqslant$$
$$\leqslant 2 \left\{ q_1^k d(x_1, F_0(x_0)) + q_1^{k-1} d(x_2, F_1(x_1)) + \ldots + \right.$$
$$\left. + q_1^0 d(x_{k+1}, F_k(x_k)) \right\} \leqslant 2 K_1 b(z)$$

e.g. $(5.7.)_{k+1}$ holds.

On the other hand,

$$\left\| y_{k+1} - \overset{o}{x}_{k+1} \right\| \leqslant \left\| y_{k+1} - x_{k+1} \right\| + \left\| x_{k+1} - \overset{o}{x}_{k+1} \right\| \leqslant$$
$$\leqslant 2K_1 b(z) + \varepsilon \leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which proves $(5.5)_{k+1}$.

By adding $(5.7)_i$ from $i = 1$ to N, we obtain

$$\| y - z \| \leqslant 2k_1 b(z) = K b(z)$$

where $y = (y_0, y_1, \ldots, y_N)$.

By virtue of $(5.6)_i$ we have $b(y) = 0$. Therefore, $y \in \{ y' : b(y') = b(\overset{o}{z}) = 0 \}$.

This completes the proof of the regularity of $\overset{o}{z}$.

Note that $\overset{o}{z}$ is a solution of Problem (5.1). We now invoke Theorem 2 in [7] to deduce a positive number r such that

$$0 \in \partial \widehat{g(\overset{o}{z})} + r \, \partial \, b(\overset{o}{z}),$$

hence

$$0 \in \partial \widehat{g(\overset{o}{z})} + r \sum_{k=0}^{N-1} \partial \, d_k(\overset{o}{x}_k, \overset{o}{x}_{k+1}).$$

This shows that conditions (3.4), (3.5), (3.6) hold for some $x_i^* \in X^*$, $i = 0, 1, \ldots, N$. To complete the proof of the Theorem, it remains to note that (3.7) is an immediate consequence of condition (3.4) and the following fact [18]:

$$F_k(\overset{o}{x}_k) - \overset{o}{x}_{k+1} \subset t_k(o).$$

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REFERENCES

1. Boltianskii V.G. *A support principle in optimal control problems.* Differensialnye uravnenya. N°8, 1973, 1363 – 1370 (in Russian).
2. Pham Huu Sach. *A support principle for discrete processes.* Differensialnye uravnenya, N°8, 1974, 1485 — 1496 (in Russian).
3. Pham Huu Sach. *A support principle for a general extremum problem. Z. Vycisl. Mat. i Mat. Fiz.*, N°2, 1978, 338 — 350 (in Russian).
4. Pham Huu Sach. *A support principle for a discrete inclusion with a vector-valued criterion function.* Acta Mathematica Vietnamica, N°2, 1979, 64 — 87.
5. Morduhovitch B.S. *Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems.* Doklady Akad Nauk. SSSR, N°5, V. 254, 1980, 1072 — 1076 (in Russian).
6. Pham Huu Sach. *Les points réguliers des applications multivoques et la commandabilité dans les systèmes discrets.* Université de Bordeaux, Analyse Appliquée et Informatique, N°8022, 1980, 1 – 47.
7. Ioffe A. D. *Necessary and sufficient conditions for a local minimum I. A reduction, theorem and first order conditions.* SIAM J. Control and Optimization, vol. 17, N°2, 1979, 245 — 250.
8. Neustadt L. W. *An abstract variational theory with applications to a broad class of optimization problems: I. General theory.* SIAM J. Control, vol. 4, N°3, 1966, 505 – 527.
9. Hoang Tuy. *On necessary conditions for optimality.* Colloquia Mathematica Sosietatis Janos Bolyai. Progress in Operations Research, Eger (Hungary), 1974, p. 921 – 939.
10. Rockafellar R.T. *Generalized directional derivatives and subgradients of nonconvex functions,* Can. J. Math., vol.32, N°2, 1980, 257 – 280.
11. Pshenitchnyi B. N. *Convex analysis and extremal problems.* Nauka, Moscow, 1980 (in Russian).
12. Ioffe A.D. Tihomirov V.M. *Theory of extremal problems.* Nauka, Moscow, 1974.
13. Pham Huu Sach. *On the control theory for processes given by set-valued maps.* Kibernetika, N°2, 1976, 107 – 116 (in Russian).
14. Dzyuba N. N., Pshenitchnyi B. N. *On the discrete maximum principle.* Kibernetika, N°2, 1975, 46 – 49 (in Russian).
15. Clarke F.H. *Generalized gradients and applications.* Trans. Amer. Math. Soc., vol.205, 1975, 247 – 262.
16. Clarke F.H. *A new approach to Lagrange multipliers.* Math. of Oper. Res. vol.1, N°2, 1976, 165 — 174.
17. Ioffe A.D. *Regular points of Lipschitz functions.* Trans. Amer. Math. Soc., vol.251, 1979, 61 – 69.
18. Pham Huy Dien. *Locally Lipschitz set-valued maps and general extremal problems with inclusion constraints.* Thesis, Hanoi, 1983.